Pg 37 (6) Assume ∇f is continuous & $p^T \nabla f(x_k) < 0$ then

HW #6

$$f(x_k + \varepsilon p) < f(x_k) \quad \text{for } \varepsilon > 0 \text{ suff. small}$$

By the Mean Value theorem

$$f(x_k + \varepsilon p) = f(x_k) + \varepsilon p^T \nabla f(\xi) \quad \text{for } \xi \text{ between } x_k \text{ \& } x_k + \varepsilon p$$

Since ∇f is continuous

$\nabla f(\xi)$ is close to $\nabla f(x_k)$ for ε small enough,

in particular we can choose it so that $p^T \nabla f(\xi)$ is still negative

Since $p^T \nabla f(x_k)$ was negative.

So $f(x_k + \varepsilon p) - f(x_k) = \varepsilon p^T \nabla f(\xi) < 0$ proving the result.

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(2) $f: \mathbb{R} \to \mathbb{R}$ smooth x_* a point s.t. $f(x^*) = 0$ $f'(x^*) \neq 0$

show $\exists M, \varepsilon$ (doesn't depend on k) s.t. if $|x_0 - x^*| < \varepsilon$ & x_k newton iterates

of f started at x_0 then $x_k \to x^*$ & $|x_{k+1} - x^*| < M |x_k - x^*|$

Since f is smooth $\Rightarrow$ f' & f'' are continuous

so we can pick $\varepsilon' > 0$ s.t. $|\xi - x^*| < \varepsilon'$ $\quad |f'(\xi)| \geq \frac{|f'(x^*)|}{2}$

$|f''(\xi)| \leq 2|f''(x^*)|$

Let $M = 2\frac{|f''(x^*)|}{|f'(x^*)|}$ & let $\varepsilon = \min(\varepsilon', 1/2M)$

I claim these will work.

We prove our claim by induction

Claim $|x_{k+1} - x^*| < \varepsilon$ & $|x_{k+1} - x^*| < M|x_k - x^*|$

For $k=0$ since $|x_0 - x^*| < \varepsilon \leq \varepsilon'$ $\Rightarrow$ $|f'(x_0)| \geq \frac{|f'(x^*)|}{2} > 0$ so $f'(x_0) \neq 0$

& $0 = f(x^*) = f(x_0) - (x_0 - x^*) f'(x_0) + \frac{1}{2}(x_0 - x^*)^2 f''(\xi)$ where ξ is between x_0 & x_*

Dividing by $f'(x_0)$ to we find

$$0 = \frac{f(x_0)}{f'(x_0)} - (x_0 - x_*) + \frac{1}{2}(x^* - x_0)^2 \frac{f''(\xi)}{f'(x_0)}$$

$$\Rightarrow \quad x_0 - \frac{f(x_0)}{f'(x_0)} - x_* = \frac{1}{2}(x^* - x_0)^2 \frac{f''(\xi)}{f'(x_0)}$$

since $|x_0 - x^*|, |\xi - x^*| < \varepsilon$

$$|x_1 - x_*| = |x_0 - x^*|^2 \left(\frac{1}{2}\frac{|f''(\xi)|}{|f'(x_0)|}\right) \Rightarrow \frac{1}{2}\frac{|f''(\xi)|}{|f'(x_0)|} \le \frac{2|f''(x_*)|}{|f'(x^*)|} = M$$

So $|x_1 - x_*| \le M|x_0 - x^*|^2$ & $|x_1 - x_*| \le M\varepsilon^2 \le M\frac{1}{2m}\varepsilon < \varepsilon$

proving its true for $k=1$

For the rest of the induction $k=N \Rightarrow k=N+1$

its exactly the same idea

$|x_N - x^*| < \varepsilon \Rightarrow |f'(x_N)| \ge \left|\frac{f'(x^*)}{2}\right| > 0$ so $f'(x_N)$ to

& $|x_{N+1} - x_*| = |x_N - x_*|^2 \frac{1}{2}\frac{|f''(\xi)|}{|f'(x_0)|}$ for ξ between x_N & x_*

Then $|x_N - x_*|, |\xi - x_*| < \varepsilon \Rightarrow |x_{N+1} - x_*| \le |x_N - x_*|^2 \frac{2|f''(x_*)|}{|f'(x_*)|} = M|x_N - x_*|^2$

& $|x_{N+1} - x_*| \le M|x_N - x_*|^2 < M\varepsilon^2 \le \varepsilon$.

proving the claim.