Combinatorial Proofs

A combinatorial proof consists of counting the same collection of things in two different ways, or alternatiely of exhibiting a bijective correspondance between two sets, and counting the number of elements of each set.

Example: A combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad 1 \leq k \leq n-1. \tag{1}$$

Proof. Recall that $\binom{n}{k}$ is the number of ways to choose a collection of k objects from a collection of n objects. We claim that the right hand side of (1) represents the same number.

Pick one of the objects. A collection of size k either includes this object or does not. Let us investigate the number of collections of each type:

- There are $\binom{n-1}{k-1}$ collections including the object, because we must choose $k-1$ of the $n-1$ other objects to complete the collection of size k.
- There are $\binom{n-1}{k}$ collections of k objects that do not include the object, because we must choose all k objects out of the $n-1$ other objects.

Therefore there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ ways to choose of k objects out of n. Therefore, identity (1) holds. □