

A theoretical digression

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Firstly, you should make liberal use of the linearity of the Laplace transform. One situation in which this came up on this week's homework (HW 3) was computing the Laplace transform of the greatest integer function, $[x]$. The way I was suggesting to do this was to write

$$[x] = \sum_{n=1}^{\infty} u(x - n)$$

where $u(x)$ is the function which is equal to 1 for positive x and 0 for negative x (sometimes called the Heaviside function—Heaviside was a genius, but he was really arrogant/egotistic/unpleasant and nobody wanted to work with him, so all he got was this stupid function).

Anyway, the formula above holds (pointwise) because when x is a number between say m and $m + 1$, the first m terms of the infinite sum are equal to 1, and the rest are 0. If you use the linearity property, this is a nice approach because earlier in the problem you already computed the $L[u(x - a)]$. However, something I glossed over about this approach is that we need to use something about the *continuity* of the Laplace transform. That is, what we want to write is

$$L\left[\sum_{n=1}^{\infty} u(x - n)\right] = \sum_{n=1}^{\infty} L[u(x - n)].$$

Recall that an infinite sum is defined to be the limit of the corresponding partial sums. In other words, we want to say

$$L\left[\lim_{N \rightarrow \infty} \sum_{n=1}^N u(x - n)\right] = \lim_{N \rightarrow \infty} L\left[\sum_{n=1}^N u(x - n)\right] = \lim_{N \rightarrow \infty} \sum_{n=1}^N L[u(x - n)].$$

The second equality is fine because of linearity (and induction), but the first is some sort of statement about continuity.

Recall that one of the equivalent definitions for continuity of a function $f : \mathbf{R} \rightarrow \mathbf{R}$ is that if you have a convergent sequence of numbers $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$, or in other words $f(\lim x_n) = \lim f(x_n)$. So continuous functions are exactly those which you can pull limits out of.

Here, instead of numbers we have functions, because L is a transform between spaces of functions. We've been pretty fast and loose about this, and mostly haven't worried about what spaces of

functions we're working on. In particular, we haven't said what we mean by a limit of functions (i.e. we haven't specified the topology of space of functions). The formula for $[x]$ that I gave (the infinite sum of $u(x - n)$) holds as a pointwise limit, but this isn't good enough to guarantee the kind of continuity that we need.

If you take something like Math 131 or Math 245, you prove lots of things of the form

$$\int \lim_{N \rightarrow \infty} g_N dx = \lim_{N \rightarrow \infty} \int g_N dx.$$

In our case, the functions g_N are the N th partial sums of u 's, multiplied by e^{-px} . i.e. We use the above as the third equality in this calculation:

$$\begin{aligned} L\left[\sum_{n=1}^{\infty} u(x - n)\right] &= \int_0^{\infty} e^{-px} \sum_{n=1}^{\infty} u(x - n) dx \\ &= \int_0^{\infty} e^{-px} \lim_{N \rightarrow \infty} \sum_{n=1}^N u(x - n) dx \\ &= \lim_{N \rightarrow \infty} \int_0^{\infty} e^{-px} \sum_{n=1}^N u(x - n) dx \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_0^{\infty} e^{-px} u(x - n) dx \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-px} u(x - n) dx \\ &= \sum_{n=1}^{\infty} L[u(x - n)]. \end{aligned}$$

Suffice it to say that there are several theorems (e.g. the dominated convergence theorem, or monotone convergence theorem, if you want to look them up on Wikipedia) that say that this is valid in the case we are considering.

If you do the problem the other way, where you break up the integral into a sum of integrals over intervals of the form $[n, n + 1]$, you avoid the above theoretical difficulty, but it's a messier calculation.