HW 8

(1) ~~Minimize~~ Find point satisfying $Ax=b$ s.t. closest to origin. A is $m \times n$ $n \geq m$ rank $= m$.

Using Lagrange multipliers

we want to minimize $\|X\|^2$ given $Ax=b$

so $$\nabla f = 2\vec{X} = A^T\lambda \qquad \text{so} \qquad X = \frac{1}{2}A^T\lambda$$ $$Ax=b$$

& $$A\,\tfrac{1}{2}A^T\lambda = b \implies AA^T\lambda = 2b$$

Now since rank $A = m =$ rank A^T then $\ker(A^T)A^T = \ker A^T = \{0\}$

so AA^T is invertible & $\lambda = (AA^T)^{-1}2b$

& $$X = \frac{1}{2}A^T(AA^T)^{-1}2b = \boxed{A^T(AA^T)^{-1}b}$$

Also note $\nabla^2 f = Id$ so $Z^T\nabla^2 f Z = Z^TZ$ which is positive definite

(2) $\bar{X} \in \mathbb{R}^n$ Z is $n \times r$ matrix $f: \mathbb{R}^n \to \mathbb{R}$

Let $\varphi: \mathbb{R}^r \to \mathbb{R}$ $\qquad \varphi(v) = f(\bar{x} + Zv)$

find $\nabla\varphi$ & $\nabla^2\varphi$

Using chain Rule $\nabla\varphi = \nabla[f(\bar{x}+Zv)] = \nabla(\bar{x}+Zv)\,(\nabla f)(\bar{x}+Zv)$

where [illegible]

if $x = \bar{x} + Zv$ $\qquad \nabla x = (\nabla x_1, \dots \nabla x_n)$

now $X_i = (\bar{X})_i + Z_i v$ so $\nabla x_i = Z_i^T$ & $\nabla x = Z^T$

so $\nabla \bar{x} + Zv = Z^T$ ~~[illegible]~~

So $$\boxed{\nabla\varphi = Z^T(\nabla f)(\bar{x}+Zv)}$$

$$\nabla^2\varphi = \nabla^2[f(\bar{x}+Zv)] = \nabla^2(\bar{x}+Zv)(\nabla f)(\bar{x}+Zv) + \nabla X(t)\,(\nabla^2 f)(\bar{x}+Zv)\,\nabla X(t)^T$$

where ~~[illegible]~~

$$\nabla^2(\bar{x}+Zv)(\nabla f)(\bar{x}+Zv) = \nabla^2(x)\,\nabla f(x) = \sum_{i=1}^{n} \nabla^2 X_i \cdot (\nabla f(x))_i$$

but $\nabla^2 X_i = (0)$ since $\frac{\partial \bar{x}+Zv}{\partial v_i \partial v_j} = 0$ & $\nabla X(t) = Z^T$ so $\boxed{\nabla^2\varphi = Z^T\nabla^2 f(\bar{x}+Zv)Z}$

(3) $x \in \mathbb{R}^n$

Z $n \times r$ basis for Nullspace of $m \times n$ Matrix A

Show $Z^T x = 0 \iff$ existence of $\lambda \in \mathbb{R}^m$ s.t. $x = A^T \lambda$

$\Leftarrow$: if $x = A^T \lambda$ then $Z^T x = Z^T A^T \lambda = (AZ)^T \lambda$

now $AZ = 0$ since columns of ~~A~~Z are a basis for $\ker A$

$\Rightarrow Z^T x = 0$

$\Rightarrow$: $Z^T x = 0$ note rows of Z^T are in $\ker A$

So $x \cdot \ker A = 0 \Rightarrow x \in (\ker A)^\perp$

So we need to show $(\ker A)^\perp \subseteq Im(A^T)$

In fact will show ~~A~~ $(\ker A)^\perp = Im(A^T)$

This is the same as $\ker A = Im(A^T)^\perp$

$$
\begin{aligned}
Im(A^T)^\perp &= \{x : x \cdot A^T \lambda = 0 \text{ for } \forall \lambda\} \\
&= \{x : Ax \cdot \lambda = 0 \text{ for } \forall \lambda\} \\
&= \{x : Ax = 0\} = \ker A
\end{aligned}
$$

for this step take $\lambda = Ax$

So their equal & $(\ker A)^\perp = Im A^T \Rightarrow x \in Im A^T$

So $x = A^T \lambda$ for some λ.

(4) Maximize $x_1 \cdots x_n$

Subject to $\frac{x_1}{a_1} + \cdots + \frac{x_n}{a_n} = 1 \quad a_i > 0$

Setting up Lagrange multipliers $A = \left(1/a_1 \;\cdots\; 1/a_n \right)$

$$\frac{\partial f}{\partial x_i} = \frac{x_1 \cdots x_n}{x_i} \quad \text{For } x_i \neq 0$$

~~Assume~~ if $x_i = 0$ $\quad x_1 \cdots x_n = 0$, won't be a minimum

Assuming $x_i \neq 0$ then $x_i \frac{\partial f}{\partial x_i} = x_1 \cdots x_n = f$

& $\nabla f = A^T \lambda = \begin{pmatrix} 1/a_1 \\ \vdots \\ 1/a_n \end{pmatrix} \lambda$ so $\begin{pmatrix} f \\ \vdots \\ f \end{pmatrix} = \begin{pmatrix} \frac{x_1 \lambda}{a_1} \\ \vdots \\ \frac{x_n \lambda}{a_n} \end{pmatrix}$

So $\frac{x_i}{a_i}$ is constant for all i & since $\sum_1^n \frac{x_i}{a_i} = 1$

$\Rightarrow \frac{x_i}{a_i} = \frac{1}{n}$ so $x_i = \frac{a_i}{n}$.

lets compute $\nabla^2 f$ as $\frac{\partial f}{\partial x_i \partial x_j} = \begin{cases} 0 & i = j \\ \frac{x_1 \cdots x_n}{x_i x_j} & i \neq j \end{cases}$

$$\text{So } \nabla^2 f\left(\frac{a_1}{n} \cdots \frac{a_n}{n}\right) = \begin{pmatrix} 0 & \cdots & \frac{a_1 \cdots a_n}{n^n a_1 a_n} \\ \frac{a_1 \cdots a_n}{n^n a_1 a_2} & \ddots 0 & \vdots \\ \frac{a_1 \cdots a_n}{n^n a_1 a_n} & \cdots & 0 \end{pmatrix} = \frac{a_1 \cdots a_n}{n^n} \begin{pmatrix} 0 & 1/a_1 a_2 & \cdots & 1/a_1 a_n \\ \vdots & \ddots & & \vdots \\ 1/a_1 a_n & \cdots & & 0 \end{pmatrix}$$

also we can check $Z = \begin{pmatrix} a_1 & \cdots & a_1 & a_1 \\ -a_2 & & & \\ & -a_3 & & 0 \\ 0 & & \ddots & \\ & & & -a_n \end{pmatrix}$

$$\& \; Z^T \nabla^2 f Z = \frac{a_1 \cdots a_n}{n^n} \begin{pmatrix} a_1 & -a_2 & & 0 \\ \vdots & & \ddots & \\ a_1 & 0 & & -a_n \end{pmatrix} \begin{pmatrix} 0 & 1/a_1 a_2 & \cdots & 1/a_1 a_n \\ \vdots & \ddots & & \vdots \\ 1/a_1 a_n & \cdots & & 0 \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_1 \\ -a_2 & & \\ & \ddots & 0 \\ 0 & & -a_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & -a_2 & & 0 \\ \vdots & & \ddots & \\ a_1 & 0 & & -a_n \end{pmatrix} \begin{pmatrix} -1/a_1 & \cdots & -1/a_1 \\ 1/a_2 & & 0 \\ 0 & \ddots & 1/a_n \end{pmatrix} = \begin{pmatrix} -2 & & -1 \\ -1 & & -2 \end{pmatrix}$$

So we just need to show $M = \begin{pmatrix} -2 & & -1 \\ -1 & & -2 \end{pmatrix}$ is negative definite

Now $\text{nullity}(A-(-1)I) = \text{nullity}\begin{pmatrix} -1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & -1 \end{pmatrix} = n-1$ since $\text{rank}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = 1$

~~then~~ so -1 is an eigenvalue multiplicity $n-1$

we have one more eigenvalue λ multiplicity 1

& $(-1)(n-1) + \lambda = \text{tr } M = -2n \implies \lambda = -n-1$

So we have eigenvalues -1 & $-n-1 < 0$ so M

is ~~[illegible]~~ negative definite $\implies$ $x_i = \frac{a_i}{n}$ is a local maximum $\square$