HW 2

Pg. 83 (2) In standard form we have

$$\begin{pmatrix}1&1&1&0\\1&2&0&1\end{pmatrix}\vec{X} = \begin{pmatrix}5\\6\end{pmatrix} \qquad \vec{X}\geq 0$$

(a) Extreme points $\Leftrightarrow$ basic feasible solutions

Basic variables $= \begin{matrix}x_1\\x_2\end{matrix} \Rightarrow B=\begin{pmatrix}1&1\\1&2\end{pmatrix} \quad B^{-1}\begin{pmatrix}5\\6\end{pmatrix} = \begin{matrix}4\\1\end{matrix} \Rightarrow \begin{pmatrix}4\\1\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}4\\1\end{pmatrix}$

$= \begin{matrix}x_1\\x_3\end{matrix} \Rightarrow B=\begin{pmatrix}1&1\\1&0\end{pmatrix} \quad B^{-1}b = \begin{pmatrix}6\\-1\end{pmatrix} \Rightarrow \begin{pmatrix}6\\0\\-1\\0\end{pmatrix} \not\geq 0$ X

$\begin{matrix}x_1\\x_4\end{matrix} \Rightarrow B=\begin{pmatrix}1&0\\1&1\end{pmatrix} \quad B^{-1}b = \begin{pmatrix}5\\1\end{pmatrix} \Rightarrow \begin{pmatrix}5\\0\\0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}5\\0\end{pmatrix}$

$= \begin{pmatrix}x_2\\x_3\end{pmatrix} \Rightarrow B=\begin{pmatrix}1&1\\2&0\end{pmatrix} \quad B^{-1}b = \begin{pmatrix}3\\2\end{pmatrix} \Rightarrow \begin{pmatrix}0\\3\\2\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0\\3\end{pmatrix}$

$\begin{matrix}x_2\\x_4\end{matrix} \quad B=\begin{pmatrix}1&0\\2&1\end{pmatrix} \quad B^{-1}b = \begin{pmatrix}5\\-4\end{pmatrix} \Rightarrow \begin{pmatrix}0\\5\\0\\-4\end{pmatrix} \not\geq 0$ X

$\begin{matrix}x_3\\x_4\end{matrix} \quad B=\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad B^{-1}b = \begin{matrix}5\\6\end{matrix} \Rightarrow \begin{pmatrix}0\\0\\5\\6\end{pmatrix} \Rightarrow \begin{pmatrix}0\\0\end{pmatrix}$

$\boxed{\text{4 extreme points are } \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}5\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}}$

(b) $Ad=0 \Rightarrow d \in \ker A = \operatorname{span}\left(\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-2\end{pmatrix}\right)$

so $d = s\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\\-2\end{pmatrix}$

& $d\geq 0 \Rightarrow \begin{matrix}s\geq 0\\ t\geq 0\\ -s-t\geq 0\\ -s-2t\geq 0\end{matrix} \Rightarrow s=0 \quad t=0 \Rightarrow d=0$

So there are no direction of unboundedness

3 (a) In standard form we have

$$\begin{pmatrix}-2&1&1&0\\-1&1&0&1\end{pmatrix}\vec{x}=\begin{pmatrix}2\\3\end{pmatrix}\qquad \vec{x}\geq 0$$

to be a direction of unboundedness

$$Ad=0$$

$$\Rightarrow \quad \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \quad \Rightarrow \quad d=\begin{pmatrix}1\\0\\2\\1\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \quad \Rightarrow \quad d=\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

(b) If d is a direction of unboundedness

$$Ad=0 \quad \& \quad d\geq 0$$

$$\Rightarrow d\in \ker A=\text{span}\left\{\begin{pmatrix}1\\0\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right\}$$

so $d = s\begin{pmatrix}1\\0\\2\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}$ for some $s,t\in\mathbb{R}$

furthermore $d\geq 0 \Rightarrow$

$$\begin{aligned}&s\geq 0\\&t\geq 0\\&2s\geq t\\&s\geq t\end{aligned} \quad\Rightarrow\quad s\geq t\geq 0$$

in particular $d = (s-t)\underbrace{\begin{pmatrix}1\\0\\2\\1\end{pmatrix}}_{d_1}+t\underbrace{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}_{d_2}$

where $s-t$ & $t\geq 0$

i.e. $d = (s-t)d_1+td_2$ a nonnegative combination of d_1 & d_2

(5) $Ax = b$ & $x \geq 0$

Prove If $d \neq 0$ satisfies $Ad = 0$ & $d \geq 0$ $\Rightarrow$ d is a direction of unboundedness

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Need to show $\forall x \in S$ & $\gamma \geq 0 \Rightarrow x + \gamma d \in S$

where S is the feasible points

Assuming $Ad = 0$ & $d \geq 0$ $d \neq 0$

for any $x \in S$ Consider $x + \gamma d$

$\gamma \geq 0$

$$A(x + \gamma d) = Ax + \gamma Ad = Ax + \gamma \cdot 0 = Ax = b$$

since $x \in S$

$x + \gamma d \geq 0$ since $x \geq 0$ (it's in S)

$d \geq 0$ & $\gamma \geq 0$

$\Rightarrow$ $x + \gamma d \in S$ $\Rightarrow$ d is a feasible direction.

page 92 (3) X is a convex combination of $\{y_1 \cdots y_k\}$

y_i is a convex combination of $\{y_{i,1} \cdots y_{i,k_i}\}$

Prove X is a convex combination of $y_{i,j}$

$$X = \sum_{i=1}^{k} \alpha_i y_i \qquad \alpha_i \geq 0 \quad \sum_{i=1}^{k} \alpha_i = 1$$

for each i $$y_i = \sum_{j=1}^{k_i} \beta_{i,j} y_{i,j} \qquad \beta_{i,j} \geq 0 \quad \sum_{j=1}^{k_i} \beta_{i,j} = 1$$

So $$X = \sum_{i=1}^{k} \alpha_i \sum_{j=1}^{k_i} \beta_{ij} y_{i,j} = \sum_{i=1}^{k} \sum_{j=1}^{k_i} \alpha_i \beta_{ij} y_{ij}$$

need to show
- $\alpha_i \beta_{ij} \geq 0$ clear since α_i & $\beta_{ij} \geq 0$
- $\sum_{i=1}^{k} \sum_{j=1}^{k_i} \alpha_i \beta_{ij} = 1$ but this is $\prod_{i=1}$

$\prod_{i}$

$$\sum_{i=1}^{k}\sum_{j=1}^{k_i} \alpha_i B_{ij} = \sum_{i=1}^{k} \alpha_i \left(\sum_{j=1}^{k_i} B_{ij}\right) = \sum_{i=1}^{k} \alpha_i = 1 \checkmark$$

(5) Let $(d_1 \dots d_k)$ be directions of unboundedness for $Ax=b$ $x \geq 0$ show a nonzero vector

$$d = \sum_{i}^{k} \alpha_i d_i \text{ with } \alpha_i \geq 0$$ is also a direction of unboundedness.

need to check $d \neq 0$ (given)

$$Ad = 0$$
$$d \geq 0$$

$$Ad = A\sum_{i=1}^{k} \alpha_i d_i = \sum_{i=1}^{k} \alpha_i A d_i = \sum_{i=1}^{k} \alpha_i 0 = 0$$

since $Ad_i = 0$

$$d = \sum_{i=1}^{k} \alpha_i d_i \geq 0$$ since $\alpha_i \geq 0$ & $d \geq 0$

So d is a direction of unboundedness