(1)

Hw 7

Q symmetric $n\times n$ matrix c, $n\times 1$ column.

Find Hessian & gradient of $w(x) = x^TQx + c^Tx$

Find necessary & sufficient conditions for a minimum at some point

$\nabla w(x) = ?$

$$w(x) = \sum_{i,j} q_{ij} x_i x_j + \sum_i c_i x_i$$

$$= \sum_i q_{ii} x_i^2 + \sum_{\substack{i,j \\ i\neq j}} q_{ij} x_i x_j + \sum_i c_i x_i$$

$$\frac{\partial}{\partial x_k} w(x) = 2q_{kk}x_k + \sum_{i\neq k} q_{ik} x_i + \sum_{\substack{j \\ k\neq j}} q_{kj} x_j + c_k$$

since $q_{ik} = q_{ki}$

$$= 2\sum_{i=1}^n q_{ik} x_i + c_k = 2(Q)_k \vec{x} + c_k$$

$$\boxed{\nabla w(x) = 2Qx + c}$$

$$\frac{\partial^2}{\partial x_\ell \partial x_k} w(x) = \frac{\partial}{\partial x_\ell}\left(2\sum_{i=1}^n q_{i,k} x_i + c_k\right) = 2q_{\ell,k}$$

So $\nabla^2 w = \left(\frac{\partial^2}{\partial x_i \partial x_j}\right) = \boxed{2Q}$

necessary conditions $\nabla w = 2Qx + c = 0$

$\nabla^2 w = 2Q$ positive semi definite

Sufficient conditions $\nabla w = 2Qx + c = 0$

$\nabla^2 w = 2Q$ positive definite

② Use Chain Rule to prove if f & g are smooth functions on $\mathbb{R}^n$

$$\nabla fg = f\nabla g + g\nabla f$$

Using Chain Rule

$$A(x_1, x_2) = x_1 x_2 \qquad \nabla A = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$x_1(t_1 \ldots t_n) = f$$

$$x_2(t_1 \ldots t_m) = g$$

$$\nabla fg = \nabla A(x_1(t), x_2(t)) =$$

$$= (\nabla x_1(t), \nabla x_2(t)) \cdot (\nabla A)(x_1(t), x_2(t))$$

$$= (\nabla f, \nabla g) \cdot \begin{pmatrix} x_2(t) \\ x_1(t) \end{pmatrix} = (\nabla f, \nabla g) \cdot \begin{pmatrix} g \\ f \end{pmatrix}$$

$$= \nabla f \cdot g + \nabla g \cdot f.$$

More Direct way

$$\frac{\partial}{\partial x_i} fg = g \frac{\partial}{\partial x_i} f + f \frac{\partial}{\partial x_i} g$$

so $(\nabla fg)_i = g(\nabla f)_i + f(\nabla g)_i$

for all i

$$\Rightarrow \nabla fg = g\nabla f + f\nabla g$$