UCLA Analysis Qual Study Guide, Version 2.2

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NOTE: I) The earlier versions of this guide were beta-tested by Anush Tserunyan.

II) This version has been edited to reflect the qualifying exams given in 2009, especially the notorious S09 exam.

Part I): Introduction

A) What this guide is NOT

1. This guide is NOT an analysis textbook. In other words, it is not self-contained (for example, there will be very few detailed proofs in here; it is expected that the reader will read this with the help of an analysis textbook, or at least a working knowledge of basic graduate level analysis.)

2. This guide does NOT contain solutions to every qual problem that could be asked (or even every qual problem that was asked); the aim of this guide is not for the reader to memorize a list of solutions to qualifying problems (which would be comparable to trying to learn differential calculus by memorizing the derivative of every possible function) but for the reader to learn which techniques in analysis are considered the most important and how to use them in the setting of an exam. Further, there were a total of 221 qual problems asked since the fall of 2001... and the present author has no interest in writing out 221 detailed solutions.

3. This guide is NOT a copy of Jon Handy's guide; although the present author (and everyone else in his generation) learned how to pass the analysis qualifying exam with the help of Handy, the present author disagrees with Handy's approach in a number of places (for example, Handy's guide followed a Schaum's Outline style approach of almost no explanation of mathematics but an attempt to work out every qual problem imaginable; further, Handy's solutions are sometimes so convoluted that the present author often found it easier to invent his own solutions!) That being said, all versions of this guide are numbered with version numbers beginning with 2 out of tribute to Handy; Handy's guide is the original and this is a new, second, version.

4. This guide is NOT intended for use for qualifying exams at schools other than UCLA. That being said, analysis is the same regardless of where one goes to school, so people may find this useful even if their analysis qual looks different from UCLA's.

B) Basic structure of the guide

The analysis qualifying exam is divided into two sections (real and complex analysis); consequently, Part II) of the guide will focus on the real analysis section and Part III) will focus on the complex analysis section.

Each of these parts will be organized as follows:

1. The set of all questions asked in the 'modern era' (i.e. since Fall 01; note that this is the first qual in the current format and also the first qual to be released online) will be divided into categories with a brief explanation for each category.

2. A detailed breakdown of all of the questions from the modern era by category (listed by number; for example: "F06 Q1".) The present author believes that this is the most important part of the guide because it shows, in tabular form, what the Department thinks is particularly important in analysis.

3. Each category will receive a more detailed explanation (with a list of what needs to be known) and some worked examples from past qualifying exams. In general, three examples of varying difficulty from the most popular categories (i.e., the ones that appear almost every qualifying exam - or even more frequently) and one or two examples from the others were worked out. Some of these solutions may be sketches or outlines instead of detailed proofs (where the reader is invited to fill in the details); on an actual qualifying exam, the reader should put in about as much explanation as he or she would provide in a regular exam for a graduate-level analysis class.

In each category, each problem will also be rated for difficulty on the following six-point scale:

E (Easy): Problems which require very little from the reader, both in terms of knowledge and in terms of computational skill.

ME (Medium easy): Problems which are fairly straightforward, but not quite as easy as those problems rated E (this is the level where most 'easy' qual problems reside).

M (Medium): Standard problems of about average difficulty (the borderline student may or may not be able to solve these problems; the well-prepared student generally should).

MH (Medium hard): Problems which are either a bit more computationally involved or a bit trickier than standard (this is the level where most 'hard' qual problems reside).

H (Hard): Problems which, due to their difficulty, are likely to provide a challenge for well-prepared students, though not quite hard enough to be considered impossible.

VH (Very hard): Problems which are sufficiently difficult that no one (not even top students) are expected to solve them. An indicator of a question of this difficulty level is that people are still talking about this problem long after the qualifying exam... or if no one still knows how to solve it, even a month after the exam. (It should be noted that only five problems fit into this last category).

C) A word to the reader

1. Each section (real or complex) normally has five to eight questions, of which the exam candidate is invited to answer ten total (of which at least four lie in each section). The goal of the reader should be to be able to completely answer at least four questions from any given qualifying exam section; no one who has done so has ever been known to fail. While people have often managed to pass with less (for example, the present author passed in S06 with only three completely correct answers in each section; however, it should be noted that the complex analysis portion of that qual was the hardest in modern times and also that the present author got enough other questions close to being correct that his total score was the equivalent of 8.5 questions completely answered), this is not guaranteed. No one wants to be the person who fails an easy exam with a score in the sixties or low seventies! (It should also be noted that anyone who performs substantially above this standard as a first-year is virtually guaranteed to win a Horn-Moez prize under ordinary circumstances.)

2. The best way to attain this goal is to be able to answer all of the easy questions (even in a hard topic such as Fourier analysis or analytic continuation, there are easy questions which may be asked) and most of the medium questions (except possibly in an isolated area of weakness or two) while being able to handle the hard questions in at least a few topics (and also understanding the solution of a hard problem). While it is theoretically possible to attain this goal by learning *most* of the topics to perfection, the reader should be advised against attempting this because it is possible to ask hard-to-impossible questions on relatively straightforward topics (for example, the hardest question asked in modern times on a complex analysis qual was in the standard 'undergraduate' topic of residue theory).

Good luck with your qual preparation!

Part II): Real Analysis

A) References

The main references for this section of the exam are Gerald Folland's book "Real Analysis: Modern Techniques and Applications" (chapters 1-8, mostly excluding chapter 7) and the book "Real Analysis" by Elias Stein and Rami Shakarchi (chapters 1-6). Every once in a while, though, H.L. Royden's book "Real Analysis" will be cited as a supplementary reference.

Although one could get by with just Folland's book (though one might want another source for Fourier analysis), Stein-Shakarchi omits too many topics (from introductory functional analysis to L^p spaces) to be a complete reference for the analysis qual.

B) List of Subtopics

The following are the broad groupings of questions which may appear on the real analysis subsection of the qualifying exam (sorted by the order in which they appear in Folland) together with the abbreviations that I use for them.

BAS: Material covered on the analysis section of the basic examination that UCLA graduate students normally take upon entrance. This material usually includes sequences, series, and metric space topology (though the present author will include point-set topology in this prerequisite material).

MT: Standard theory of Lebesgue measure and integration, which is typically covered in the introductory graduate course Math 245A at UCLA.

FUN: General functional analysis; this primarily means the theory of Banach spaces (but excluding L^p spaces and Hilbert spaces). This abbreviation should NOT be considered an endorsement by the present author of the topic!

HIL: The basic theory and geometry of Hilbert spaces; even though this topic is covered in a mere section of a chapter of Folland's book, it appears more often than the rest of functional analysis combined.

LP: The theory of L^p spaces of functions (i.e. those functions f such that $|f|^p$ are Lebesgue integrable).

FA: Fourier analysis, which includes the basic properties of both the Fourier transform and Fourier series.

FA/HIL: Fourier analysis questions whose solutions require nontrivial properties of Hilbert spaces (the abbreviation stands for

"Fourier analysis questions with Hilbert space theory").

MISC: The small handful of difficult questions that do not fall into the preceding seven categories.

C) Breakdown of the Real Analysis Quals by Category

Since real analysis and complex analysis were combined into a single qualifying exam in Fall 2001, there have been 18 qualifying exams in analysis, in which there were 122 questions in the real analysis section. They break down as follows:

F01 Q1: BAS Q2: BAS Q3: BAS Q4: FA/HIL Q5: FUN Q6: FUN

W02 Q1: BAS Q2: BAS Q3: MT Q4: MT Q5: LP Q6: FUN Q7: FUN Q8: HIL

S02 Q1: MISC Q2: BAS Q3: FUN Q4: MT Q5: LP Q6: MT Q7: FA/HIL Q8: HIL

F02 Q1: MT Q2: LP Q3: LP Q4: LP Q5: MISC Q6: MISC Q7: HIL Q8: FUN Q9: FA/HIL

W03 Q1: MT Q2: MISC Q3: LP Q4: MT Q5: FUN Q6: MT Q7: LP Q8: HIL

F03 Q1: MT Q2: MT Q3: LP Q4: HIL Q5: HIL Q6: LP Q7: HIL

W04 Q1: BAS Q2: BAS Q3: MT Q4: LP Q5: FA Q6: HIL

F04 Q1: MT Q2: LP Q3: MT Q4: MT Q5: FUN Q6: MISC Q7: BAS

W05 Q7: BAS Q8: LP Q9: MT Q10: FA Q11: MT Q12: HIL

F05 R1: MT R2: MT R3: BAS R4: FUN R5: FA

S06 Q1: MT Q2: MT Q3: MT Q4: BAS Q5: LP Q6: HIL

F06 Q1: FA Q2: MT Q3: LP Q4: FA Q5: HIL Q6: HIL

W07 Q1: MT Q2: BAS Q3: FA (barely) Q4: LP Q5: FUN Q6: FUN

F07 Q1: FA Q2: BAS Q3: FUN Q4: MT Q5: BAS Q6: BAS Q7: HIL Q8: FA

S08 Q1: FA Q2: MT Q3: BAS Q4: LP Q5: HIL Q6: MT Q7: MISC Q8: HIL

F08 Q1: LP Q2: FUN Q3: MT Q4: BAS Q5: FA Q6: MT

S09 Q1: MT Q2: HIL Q3: FUN Q4: MT Q5: MT Q6: MISC

F09 Q1: HIL Q2: HIL Q3: MT Q4: MT Q5: MT Q6: MISC

In descending order by popularity, we therefore have MT (34 questions), BAS (18 questions), HIL (18), LP (16), FUN (14), FA (11), MISC (8), and FA/HIL (3).

In other words, the most popular topic (basic Lebesgue measure theory) appears nearly twice every exam. The next three topics (basic exam material, Hilbert spaces, and L^p spaces) are sufficiently popular to appear nearly exactly once every exam. General functional analysis and Fourier analysis appear on most, but not all, exams (with an average of .824 and .647 questions per exam, respectively), whereas a miscellaneous question will appear less than once every other exam (and questions in the hybrid category of 'FA/HIL' have not appeared since 2002).

D) A Look at the Categories

NOTE: The questions below are graded by difficulty on the following scale; E = easy, ME = medium-easy, M = medium, MH = medium-hard, H = hard, and VH = very hard. However, it should be noted that this merely reflects the present author's opinion!

BAS (Basic Exam questions)

Reference: None needed (the reader should already have learned this stuff when studying for the Basic Exam). That being said, Folland covers some of this topic in Chapter 0, while the standard reference for the Analysis portion of the Basic Exam is the first nine chapters of Principles of Mathematical Analysis (by Walter Rudin).

List of questions: F01 Q1 (ME), F01 Q2 (E), F01 Q3 (ME), W02 Q1 (ME), W02 Q2 (ME), S02 Q2 (ME), W04 Q1 (ME), W04 Q2 (M), F04 Q7 (H), W05 Q7 (M), F05 R3 (ME), S06 Q4 (M), W07 Q2 (M), F07 Q2 (ME), F07 Q5 (ME), F07 Q6 (M), S08 Q3 (ME), F08 Q4 (ME)

Frequency: 1 question per exam (common; further, these questions often provide the easiest points on the exam!)

Description: A number of questions asked on the Analysis Qual require no special graduate-level knowledge; rather, they can be solved with nothing more than an honors undergraduate-level analysis course (such as the Bootcamp offered to incoming students at UCLA). As this is a guide to prepare for the Analysis Qual rather than the Basic Exam, this topic will not be covered in detail in this guide (the reader should already know the topics of sequences and series, derivatives of real-valued functions, Riemann integrals, and basic metric space topology). Instead, this guide will discuss the hardest question asked in the modern era on an Analysis Qual which falls into this category.

F04 Q7: Suppose f_n is a differentiable real-valued function on $[0, 1]$ such that $\{f_n\}$ converges pointwise and $\{f'_n\}$ converges uniformly. Prove that $\{f_n\}$ converges uniformly to an everywhere-differentiable limit.

Proof: This is Theorem 7.17 in Rudin's book. To proceed we let f be the pointwise limit of $\{f_n\}$ and g be the uniform limit of $\{f'_n\}$; we seek to show I) $f_n \to f$ uniformly and II) $f' = g$.

I): Pick $\epsilon > 0$ and set N such that $||f'_m - g|| < .1\epsilon$ (in the uniform norm) for $m > N$. Also suppose $|f_m(0) - f(0)| < .1\epsilon$ for such m. If $m_1, m_2 > N$ then for each $x \in [0, 1]$,

$$|f_{m_1}(x) - f_{m_2}(x)| \leq |f_{m_1}(0) - f_{m_2}(0)| + |(f_{m_1} - f_{m_2})(x) - |(f_{m_1} - f_{m_2})(0)|$$

$$< .2\epsilon + |(f_{m_1} - f_{m_2})'(t)| * |x|$$

for some $t \in [0, x]$ (the first term comes from the fact that $f_{m_1}(0)$ and $f_{m_2}(0)$ are both with $.1\epsilon$ of $f(0)$; the second term comes from applying the Mean Value Theorem to $f_{m_1} - f_{m_2}$)

$$\leq .2\epsilon + |f'_{m_1}(t) - f'_{m_2}(t)| < .2\epsilon + .1\epsilon < .3\epsilon$$

(this is independent of x) which tells us that $\{f_n\}$ is uniformly Cauchy and therefore converges uniformly.

II) Pick $x \in [0,1]$; to show $f'(x) = g(x)$ we consider the function $\phi_n(t)$ (for each n) defined to equal $\frac{f_n(t)-f_n(x)}{t-x}$ for $t \neq x$ and $f'_n(x)$ at $t = x$. As the ϕ_n are continuous and converge uniformly (if $t \neq x$ the mean value theorem enables us to write the numerator of $\phi_n - \phi_m$ as $(f'_n - f'_m)(s) * |t - x|$ for some s between t and x; at $t = x$ we already have the pointwise convergence to $g(x)$) the limiting function ϕ (which sends $t \neq x$ to $\frac{f(t)-f(x)}{t-x}$ and x to $g(x)$) is continuous which, by the definition of derivative, says f is differentiable with $f'(x) = g(x)$ for all x.

MT (Measure Theory)

Reference: Folland Chapters 1-3 (with the heaviest emphasis on Chapter 2), OR Stein-Shakarchi Chapters 1-3 and 6. Those who don't like Folland's treatment of functions of bounded variation can look at Chapter 5 of Royden's book (Real Analysis) for a clearer presentation of this topic.

List of questions: W02 Q3 (ME), W02 Q4 (M), S02 Q4 (ME), S02 Q6 (ME), F02 Q1 (ME), W03 Q1 (ME), W03 Q4 (ME), W03 Q6 (ME), F03 Q1 (ME), F03 Q2 (M), W04 Q3 (E), F04 Q1 (ME), F04 Q3 (MH), F04 Q4 (M), W05 Q9 (MH), W05 Q11 (M), F05 R1 (ME), F05 R2 (MH), S06 Q1 (MH), S06 Q2 (M), S06 Q3 (M), F06 Q2 (MH), W07 Q1 (ME), F07 Q4 (E), S08 Q2 (ME), S08 Q6 (M), F08 Q3 (ME), F08 Q6 (M), S09 Q1 (M), S09 Q4 (M), S09 Q5 (MH), F09 Q3 (M), F09 Q4 (MH), F09 Q5 (M)

Frequency: 1.889 questions per exam (very common; further, these questions are more likely to be easy or 'standard' than hard)

Description: The most important component of the real analysis qual (and the topic which separates graduate level analysis from undergraduate level analysis) is the theory and basic constructions of Lebesgue measure and integration. In addition to appearing by itself on nearly two questions per exam, two other subcategories (Fourier analysis and L^p spaces) build on the material in this section.

Basically, this section emphasizes everything covered in the introductory analysis course Math 245A (definition and basic properties of Lebesgue measure and the Lebesgue integral, the main convergence results: monotone convergence, Fatou's Lemma, and dominated convergence, convergence in measure/probability versus 'almost everywhere' or uniformly, Fubini's theorem with product measures). Further, the Lebesgue-Radon-Nikodym theorem (breaking up a measure into 'absolutely continuous' and 'singular' parts), Lebesgue differentiation, and functions of bounded differentiation (all of which are found in Chapter 3 of Folland's book and may appear in either 245A or 245B) have all been known to be tested on the analysis qual, albeit emphasized a little less than the 'core 245A' material.

Because of the importance of this category of questions, FOUR questions will be discussed.

F07 Q1: Let $f_n : \mathbf{R} \to \mathbf{R}$ be nonnegative integrable functions with L^1 norm 1. Suppose $f_n \to f$ pointwise a.e. with $||f||_1 = 1$ and show $\int_A f_n(x)dx \to \int_A f(x)dx$ uniformly in the choice of Borel set $A \subset \mathbf{R}$.

Proof: It suffices to show $f_n \to f$ in L^1 as

$$|\int_A f_n(x)dx - \int_A f(x)dx| \leq \int_A |f_n(x) - f(x)|dx \leq \int |f_n(x) - f(x)|dx$$
$$= ||f_n - f||_1.$$

However, $||f_n - f||_1 \to 0$ (apply Fatou's lemma to $-|f_n - f| + |f_n| + |f|$ to get that the lim inf of $\int(-|f_n - f| + |f_n| + |f|)$ is bounded below by $2 = \lim ||f_n||_1 + ||f||_1$) as desired.

F08 Q6: For $n \geq 1$ we create the sets C_n as follows: at the first stage we remove the central open interval of length $\frac{1}{2^n} * \frac{1}{3}$ and at the jth stage, for each of the 2^{j-1} untouched intervals of length $3^{-(j-1)}$ we remove the middle segment of length $\frac{1}{2^n} 3^{-j}$. (For $n = 0$ this would construct the regular Cantor set, which we call C_0).

(a) With μ = Lebesgue measure, show that $\mu([0,1] - \bigcup_{n=1}^{\infty} C_n = 0$

Proof: To construct C_n, we remove intervals of total length $2^{-n}(\frac{2}{3})^{j-1} * 3^{-1}$ at the jth stage for a total length removed of 2^{-n}. Therefore, as C_n has measure $1 - 2^{-n}$, $\bigcup C_n$ is a subset of $[0,1]$ with full measure giving the desired result.

(b) Show that if $E \subset [0,1]$ is not Lebesgue measurable there exists $n \geq 1$ with $E \cap C_n$ not Lebesgue measurable.

Proof: There exists $\epsilon > 0$ and some set A with $\mu(E) < \mu(E \cap A) + \mu(E \cap A^c) - \epsilon$ (we suppose $A \subset [0,1]$); letting n be such that $\mu(C_n) > 1 - .1\epsilon$,

$\mu(E \cap C_n) < \mu(E) < \mu(E \cap C_n \cap A) + \mu(E \cap C_n \cap A^c) - .8\epsilon$ so $E \cap C_n$ is not measurable.

(c) Use (b) to show there is a continuous, strictly increasing function $f : \mathbf{R} \to \mathbf{R}$ with full image and a Lebesgue measurable A with $f(A)$ nonmeasurable.

Proof: With n fixed as in the last part, we note from point-set topology that C_0 is homeomorphic to C_n for each n (by an increasing map which can be thought of as sending an interval of a given scale in the construction of C_0 for the corresponding interval in the construction of C_n) and let f be this homeomorphism, which we extend to the entire line by linear interpolation on the rest of $[0,1]$ and by defining $f(x) = x$ outside of $[0,1]$. Letting $E \cap C_n$ be as in the preceding part, it is nonmeasurable; however, $f^{-1}(E \cap C_n)$ is a subset of C_0, which is a nullset, so letting $A = f^{-1}(E \cap C_n)$, A is measurable but $f(A)$ is not.

W05 Q9 (a): Let f be an L^1 function on the real line and $\{a_n\}$ be a sequence of real numbers with $\Sigma \frac{1}{|a_n|} < \infty$; show that $g(x) = \Sigma f(a_n x)$ converges a.e. and g is L^1.

Proof: Note that $\int \Sigma |f(a_n x)| = \int \Sigma |a_n|^{-1} |f(x)| = ||f||_1 \Sigma |a_n^{-1}| < \infty$, so $\Sigma f(a_n x)$ converges absolutely a.e. (otherwise, the integral would be infinite); its limit is in L^1 by dominated convergence.

(b) Let $\{a_n\}$ be such that $\Sigma \frac{1}{|a_n|} = \infty$ and show the result from the preceding part is false for f the characteristic function of $[0,1]$.

Proof: The sum of the first N terms has integral $\Sigma_{n<N} |a_n|^{-1}$ (note that each summand is the characteristic function of an interval of length $|a_n|^{-1}$) which goes to infinity with N.

S09 Q5: Let $I = I_{0,0} = [0,1]$; for $n \in \mathbf{Z}_{\geq 0}$ and $0 \leq j \leq 2^n - 1$ let $I_{n,j} = [j2^{-n}, (j+1)2^{-n}]$.

For $f \in L^1(I, dx)$ define $E_n f(x) = \Sigma_{j=0}^{2^n-1} 2^n \int_{I_{n,j}} f\,dt\, \chi_{I_{n,j}}$.

Show that $E_n f \to f$ almost everywhere on I.

Proof: Let S be the class of all functions $f \in L^1(I, dx)$ such that $E_n f \to f$ a.e. on I. Clearly S contains the characteristic functions of dyadic intervals as $E_m \chi_{I_{n,j}} = \chi_{I_{n,j}}$ for $m > n$. Approximating a positive L^1 function f by a sequence of positive linear combinations of characteristic functions of dyadic intervals gives that such an S lies in f by monotone convergence; as S is clearly a vector space, writing a general L^1 function as the difference of its positive and negative parts gives the desired result.

FUN (General functional analysis)

Reference: Folland Sections 5.1-5.4, OR Chapter 10 of "Real Analysis" by Royden (note that this topic does NOT appear in Stein-Shakarchi, though it is due to appear in the rumored fourth volume of their series). Those who need a review of metric and point-set topology can consult Chapter 4 of Folland or Chapters 7-9 of Royden before proceeding to the main references.

List of questions: F01 Q5 (H), F01 Q6 (ME), W02 Q6 (ME), W02 Q7 (M), S02 Q3 (M), F02 Q8 (ME), W03 Q5 (E), F04 Q5 (MH), F05 Q4 (ME), W07 Q5 (E), W07 Q6 (false as written), F07 Q3 (M), F08 Q2 (M), S09 Q3 (ME)

Frequency: .778 questions per exam (common; these questions vary wildly in difficulty, and one question in particular has appeared four times)

Description: Now the guide moves to "245B" topics, starting with general functional analysis, i.e. those functional analysis topics which involve neither Hilbert spaces nor general L^p function spaces. Usually, these questions involve the theory of general Banach spaces: one should know the Hahn-Banach Theorem (extension of functionals defined on a subspace to the whole space) and consequences of the Baire Category Theorem (namely: the Open Mapping Theorem - stating that continuous surjective linear Banach space maps send open sets to open sets, the Closed Graph Theorem - saying that linear Banach space operators are continuous iff their graphs are closed - and the Principle of Uniform Boundednes for linear maps). It should be noted that Alaoglu's theorem (pre-compactness of the unit ball in the weak-star topology) belongs in this area; although it has never been tested directly, it has been known to be useful for other questions.

The following questions shall be discussed in this section.

F08 Q2 (also asked as W02 Q7, S02 Q3, and F07 Q3)

Is every vector space isomorphic as a vector space to some Banach space? Prove your answer.

Proof: We show the vector space V which consists of all infinite sequences which have only finitely many terms nonzero cannot be given a Banach space structure (we write elements as $(v_1, v_2, \ldots, v_n, \ldots)$.)

Assuming to the contrary we let V_n be the subspace of V consisting of those elements which have all coordinates zero after the nth one; V_n is closed (finite-dimensional normed vector spaces are isomorphic as Banach spaces to $\mathbf{R}^n$) but contains no open balls so the Baire category theorem says that the union of the V_n cannot equal V; however, V is the union of the V_n producing a contradiction.

F01 Q5:

(a) Which of the following vector spaces: $l^1(2), l^2(2), l^\infty(2)$ are isometrically isomorphic? (these spaces are $\mathbf{R}^2$ equipped with the corresponding norms)

Proof: Recalling that an extreme point of a set is a point that is in the interior of no nontrivial line segments in the set, we note that Banach space isomorphisms send extreme points of unit balls to extreme points of unit balls. However, $l^2(2)$ has a circle for its unit ball; the other two spaces have squares as unit balls (which have extreme points) so it can't be isomorphic to the others. An isometric isomorphism from l^1 to l^∞ can be defined as $f((x, y)) = (x+y, x-y)$.

(b) Same question but for three-dimensional space.

Proof: The unit balls here are an octahedron, a sphere, and a cube respectively (with six, zero, and eight extreme points respectively) so none of these spaces are isometrically isomorphic.

W07 Q6: Let X be a Banach space and let $A : X \to X$ be a linear map; set $\rho(A)$ be the set of complex λ with $\lambda - A$ surjective and show $\rho(A)$ is an open subset of $\mathbf{C}$.

NOTE: This question is false as written because of the fact that the operator need not be bounded and the highly nonstandard definition of resolvent used; the question in fact deals with surjectivity of non-bounded operators on Banach spaces!

In fact, let V be the space of infinite sequences of the form $(v_1, \ldots, v_n, \ldots)$; as V is isomorphic as a **vector space** to $l^\infty(\mathbf{N})$ (their bases have the same cardinality), it can be given a Banach space structure induced by this isomorphism. Letting $A : V \to V$ send $(v_1, \ldots, v_n, \ldots)$ to $(\frac{v_1}{1}, \ldots, \frac{v_n}{n}, \ldots)$ we have that $\rho(A)$ contains 0 but no number of the form $\frac{1}{n}$ for n an integer and therefore is not open.

That being said, the standard definition of the resolvent requires that $\lambda - A$ have a bounded inverse; in this case, the resolvent is indeed open. To see this, it suffices to note that if A has a bounded inverse B and $\epsilon < ||B||^{-1}$ then $(A + \epsilon I)^{-1} = B\Sigma_{i=0}^{\infty}(-\epsilon B)^i$.

HIL (Hilbert space theory)

Reference: Folland Section 5.1-5.5, OR Chapter 4 of Stein-Shakarchi.

List of questions: W02 Q8 (ME), S02 Q8 (ME), F02 Q7 (ME), W03 Q8 (M), F03 Q4 (ME), F03 Q5 (ME), F03 Q7 (E), W04 Q6 (E), W05 Q12 (ME), S06 Q6 (ME), F06 Q5 (ME; note that the problem assumes T is extended to general elements as follows: $T(\Sigma_n a_n e_n) = a_n \Sigma_n (Te_n))$, even for infinite sums) F06 Q6 (ME), F07 Q7 (M), S08 Q5 (ME), S08 Q8 (ME), S09 Q2 (M), F09 Q1 (ME), F09 Q2 (M)

Frequency: 1 question per exam (common; while usually not trivial, these questions are nearly always either straightforward or standard)

Description: This is the last topic which is definitely covered in 245B (the next three topics are sometimes covered in 245B and covered other times in 245C). As Hilbert spaces are covered in a single section of Folland's book (which mentions the standard results like Cauchy-Schwarz, continuity of inner products, the parallelogram law, points with minimal norm, self-duality, Bessel's inequality and Parseval's identity), the questions tend to require little more than the basics of Hilbert space theory. That said, the Cauchy-Schwarz inequality (that $| < a, b > | \leq ||a||||b||$) is often called upon in fairly clever ways; for example, the trick to solving F09 Q2 is to view the sum $\Sigma a_{m,n}$ as an inner product of $(a_{m,n}(1 + m^2 + n^2))$ and $((1 + m^2 + n^2)^{-1})$ (both of which are vectors ranging over m and n).

The following two questions are representative of what could be asked:

F06 Q6: Let D be the unit disc in the complex plane, endowed with the usual Lebesgue measure. Let $H = L^2(D)$ be the space of square-integrable complex-valued functions on D.

(1) Show that $\{z^n : n \geq 0\}$ are orthogonal in H.

Proof: If $n \neq m$ then the inner product of z^n and z^m is equal to the integral of $z^n \bar{z^m}$ with respect to the disc. Integrating around a circle of radius r (substitute $z = re^{i\theta}$, $dz = izd\theta$) gives $\int_0^{2\pi} r^m e^{i(n-m)\theta} d\theta = 0$; therefore, the same holds for integration over the disc.

(2) Is $\{||z^n||_H^{-1} z^n : n \geq 0\}$ an orthonormal basis for H?

No: consider the function $\bar{z}$; this function is orthogonal to every element in this set (the inner product of z^n and $\bar{z}$ is the integral of z^{n+1} over the disc; use an identical computation to (1) for the result) so the closed span of the set above is contained in the complement of $\bar{z}$ and therefore not equal to all of H.

NOTE: If H referred to analytic functions (not just complex-valued ones) the answer would be yes.

F07 Q7 (this was also asked as W03 Q8):

(a) Prove: If $T : H \to H$ is a linear map with $||I - T|| < 1$ (where I is the identity map) then T is invertible.

Proof: The series $\Sigma_{n=0}^{\infty}(I - T)^n$ converges in the norm topology to a map which is seen by direct computation to be the left and right inverse of $I - (I - T) = T$.

(b) Suppose $\{e_n\}$ and $\{f_n\}$ are orthonormal sets in H with $\Sigma||e_n - f_n||^2 < 1$ such that $\{e_n\}$ is a Hilbert space basis (i.e. a complete orthonormal set). Show that $\{f_n\}$ is a Hilbert space basis as well.

Proof: Define the map $T : H \to H$ to send an element $\Sigma a_n e_n$ to $\Sigma a_n f_n$. We represent T by an infinite 'matrix' as follows: because we want the jth column to represent $f_j = T(e_j)$, the entry in the ith row and jth column is $< f_j, e_i >$. Under this procedure, I would be represented as a matrix with 1's on the diagonals and zeroes everywhere else and $I - T$ is represented as a matrix (called d_{ij}) such that the sum of the squares of the norms of all the entries is strictly less than 1 (the jth column would represent $e_n - f_n$ in the usual manner). Letting α_i be the sum of the squares of the elements in the ith row, we note that if $x = \Sigma a_n e_n$ and $(I - T)x = \Sigma b_n e_n$ then $b_n = \Sigma_j d_{nj} a_j e_j$ and $||b_n||^2 =\leq (\Sigma_j |d_{nj}|^2)(\Sigma_j |a_j|^2) = \alpha_i ||x||^2$ (by Cauchy-Schwarz) so $||(I - T)x||^2 \leq ||x||^2 \Sigma_i \alpha_i$ which implies that $||T||$ (in operator norm) is at most $\sqrt{\Sigma_i \alpha_i} < 1$. We then appeal to (a) to show that T is invertible, which implies that T sends Hilbert space bases to Hilbert space bases and therefore $\{f_n\}$ is a Hilbert space basis.

LP (L^p spaces)

Reference: Folland Chapter 6 (Section 1 is emphasized; and Sections 2-4 appear rarely while Section 5 has never been tested). It should be noted that Stein-Shakarchi does not mention this topic (again, it could end up in the rumored fourth volume of that series); those who have a hard time with Folland's treatment may want to begin with Chapter 6 of Royden (which provides a clear treatment of the function spaces on $[0, 1]$ as well as the space of sequences) or read the present author's L^{primer}, which can be found at www.math.ucla.edu/ meyerson/lprimer.pdf .

List of questions: W02 Q5 (M), S02 Q5 (VH), F02 Q2 (H), F02 Q3 (M), F02 Q4 (M), W03 Q3 (MH), W03 Q7 (M), F03 Q6 (E), W04 Q4 (MH), F04 Q2 (H), W05 Q8 (M), S06 Q5 (M), F06 Q3 (E), W07 Q4 (MH), S08 Q4 (ME)

Frequency: .889 question per exam (common; these questions are more likely to be challenging than easy or standard)

Description: This can be thought of as the last of the 'functional analysis' topics (broadly speaking) which appear on the analysis qual. The main idea is as follows: if a measure space is fixed, the L^p functions (for $1 \leq p < \infty$) are those functions f such that $|f|^p$ is Lebesgue integrable (with norm $(\int |f|^p)^{1/p}$) while the L^∞ functions are those functions which are bounded outside a nullset (with norm equal to the supremum of $\{r : |f| \leq r a.e.\}$). Minkowski's inequality (the triangle inequality) shows that these are indeed norms (further, as the L^p spaces are complete, they are Banach spaces); the majority of the questions on this topic can be done using only Minkowski's inequality and its cousin, Holder's inequality (which says that if $p^{-1} + q^{-1} = 1$ then $||fg||_1 \leq ||f||_p||f||_q$). Sometimes, other concepts are needed to handle these questions (questions in the past have required, among other things, Stirling's formula, the mean ergodic theorem, distribution functions, and Minkowski's inequality for integrals), making a topic which was already challenging even harder.

The following questions demonstrate the type of reasoning one would generally use to solve these types of questions.

F02 Q3: Let (X, M, μ) be a finite measure space and let $1 \leq p < \infty$. Let $f_1, f_2, \ldots$ be a sequence in $L^p(X, M, \mu)$ which converge pointwise a.e. to $f \in L^p(X, M, \mu)$. Show that $\lim_{n\to\infty} ||f_n - f||_p = 0$ if and only if $\lim_{n\to\infty} ||f_n||_p = ||f||_p$.

Proof: "only if": Trivial (by Minkowski's inequality which shows that the p-norm is a norm).

"if": The case $p = 1$ follows from applying Fatou's lemma to $|f_n| + |f| - |f - f_n|$. In general, if the p-norm limit condition holds, then the $p = 1$ case tells us that $sgn(f) * |f|^p - sgn(f_n) * |f_n|^p$ converges to 0 in L^1 (the sgn function sends positive numbers to 1 and negative numbers to -1).

Now, if $x \geq y \geq 0$ then $x^p - y^p \geq p(x - y)x^{p-1} \geq p(x - y)^p$,

If $x \geq 0 \geq y$ and $|x| \geq |y|$ then $x^p + y^p \geq x^p \geq 2^{-p}(x + y)^p$.

As the other cases can be treated similarly, there exists a constant C such that $|sgn(f) * |f|^p - sgn(f_n) * |f_n|^p| \geq C|f - f_n|^p$; therefore, $|f - f_n|^p \to 0$ in L^1 so $|f - f_n| \to 0$ in L^p as desired.

W04 Q4: Suppose f is L^∞ wrt the finite measure space $< X, \mu, B >$; let $\alpha_n = \int |f|^n$ and show that the limit of $\frac{\alpha_{n+1}}{\alpha_n}$ as n goes to ∞ is $||f||_\infty$.

Proof: Clearly $\alpha_{n+1} \leq ||f||_\infty \alpha_n$; to show the other direction use Holder's inequality on α_n (noting $|f|^n$ is in $L^{(n+1)/n}$ and 1 is in L^{n+1}) to conclude

$$\alpha_n = ||f^n||_1 \leq ||f^n||_{(n+1)/n}||1||_{n+1} = \alpha_{n+1}^{n/(n+1)}\mu(X)^{1/(n+1)}.$$

Consequently, $\frac{\alpha_n}{\alpha_{n+1}} \leq (\frac{\mu(X)}{\alpha_{n+1}})^{1/(n+1)}$, from which

$$\frac{\alpha_{n+1}}{\alpha_n} \geq (\alpha_{n+1})^{1/(n+1)}(\mu(X))^{1/(n+1)} = ||f||_{n+1}(\mu(X))^{1/(n+1)}$$

which approaches $||f||_\infty$ as n goes to ∞, giving the desired lower bound.

S02 5. Let (X, M, μ) be a measure space with $\mu(X) = 1$.

Let $f \in L^1(X, M, \mu)$. Prove

$$\lim_{p\to 0}(\int |f|^p d\mu)^{1/p} = \exp(\int \log|f| d\mu)$$

where $e^{-\infty}$ is defined to be 0. Hint: Jensen's inequality.

Note: This is actually Question 8c in Section 6.1 in Folland's Real Analysis book, made harder by the fact that Questions 8a and 8b, which provide an outline of the proof, are not shown (in fact, this is generally considered the hardest question asked in modern times on the real analysis section; even if 8a and 8b were given at hints, it would still be an MH or H-level problems!).

Proof: The best way to proceed is to show that the expression on the left is I) bounded below and II) bounded above by the expression on the right.

Before beginning, though, we note that if we take $\log(0) = -\infty$, log is continuous from $[0, \infty)$ to $[-\infty, \infty)$. Consequently, we can simplify the desired inequality by taking logarithms of both sides: our result is equivalent to

$$\lim_{p\to 0} \frac{1}{p} \log(\int |f|^p d\mu) = \int \log|f| d\mu.$$

I) Jensen's inequality states that if ϕ is a convex function, (X, M, μ) is as in the problem, and g is integrable with respect to μ, $\int \phi \circ g d\mu \geq \phi(\int g d\mu)$.

Observing that $|f|^p = e^{p\log|f|}$, we therefore have that

$$\log(\int |f|^p d\mu) = \log(\int e^{p\log|f|} d\mu) \geq e^{\log(\int p\log|f| d\mu)} = p\int \log|f| d\mu$$

so

$$\frac{1}{p}\log(\int |f|^p d\mu) \geq \int \log|f| d\mu;$$

letting p go to zero gives us the desired result in this direction.

II) Our goal is to bound $\frac{1}{p}\log(\int |f|^p d\mu)$ above by some function $\phi(p)$ which approaches $\exp(\int \log|f| d\mu)$ as p approaches zero.

We do this in stages; first, we get rid of the logarithm by noting from calculus that $\log x \leq x - 1$ for all x (with equality for $x = 1$); this tells us

$$\frac{1}{p}\log(\int |f|^p d\mu) \le \frac{1}{p}(\int |f|^p d\mu - 1) = \int \frac{|f|^p - 1}{p} d\mu$$

(in the last inequality, we remember that $\int 1 d\mu = \mu(X) = 1$).

Our candidate function $g(p)$ is $\int \frac{|f|^p-1}{p} d\mu$; what happens as p goes to zero?

One begins by looking at the behavior of the integrand in the limit; as p approaches 0, $\frac{|f|^p-1}{p}$ (which is differentiable in p for $p \neq 0$) approaches the indeterminate form 0/0 so L'Hopital's rule says that the limit is equal to the limit of $\frac{\log|f| * |f|^p}{1}$ (differentiating with respect to p) as p goes to zero, which is $\log|f|$). (One could also see this by looking at the defnition of derivative).

To show that the integrals indeed converge, all we need is to use an appropriate convergence result. To that end, we state the following version of the Monotone Convergence Theorem:

If, for each n, h_n is an L^1 function on X such that $h_j \ge h_k$ for $j < k$, while the h_n approach some limit function pointwise h on X (where h is measurable and may attain $-\infty$ but not ∞) then $\int h_n d\mu \to \int h d\mu$.

To use this theorem (and finish the problem), it therefore suffices to show that $\frac{|f|^p-1}{p}$ decreases as p approaches 0, i.e. that $\frac{|f|^p-1}{p}$ is an INCREASING function in $p > 0$! This is trivial for $|f| = 0$; we assume this is not the case.

To proceed, we take its p-derivative, using the Quotient Rule, to come up with

$\frac{p|f|^p \log|f| - |f|^p + 1}{p^2}$ and seek to show that it is nonnegative; this is equivalent to showing

$p|f|^p \log|f| - |f|^p + 1 \ge 0$ for all $|f|, p > 0$.

The final trick is to treat this last expression as a function of $|f|$ (noting that it vanishes for $|f| = 1$). Taking its derivative with respect to $|f|$ gives us

$$p^2|f|^{p-1}\log|f| + p|f|^{p-1} - p|f|^{p-1} = |f|^{p-1}(p^2\log|f| - p + p) = p^2|f|^{p-1}\log|f|,$$

which is clearly positive for $|f| > 1$ and negative for $|f| < 1$, so

$p|f|^p \log|f| - |f|^p + 1 \ge 0$ for all $|f|, p > 0$ as desired, which was exactly the result we needed to complete II) and therefore establish the limit.

FA (Fourier Analysis)

Reference: Folland 8.1-8.6 or Stein-Shakarchi 5.1 with Stein-Shakarchi's other book "Fourier Analysis" (chapters 1-6). A clearer treatment of Fourier series than can be found in Folland (Folland leaves most of the major results as exercises) can be found in T.W. Koerner's "Fourier Analysis"; also,

www.math.ucla.edu/ sgautam/245b.06w/fourier.pdf is a clearer guide of the key results for Fourier transforms (the present author used this guide to study for his qual, but must admit he was pleasantly surprised to see no Fourier analysis on his analysis qual.)

List of questions: F03 Q3 (MH), W04 Q5 (MH), W05 Q10 (ME), F05 Q5 (M), F06 Q1 (ME), F06 Q4 (H), W07 Q3 (ME; note that (c) is the only part which uses Fourier analysis), F07 Q1 (M), F07 Q8 (ME), S08 Q1 (H), F08 Q5 (H)

Frequency: .611 questions per exam (fairly common; while many of these questions are hard, there are quite a few fairly easy, if nontrivial, questions asked in this subject)

Description:

1) Fourier series: The setting is a function f defined on the torus, i.e. $[0,1]$ with 0 and 1 identified (or, equivalently, a 1-periodic function where we only care about f on a single period). Then the nth Fourier coefficient, $\hat{f}(n)$, is equal to $\int fe^{-2\pi int}$; the Fourier series representing f is $\Sigma_n \hat{f}(n)e^{2\pi int}$. The main thing that needs to be known is that the Fourier transform preserves L^2 norms: if f is L^2 then $\{\hat{f}\}$ (as a sequence) is L^2 with respect to counting measure; the two functions have the same norm (this is called Parseval's identity).

The other thing (which shows up on particularly hard Fourier series questions) involves integration kernels which turn functions into trigonometric polynomials (i.e. finite Fourier series, 'polynomials' in $\pm e^{in\theta}$.) The main idea is that if $g = \Sigma_n c_n e^{in\theta}$ is a finite trigonometric sum, the convolution of f with g is $\Sigma_n c_n \hat{f}_n e^{in\theta}$. The three most popular kernels are the Dirichlet kernel $D_n = \Sigma_{|m|\leq n} e^{in\theta} = \frac{\sin(.5(m+1)\theta)}{\sin .5\theta}$ (which, in convolution, sends functions to partial Fourier sums S_n), the Fejer kernel K_n which is the average of the K_m for $m \leq n$ ($\sigma_n = \frac{S_n^2}{n+1}$; the resultant convolution sends a function to σ_n, which converges pointwise to f if f is continuous), and the De La Vallee Poussin kernel $V_{kn,(k+1)n} = n^{-1}(((k+1)n + K_{(k+1)n} - (kn+1)K_{kn})$ (the resulting partial sums are denoted by $\sigma_{kn,(k+1)n}$).

2) Fourier transforms: The setting is now a function f on the real line; the Fourier transform is $\hat{f}(t) = \int f(x)e^{-2\pi ixt}dx$. Like the Fourier series, this map preserves the L^2 norm (only now it sends functions to functions) and there is an inversion formula that works if $f, \hat{f}$ are in L^1: $f(x) = \int \hat{f}(t)e^{2\pi ixt}dt$. People should know how Fourier transforms behave with respect to derivatives and complex conjugation (for example: $\hat{\bar{\bar{f}}}(t) = \hat{f}(-t)$.) Surprisingly, even though Fourier transforms are considered a 'graduate' topic and Fourier series are considered 'undergraduate', the hardest questions tend to involve Fourier transforms, not series.

The last thing to note (for both Fourier series and Fourier transforms) is that if h is the convolution of f and g, $\hat{h}$ is the pointwise product of $\hat{f}$ and $\hat{g}$.

Below are two recent questions: one using Fourier transforms and one using Fourier series.

W07 Q3: (1) Show that A_g (sending f to the convolution $f * g$ is a bounded map from $L^1(\mathbf{R})$ to itself.

(2) Suppose $g \geq 0$; find $||A_g||$.

(3) Show that if $f \in L^1$ and $f * f = f$ then $f = 0$.

Proof: (1) As $f * g(x) = \int f(x-y)g(y)dy$, integrating and using Fubini's theorem gives $||A_g(f)||_1 \leq ||f||_1||g||_1$.

(2) If $g \geq 0$ note that $A_g(g)[x] = \int g(x-y)g(y)dy$; integrating with respect to x gives $\int g(x-y)g(y)dxdy = \int_y g(y) \int_x g(x-y)dxdy$; as the inner integral is $||g||_1$ (all functions involved are positive) one concludes $||A_g(g)||_1 = ||g||_1^2$ so $||A_g|| \geq ||g||_1$ (giving equality by the previous result).

(3) NOTE: This is the only part that uses Fourier analysis.

As $\hat{f * f} = \hat{f}^2$, applying Fourier transforms gives that $\hat{f}^2 - \hat{f} = 0$ a.e., i.e. $\hat{f} = 0$ or 1 a.e. However, as $f \in L^1$, $\hat{f}$ is continuous and goes to 0 at infinity (this is the Riemann-Lebesgue theorem; its proof is actually fairly trivial) so $\hat{f} = 0 \in L^1$; Fourier inversion now says $f = 0 \in L^1$.

F06 Q4: Suppose f is continuously differentiable and 2π-periodic: show that its Fourier series is absolutely convergent.

(NOTE: Here the torus is considered to be based on $[0, 2\pi]$ instead of $[0, 1]$; the Fourier coefficients are $\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ and the integral kernels shall be changed accordingly).

Note also that although this question is considered part of the real analysis section, it was actually found in Gamelin's complex analysis book as exercise VI.6.15.

Proof: Because f is continuously differentiable, one can differentiate under the integral for $\hat{f}(n)$ and note that $\hat{f'}(n) = -in\hat{f}(n)$. One can therefore write $\hat{f}(n) = a_n b_n$ where $a_n = \frac{1}{in}$ and $b_n = \hat{f'}(n)$.

The aim is to show that $\infty > \Sigma_n|hatf(n)| = \Sigma_n|a_n b_n|$; the Cauchy-Schwarz inequality gives that

$\Sigma_n|a_n b_n| \leq \sqrt{\Sigma_n|a_n|^2}\sqrt{\Sigma_n|b_n|^2}$; the first term is finite by the convergence of $\Sigma_n n^{-2}$ and the second term is merely the L^2 norm of f, giving the desired result.

NOTE: A tempting way to proceed (for someone who has just learned the De La Vallee Poussin kernel) is as follows:

We know that $\sigma_n(f)$ converges pointwise (in fact, uniformly) to f; we seek to show the same for S_n. To this end, one can fix k and look at

$$\sigma_{kn,(k+1)n}(f) = n^{-1}(\sigma_{(k+1)n}(f)((k+1)n+1) - \sigma_{kn}(f)(kn+1)$$

$$= \sigma_{kn+k}(f)(k+1-n^{-1}) - \sigma_{kn}(f)(k+n^{-1})$$

which inherits uniform convergence to f from the σ_n (note that the "error terms" are bounded by $n^{-1}||f||_\infty$.)

However, $\sigma_{kn,(k+1)n}(f)$ has all the same Fourier coefficients as f (up to the first kn in either direction, has no Fourier coefficients past the $(k+1)n$th, and has Fourier coefficients equal to some fraction of the corresponding coefficient of f in between.

In other words, if one were to look at the Fourier expansion of $\sigma_{kn,(k+1)n}(f) - S_z(f)$ for some $z \in [kn, (k+1)n]$, the only terms they would find would be the terms of order greater than kn but less than $(k+1)n$, and they would all have magnitude smaller than the corresponding terms in the Fourier expansion of f. This gives a uniform bound of Ck^{-1} for $\sigma_{kn,(k+1)n}f - S_z(f)$ (as the mth coefficient is $O(m^{-1})$, there are only $O(n)$ coefficients that appear here all of which are $O((kn)^{-1})$). By choosing k and then n sufficiently large, one can conclude that the Fourier series converges uniformly.

However, this proof does not show absolute convergence; only uniform convergence (which actually is implied by absolute convergence).

FA/HIL (Fourier Analysis using Hilbert space theory)

Reference: None needed (the previous references given for FA and HIL should suffice; the only new thing here is that these topics are combined).

List of questions: F01 Q4 (MH), S02 Q8 (ME), F02 Q9 (H)

Frequency: 1 question every 6 exams (very rare; no questions on this topic have been asked in over six years as the examiners generally now prefer to test Fourier analysis and Hilbert spaces on separate questions)

Description: Historically, one of the major breakthroughs of the Fourier transform was an identification of the 'complicated' space of periodic functions with the 'simple' space of infinite sequences under L^2 norms. This 'simple' space could be further analyzed by standard Hiblert space techniques. In general, Fourier analysis often uses the following two-step heuristic:

I) View a property of functions as a property of elements of a Hilbert space which is easier to understand.

II) Use known facts about Hilbert spaces to draw conclusions about this function.

Therefore, it should not be surprising that forms of this heuristic should be found on qualifying exam questions; the most recent (and hardest) question of this type is below.

F02 Q9: Let $T : L^2([0,1]) \to L^2([0,1])$ be the operator $(Tf)(x) = \int_0^x f(y)dy$.

(a) Show that

$$|\hat{Tf}(n)| \leq \frac{C||f||_2}{|n|}$$

(where the nth Fourier coefficient is $\int_0^1 e^{-2\pi inx} Tf(x)dx$).

Proof: Because T is a continuous linear operator (as $|Tf(x)|^2 \leq \int_0^x |f(y)|^2 \leq ||f||_2^2$ by Cauchy-Schwarz when $x = 1$) and the map sending a function to the nth Fourier coefficient is a continuous functional on L^2 with operator norm at most 1 (by Cauchy-Schwarz again; in fact, equality holds) it suffices to show the bound for a dense subclass of L^2, say, the smooth functions. (Also note that the bound is trivial for $n = 0$ so one may assume this is not the case)

If g is smooth, then denoting Tg by h, integration by parts yields

$$\hat{h}(n) = \frac{1}{-2\pi in}(h(1) - h(0)) + \int_0^1 \frac{1}{2\pi n} e^{-2\pi inx} g(x)dx;$$

as $h(1) \leq ||g||_1 \leq ||g||_2$, the first term is $O(n^{-1}||g||_2)$ uniformly in n, as is the second term by Cauchy-Schwarz, proving the desired result .

NOTE: One could also compute $\hat{Tf}$ by direct computation using Fubini's theorem.

(b) Show that T is a continuous, compact operator.

Proof: Continuity was already shown; for compactness, it suffices to show that the subset of l^2 with the nth coefficient at most n^{-1} is compact. This follows by sequential compactness; for each sequence in this space, extract a subsequence which converges pointwise and note that for each $\epsilon > 0$ there exists N such that the 'N-tail' of the ball (i.e. the points where the coefficients from $-N$ to N are zero) has diameter less than ϵ.

(c) Show that for any complex non-zero λ, $T - \lambda$ has no kernel.

Proof: If $(T-\lambda)f = 0$ then $Tf = \lambda f$ so $f = \lambda^{-1}Tf$. Iterating this equation n times shows that f is $n-1$ times continuously differentiable (by construction of T) so by induction on n, f is smooth. However, taking derivatives yields $f = \lambda f'$ from which standard ODE theory yields $f = Ce^{-\lambda t}$, though $f(0) = 0$ as $Tf(0) = 0$) giving that $C = 0$ so $f = 0$.

(d) Show $T - \lambda$ is invertible on $L^2[0,1]$ if λ is non-invertible.

Proof: Observe that there exists $\epsilon > 0$ such that $||(T-\lambda)f||_2 \geq \epsilon ||f||_2$ for all f (otherwise pick a sequence $\{f_n\}$ of unit vectors with $(T-\lambda)f_n \to 0$; compactness of T as in (b) allows one to pass to a subsequence where the Tf_n converge to some g, which yields that $\lambda f_n \to g$ so $f_n \to \lambda^{-1}g$ is a λ-eigenvector for T contradicting the preceding part).

As the same holds for $(T-\lambda)^* = -T - \bar{\lambda}$ ($T^* = -T$ by integration by parts; as before, it suffices to check this for smooth functions) one concludes that $T - \lambda$ is a continuous bijection and therefore is invertible (by the Open Mapping Theorem, for example).

MISC (Miscellaneous questions)

Reference: Provided after each question.

List of questions: S02 Q1 (M), F02 Q5 (VH), F02 Q6 (H), W03 Q2 (MH), F04 Q6 (MH), F07 Q7 (MH), S09 Q6 (MH), F09 Q6 (MH)

Frequency: 1 question every 2.25 exams (uncommon, but each miscellaneous topic by itself would be 'very rare' or 'extremely rare')

Description: Of the 122 questions asked on the real analysis section of the analysis qualifying exams, eight of them do not fit into the broad categories previously listed. In determining whether or not to create a new category for a broad method (or combination of methods), the following rule of thumb was used: if a topic has had three or more questions asked (such as Hilbert-space intensive Fourier analysis questions, i.e. FA/HIL), it is important enough to warrant its own category, while if questions related to the topic have been asked only once or twice (such as general compact metric spaces) it is placed in the 'miscellaneous' section.

While (usually) not impossible, these questions tend to be fairly difficult; often these questions arise because the author of the qualifying exam is interested in a particular non-standard topic. In these cases, as it is not necessary to answer every question (in fact, the instructions explicitly tell the student to leave two to three questions blank), it is often intended that this will be the question that the students skip.

The best way to deal with these questions is to list them and give hints or a brief outline on how to solve them (along with a reference for the corresponding area of mathematics).

S02 Q1: Let V be a finite-dimensional real vector space and let $||||_V$ be a norm on V. Let P be the set of one-dimensional linear subspaces of V; for $W_1, W_2 \in P$ define $d(W_1, W_2) = \inf\{||v_1 - v_2||_V : v_j \in W_j, ||v_j||_V = 1$.

Prove that d is a metric on P and that P is compact with respect to this metric.

Hint: Showing that d is a metric is trivial. For compactness, note that there is a natural map from the unit sphere in V to P (sending a unit vector to the subspace containing it); use this map and the compactness of the unit sphere.

Reference: This question involves metric topology; the best reference for this topic is chapter 7 of Royden.

F02 Q5: If $0 < \alpha < 1$, show $f(x) = \Sigma_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x)$ is Holder continuous of order α (i.e. there exists C such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all x, y) but nowhere differentiable on $[0, 1]$.

NOTE: If not for the Holder continuity part (which, ironically, is the easier part), this would merely be a (difficult) Fourier analysis question instead of a miscellaneous question. As it is, this is probably the second most difficult real analysis question asked in the modern era.

Hints:

I) Holder continuity: Pick x, y and let n be the largest number such that $2^{-n} \geq |x - y|$.

For the terms $2^{-j\alpha} \cos(2^j x)$, $j \leq n$, note $|2^{-j\alpha} \cos(2^j x) - 2^{-j\alpha} \cos(2^j y)| \leq 2^{j(1-\alpha)}|x - y|$. If $j > n$, note that $|2^{-j\alpha} \cos(2^j x) - 2^{-j\alpha} \cos(2^j y)| \leq 2 * 2^{-j\alpha}$;

use these to bound $|f(x) - f(y)|$ by $C2^{-n\alpha}$.

II) Nowhere differentiability: By observing how translation interacts with Fourier coefficients, it suffices to show nondifferentiability at 0 (and the idea is to show that if f only has an nth Fourier coefficient when n is a power of 2^k, $f(0) = 0$, and $f'(0)$ exists, then the nth Fourier coefficient is $o(2^{-n\alpha})$. One goes about this by noting that letting K_{2^n-1} denote the $(2^n - 1)$st Fejer kernel, integrating $fK_{2^n-1}e^{2^n i\theta}$ gives the 2^nth Fourier coefficient of f here; bounds on this coefficient are found by observing that $f(x) \leq cx$ and treating the regions $|x| < 2^{-n}$ and $|x| \geq 2^{-n}$ separately.

Reference: The best reference for an introduction to nowhere differentiable functions and Fourier analysis is Chapter 11 of "Fourier Analysis" by T.W. Korner.

F02 Q6: Let $n > 1$ be an integer and B be the open unit ball in $\mathbf{R}^n$. Show there exists a constant $C < \infty$ depending only on n such that whenever $u : B \to \mathbf{R}$ is smooth and compactly supported,

$$\int_B |u(x)|^2 dx \leq C \int_B \int |\nabla(x)|^2 dx.$$

Hint: Fixing a point p, one notes that $|u(p)|^2 \leq (\int |u_x(p)|)^2$ (the integral being taken over the horizontal line through p; note that the integral is taken over a line of length at most 2) $\leq 4 \int |u_x(p)|^2$ by Cauchy-Schwarz. Integrating with respect to x, one notes that for each $w \in \mathbf{R}^{n-1}$, setting $\alpha(w) = \int |u(x, w)|^2 dx$ and $\beta(w) = \int |u_x(x, w)|^2 dx$, $\alpha(w) \leq 8\beta(w)$, from which integrating with respect to the other coordinates completes the problem.

Reference: The outline above gives a solution 'by hand' (the result in this problem is called the Sobolev embetting theorem); for a fancier method involving Fourier analysis one can consult the proof of Lemma 5.3.3 of Stein-Shakarchi.

W03 Q2: Prove there is a constant C such that for every closed bounded interval $I = [a, b] \subset \mathbf{R}$ there is a constant α_I such that

$$\int_I |\log |x| - \alpha_I| dx \leq C(b - a).$$

Hint: Take α_I to be the maximal value of $\log |x|$ in the interval (if $0 < a < b$, use $\log b$) so that the expression is always of the same sign; note that for $b > a > 0, \log b - \log a \leq \frac{b-a}{a}$ by the mean value theorem.)

Reference: None needed; the mathematics needed is nothing more than Riemann integration (even if the trick used is highly nonstandard). Formally, this question asks one to say that $\log |x|$ is of "bounded mean oscillation" (BMO); the most elementary exposition of this that the present author could find is from Terence Tao's harmonic analysis lecture notes, at

http://www.math.ucla.edu/ tao/247a.1.06f (section 4.3).

F04 Q6: Let (X, d) be a compact metric space and let F be the set of nonempty compact subsets of X. Define the Hausdorff distance $d_H(K_1, K_2)$ on F to be

$$\max(\sup_{x\in K_1}\inf_{y\in K_2} d(x,y), \sup_{y\in K_2}\inf_{x\in K_1} d(x,y).$$

Assuming (F, d_H) is a metric space show it is complete. (Hint given by the problem: If $d_H(K_1, K_2) \le \epsilon, K_1$ lies in the ϵ-neighborhood of K_2 and K_2 lies in the ϵ-neighborhood of K_1).

Hint: Looking at a Cauchy sequence $\{K_n\}$, one may, by passing to a subsequence, consider the case where $d(K_j, K_n) < 10^{-j}$ for $j < n$.

Letting L_j be the closure of the set of all points within $10 * 10^{-j}$ of K_j, note that $\{d(K_j, L_j)\} \to 0$ and, setting $L = \cap L_j$, $L_j \to L$ so $K_j \to L$.

Reference: This is another metric topology question; as before, the best reference for this topic is chapter 7 of Royden.

S08 Q7: Let $u : \mathbf{R}^n \to \mathbf{R}$ be bounded smooth and suppose its Laplacian is rotationally symmetric, i.e. for each rotation R, $\Delta u \circ R = \Delta u$. Show that u is also rotationally symmetric. (Hint given by the problem: You may use without proof that for each rotation R, $\Delta u \circ R = \Delta(u \circ R)$.)

Hint: As $\Delta u = \Delta u \circ R = \Delta(u \circ R)$, $u - u \circ R$ is bounded harmonic (and therefore constant by the maximum principle, so it must be 0 as it is zero at 0).

Reference: Looking at a general PDE book (such as the one written by L.C. Evans) is overkill; the best reference for the two-dimensional Laplacian is Gamelin's book (whose treatment is, naturally, very complex-analytic in nature). S09 Q6: For $I_{n,j} = [j2^{-n}, (j+1)2^{-n}]$ define the Haar function $h_{n,j} = 2^{n/2}(\chi_{I_{n+1,2j}} - \chi_{I_{n+1,2j+1}})$.

a) Carefully draw $I_{2,1}$ and graph $h_{2,1}$.

b) Prove that if $f \in L^2(I)$ (recall $I = [0,1]$) with respect to Lebesgue measure with mean zero, then

$$\int_I |f|^2 = \Sigma_{n,j} |\int f h_{n,j}|^2$$

(where n ranges over the nonnegative integers and j is an integer in $[0, 2^n - 1]$).

c) Prove that if $f \in L^1(I)$ with mean zero then

$$f = \Sigma_{n=0}^{\infty} \Sigma_{j=0}^{2^n-1} (\int f h_{n,j}) * h_{n,j}(x) \text{ a.e.}$$

(NOTE: The outer sum in the actual question statement starts from $n = 1$, but this is false; to see this, look at $h_{0,0}$).

Hints: a) The present author considers this part to be nothing more than free points. (The interval in question is $[1/4, 1/2]$; the function in question is equal to 2 on the left half of the interval, i.e. $[1/4, 3/8)$, -2 on the right half, i.e. $(3/8, 1/2]$, and 0 everywhere else, including $3/8$).

b) It suffices to show that the $h_{n,j}$ form an orthonormal basis (i.e. spans a dense subset) of the set of mean-zero L^2 functions (which, being the orthogonal complement of the constant functions, is itself a Hilbert space).

Orthonormality can be checked directly with little effort (though one must distinguish between the case of disjoint intervals and the case of nested intervals). For density, one merely notes that the set of mean-zero linear combinations of characteristic functions of dyadic intervals is dense in the set of mean-zero L^2 functions.

c) The hint given is to compare the Nth partial sum (i.e. replace ∞ in the the outer partial sum with N) with $E_N f$. The two can be seen to be equivalent (when f is mean-zero) by induction on N. The result then follows by the preceding problem, S09 Q5, which was discussed in detail in the MT section of this guide.

NOTE: This problem, together with S09 Q5, serves as a nice introduction to the Haar wavelets, which are the $h_{n,j}$ from this problem. The present author likes working with Haar wavelets (in fact, a generalized version of them were a major tool in his ATC result) because they allow one to use the methods and ideas of harmonic analysis without the complicated computations associated with Fourier series and Fourier transforms.

Reference: As part b) of this question falls into the HIL category and part c) falls into MT, no references are needed other than the ones the reader has already used for MT and HIL (e.g. Folland 1-3 and 5.5). The reason this question is listed as MISC is that it combines these two categories in a way never seen before on a qualifying exam at UCLA (also, it is the only 'true' harmonic analysis question asked in the real analysis section which uses neither Fourier series nor Fourier transforms).

F09 Q6: The poisson kernel for $0 \leq \rho < 1$ is the 2π periodic function on $\mathbf{R}$ defined by

$$P_\rho(\theta) = \text{Re})(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}})$$

For functions h continuous and harmonic inside the closed disc of radius R about the origin one has (you need not prove this)

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\eta-\theta)h(Re^{i\theta})d\theta$$

Assume that h is harmonic and positive on the open unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$. Prove that there exists a positive Borel measure μ on $[0, 2\pi]$ such that for all $re^{i\theta} \in D$ one has

$$h(re^{i\theta})) = \int_0^{2\pi} P_r(\eta-\theta)d\mu(\theta).$$

Hint: Let $h_\lambda(z) = h(\lambda z)$ and use weak convergence on the measures associated with $h(\lambda e^{i\theta})$ as λ goes to one.

Reference: To learn about the Poisson kernel (which is in the complex analysis section of the qual syllabus), one should consult Gamelin's book (Chapter X.1) or Ahlfors's book (Chapter 4.6.3-4) on complex analysis.

Part III): Complex Analysis

A) References

The main references for this section of the exam are Theodore Gamelin's book "Complex Analysis" (chapters I-XI, with heaviest emphasis on chapters III-VIII) and Lars Ahlfors's book "Complex Analysis" (chapters 1-5, with heaviest emphasis on chapters 1-4).

The present author finds Gamelin's book to be a better reference for nearly every topic (with the possible exception of advanced conformal mappings) than Ahlfors because of the more modern teaching style (favoring algebra instead of geometry for intuition and reasoning) and the fact that Gamelin simply has better pictures. One can also note that Ahlfors was writtin in 1953 with students of Gamelin's generation in mind, whereas Gamelin was written in 2001 with the current generation of students in mind. Even though Gamelin is used as an undergraduate textbook at UCLA, it covers the full range of graduate topics in complex analysis (much as Dummit and Foote does for algebra).

B) List of Subtopics

The following are the broad groupings of questions which may appear on the complex analysis subsection of the qualifying exam (sorted by the order in which they appear in Gamelin) together with the abbreviations that I use for them.

RAID: Short for "real analysis in disguise". Although these (usually easy) questions appear in the complex analysis section of the exam, the methods used to solve them require almost no complex variable theory. A question of this type might ask the test-taker to sum up a series (which just happens to consist of a complex-valued function) or estimate the L^p norm of a function.

FUND: Stands for "fundamentals". These are the questions that use the most basic properties of complex derivatives, directly from the definition. In particular, these questions do not require the use of complex integrals.

INT: Questions involving the most basic properties of complex integrals, such as Cauchy's integral formula and Taylor's theorem for analytic functions.

AC: Analytic continuation; this generally involves taking a function which is defined on a curve and extending it to a neighborhood of this curve, where it is differentiable.

LAUR: The Laurent series of complex analytic functions; questions in this small category ask the test-taker to directly compute this series.

SING: Questions dealing with the three types of isolated singularities of meromorphic functions: removable singularities (where the function can be extened to the missing point), poles (where the functions look like z^{-n} near that point), and essential singularities.

RES: Questions which require the direct use of the Residue Theorem (as opposed to merely the Cauchy Integral Formula) as their basic tool; the (admittedly subtle) distinction between questions in this category and questions in INT is that INT questions are in Chapter IV of Gamelin's book (or Chapter V for Taylor series) whereas RES questions fall in Chapter VII.

ARG: Questions involving the Argument Principle (which uses a contour integral to determine the number of zeroes and poles of a meromorphic function in a region).

DISC: Questions which deal with specific properties of the analysis of functions from the unit disc to itself.

CM: Questions which require the test-taker to use a conformal mapping (i.e. analytic invertible map with derivative nowhere zero) to transform one region into another region.

NORM: Questions involving normal families of analytic functions (which are those collections of analytic functions which are uniformly bounded on any given compact set).

HAR: Questions involving Harnack's inequality (which states that if f is a positive harmonic function defined on the unit disc with $f(0) = 1$ then $\frac{1-|z|}{1+|z|} \leq f(z) \leq \frac{1+|z|}{1-|z|}$).

MISC: The very small handful of difficult questions that do not fall into the preceding twelve categories.

C) Breakdown of the Complex Analysis Quals by Category

Since real analysis and complex analysis were combined into a single qualifying exam in Fall 2001, there have been 18 qualifying exams in analysis, in which there were 99 questions in the complex analysis section. They break down as follows (where stars denote questions using Taylor series):

F01 C1: CM C2: RES C3: FUND C4: INT C5: RAID

W02 Q9: CM Q10: NORM Q11: RES Q12: SING

S02 Q9: CM Q10: CM Q11: RAID Q12: RES Q13: NORM Q14: AC

F02 Q10: RAID Q11: CM Q12: DISC Q13: HAR Q14: RES Q15: RAID

W03 Q9: LAUR Q10: RES Q11: RAID Q12: SING Q13: AC

F03 Q8: CM Q9: CM Q10: RES Q11: ARG Q12: RAID

W04 CA1: FUND CA2: INT CA3: INT CA4: CM CA5: RES CA6: ARG

F04 Q8: RES Q9: ARG Q10: CM Q11: NORM Q12: INT

W05 Q1: RAID* Q2: HAR Q3: SING Q4: CM Q5: RES Q6: AC

F05 C1: ARG C2: CM C3: LAUR C4: RES C5: RAID*

S06 Q7: INT Q8: RES Q9: MISC Q10: SING Q11: RES Q12: RAID

F06 Q7: SING Q8: DISC Q9: RES Q10: CM Q11: ARG Q12: NORM

W07 Q7: RES Q8: ARG Q9: SING Q10: CM Q11: SING Q12: HAR

F07 Q9: MISC Q10: RAID Q11: INT Q12: CM Q13: INT*

S08 Q9: RES Q10: RAID Q11: DISC Q12: RAID Q13: ARG

F08 Q7: INT Q8: SING Q9: HAR Q10: NORM Q11: DISC Q12: RES

S09 Q7: MISC Q8: MISC Q9: MISC Q10: NORM Q11: ARG Q12: MISC

F09 Q7: MISC Q8: INT Q9: SING Q10: FUND Q11: DISC Q12: SING In descending order by popularity, we therefore have

RES (16 questions), CM (14 questions), RAID (12), SING (10), INT (9), ARG (8), MISC (7), NORM (6), DISC (5), HAR (4), AC (3), FUND (3), and LAUR (2).

In other words, the most popular topic (residue theory) appears almost exactly once (.889 times) every exam on average. The next two topics (conformal mappings and real analysis in disguise) are sufficiently popular to appear on most exams (an average of .778 and .667 questions per exam respectively). Singularities, integration theory, and the argument principle seem to appear approximately once every other exam (.556, .5, and .444 questions per exam, respectively). Normal families of functions appear about once every three exams, whereas questions involving Harnack's inequality or anlaysis on the unit disc seem to appear once every four exams each; the other topics (analytic continuation, fundamentals, and Laurent series) appear even less frequently than that. While miscellaneous questions appear about once every three exams, it should be noted the majority of the miscellaneous questions come from a single qualifying exam (Spring 2009); the present author is considering creating a new category for the two harmonic analysis questions (in which case there will be 5 miscellaneous questions, all but one from 2009). Finally, Taylor series (as part of other questions) also have shown up only three times (i.e. once every six exams).

D) A Look at the Categories

RAID (Real Analysis In Disguise)

Reference: None needed (as the questions, despite falling in the complex analysis section, do not test complex analysis.)

List of questions: F01 C5 (M), S02 Q11 (ME), F02 Q10 (M), F02 Q15 (ME), W03 Q11 (ME), F03 Q12 (ME), W05 Q1 (ME), F05 C5 (MH), S06 Q12 (ME), F07 Q10 (M), S08 Q10 (M), S08 Q12 (ME)

Frequency: .667 questions per exam (fairly common)

Description: A number of questions that nominally appear in the Complex Analysis section of the qualifying exam are best solved by real analysis methods instead (usually, this involves either summing up some series using standard undergraduate analysis, estimating L^p norms of complex-valued functions, or computing Fourier coefficients of such functions). The best way to describe how to solve these types of questions is with the following two examples: the first is a series question and the second is a Fourier coefficients question.

S06 Q12 (the question that defined the category in the first place):

Prove that the infinite product $\Pi_{n=0}^{\infty}(1+z^{2^n})$ converges and equals $(1-z)^{-1}$ for all z in the open unit disc.

Proof: Note that by induction on N, $\Pi_{n=0}^{N}(1+z^{2^n}) = 1 + z + \ldots + z^{2^N-1}$; this converges to $\Sigma_{n=0}^{\infty} z^n = \frac{1}{1-z}$ as desired by basic calculus.

S08 Q10: Let the power series $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ have radius of convergence $r > 0$. If $0 < \rho < r$ set $M_f(\rho) = \sup\{|f(z)|; |z| = \rho\}$ and show that for each such ρ, $\Sigma_{n=0}^{\infty} |a_n|^2 \rho^{2n} \leq M_f(\rho)^2$.

Proof: Set $\phi : S^1 \to 1$ as follows: $\phi(z) = f(\rho z)$ (under the convention that the Lebesgue measure of S^1 is 1). Note that the LHS is the square of the L^2 norm of f (as its Fourier expansion is $\Sigma_{n=0}^{\infty} a_n \rho^n z^n$) and its RHS is the square of the L^∞ norm of f; the result follows by direct computation (on a probability space, $\int |f|^2$ is clearly bounded above by the square of the supremum norm of f).

FUND (Fundamentals)

Reference: Gamelin Chapters I and II or Ahlfors Chapters 1 and 2.

List of questions: F01 C3 (E), W04 CA1 (E, though the question is false unless you assume f' never equals zero in D), F09 Q10 (M)

Frequency: 1 question every 6 exams (very rare; these very easy questions should be considered as ten 'gift points' when they appear)

Description: Every once in a while, a complex analysis question appears on the Analysis Qual that requires nothing more than the most basic properties of complex analysis: the definition of the complex derivative, the Cauchy-Riemann equations, the fact that $f' \neq 0$ at z implies f conformal (i.e. angle-preserving) at z, the fact that analytic functions are harmonic ($f_{xx} + f_{yy} = 0$), and, at worst, an occasional real-valued integral when dealing with harmonic functions. (In particular, the complex integral never appears in these types of questions).

Although a question of this type was actually asked on the most recent qualifying exam (phrased in the form 'give an explicit formula for a conformal map of the complex plane onto the unit sphere' - which merely asked you to come up with the inverse formula for the standard stereographic projection, as in Gamelin I.3), no question of this type had appeared in five years before that. One question of this type (which is actually false as written) is discussed here for completeness.

W04 CA1: Let $f(z) = u + iv$ with u, v real-valued be a nonconstant analytic function on some open domain D. Show that at each point of D, the level curves $u(x, y)$ constant and $v(x, y)$ constant intersect at right angles.

Proof: This question is false as written (for a counterexample, look at $f(z) = z^2$ and note that $u(x, y) = 0$ is the coordinate axes and $v(x, y) = 0$ are the lines $y = \pm x$; these curves intersect at 45-degree angles, not right angles).

The authors meant to assume $f' \neq 0$; in this case, the result follows directly from conformality: if γ, η are level curves for u, v respectively with $\gamma(0) = \eta(0) = z$ parametrized by arclength, we let $\alpha = \gamma'(0), \beta = \eta'(0)$. As $f'(z) * \alpha$, the derivative for the real part of $f(\gamma)$ at $z = 0$, is pure imaginary, and $f'(z) * \beta$, the derivative for the real part of $f(\gamma)$ at $z = 0$, is pure real, we have $\alpha = ri\beta$ for some real constant r (which equals ± 1, but showing this is unnecessary for the problem). As α, β are the tangents to γ, η respectively at $z = 0$, it follows that γ, η intersect at z at right angles.

INT (Basic integration theory)

Reference: Gamelin Chapters III (excluding the last two sections about physics applications) and IV (with IV emphasized) OR Ahlfors Chapter 4, sections 1-3. For Taylor series, read Gamelin Chapter V (excluding V.8) OR Ahlfors subsection 5.1.2.

List of questions: F01 C4 (ME), W04 CA2 (ME), W04 CA3 (MH), F04 Q12 (ME), S06 Q7 (M), F07 Q11 (MH), F07 Q13 (ME), F08 Q7 (MH), F09 Q8 (ME)

Frequency: .5 questions per exam (fairly common; while not trivial, these questions still tend to be fairly doable)

Description: The next topic in complex analysis consists of the results which follow directly from the basic properties of complex integrals, along with path integrals of harmonic functions on the plane. The main results are Cauchy's theorem, Cauchy's integral formula, the mean value property (and maximum modulus principle) for analytic and harmonic functions, and Liouville's theorem concerning bounded analytic functions on the plane. Taylor's theorem also fits into this category as the power series decomposition is nothing more than a consequence of Cauchy's integral formula.

The following two questions are fairly representative of what can be asked during this category.

W04 CA2: Let K be a real-valued function of the complex variable z defined in some open domain $D \subset \mathbf{C}$; then K is strictly subharmonic if $K(z_0) < \frac{1}{2\pi}\int_0^{2\pi} K(z_0\rho e^{i\phi} d\phi$ holds for all ρ smaller than the distance from z_0 to the boundary of D. If f is a nonconstant analytic function on D show $|f(z)|$ is strictly subharmonic.

Proof: Fixing z_0, ρ, the standard maximum modulus principle almost gives the desired inequality (but with $\leq$ instead of strict inequality). We may suppose that $f(z)$ is a positive real number by rescaling (if $f(z) = 0$ either the inequality holds or $f = 0$ on a circle; as zeroes of nonconstant analytic functions are isolated, f would be constant in this latter case); also, we may suppose $z = 0$ and $\rho = 1$. By the mean value property (which states that $f(z_0)$ is the average of f along the circle of radius ρ centered at z_0), f must be real on the unit circle (because $f(z_0)$ is the average of the magnitudes of f along the circle, but also the average of the real parts of f).

However, letting z^* denote the point $\overline{z^{-1}}$ (this is z on the unit circle), we can define a function g equal to f inside the closed unit circle and $g(z) = \overline{f(z^*)}$ outside the circle; g is bounded and analytic and therefore constant (by Liouville) so f, equal to this constant on the unit circle, is itself constant, producing the desired contradiction.

S06 Q7. Show that every non-negative harmonic function on $\mathbf{R}^2$ is constant.

Proof (Observe this appears in Gamelin as exercise IV.5.1): Let f be a nonnegative harmonic function on $\mathbf{R}^2$ and g be its harmonic conjuate (a real-valued function with $f + ig$ analytic). A specific construction of g can be done by (for $z = x + iy$ fixed with x, y real) setting γ_1 to be the horizontal line from 0 to x and γ_2 be the vertical line from x to z; then set $g = -\int_{\gamma_1} f_y dx + \int_{\gamma_2} f_x dy$.

As $h := f + ig$ has real part nonnegative, its image is always at least one unit away from -1 so $\frac{1}{h+1}$ is a bounded analytic function and therefore constant by Liouville's theorem so h, and therefore its real part f, is constant.

NOTE: This problem appeared on the current author's analysis qual; he left out the γ_2 term for the harmonic conjugate and therefore was awarded only two points out of ten for this question (worse than most of the students he had TA'd in Math 132 the preceding term had done on the same question as homework). This is another reason why test-takers should think in terms of completely solving at least four questions in each section of the test; if one solution is deemed incorrect, the test-taker still has a solid chance of passing.

AC (Analytic Continuation)

Reference: Gamelin V.8 (for power series methods), AND either Gamelin VIII.6-8 or Ahlfors Chapter 4, section 4 (for methods involving integration). The more masochistic reader (or the reader with an interest in algebraic geometry) may also prefer to read Ahlfors Chapter 8, although this is not necessary for the exam.

List of questions: S02 Q14 (MH), W03 Q13 (M), W05 Q6 (M) (it should be noted that the hardest question asked in modern times on a complex analysis qual, S06 Q11, looks like an analytic continuation question, but is better described as a residue computation)

Frequency: 1 questions every 6 exams (very rare; in fact, Jon Handy refused to include these questions in his guide)

Description: The main idea behind analytic continuation is to take a multivalued function f (such as a power function or a logarithm) on the complex plane and find a way to define a unique branch of this function along a given curve so that the function can be extended to be analytic at each point of the curve. There are two standard ways to do this:

I) Power series methods: create a Taylor series for the function about each point of the curve so that the coefficients vary continuously as one moves along the curve (this is what is to be done in W03 Q13)

II) Integration-based methods: if the curve starts at x_0, define the function to be the integral of f' along the curve (plus a constant to attain the right value at f_0); this is what is to be done in W05 Q6.

However, the hardest question of this type does not fall into either of these categories; it shall be discussed below in detail.

S02 Q14: Let S be the unit square in $\mathbf{C}$ ($[0,1]$ x $[0,1]$) and let $f : S \to \mathbf{C}$ be a continuous nonzero function. Prove there exists continuous $g : S \to \mathbf{C}$ with $f = e^g$.

Proof: Note that if f were analytic, one could use method II) by creating a primitive of $\frac{f'}{f}$ which equaled an element of $\log g(0)$ at the origin. However, because f' does not exist for a general continuous function, one is forced to use other (topological) methods.

To this end, set ϵ to be the minimum value of $|f|$ on S and let δ be sufficiently small that if $x, y \in S$ with $|x-y| < \delta$, $|f(x)-f(y)| < .5\epsilon$ (continuous functions on compact sets are uniformly continuous). Let N be an integer which is greater than $10\delta^{-1}$; divide S into closed subsquares $S_{j,k}$ for j,k integers with $0 \leq j,k < N$; $S_{j,k}$ shall be those points whose x coordinates lie in $[j/N,(j+1)/N]$ and whose y coordinates lie in $[k/N,(k+1)/N]$. Order the $S_{j,k}$ under the lexicographic order (start with $S_{0,0}$, then $S_{0,1}$, ..., then $S_{0,(n-1)}$ followed by $S_{1,0}$ and so on).

After fixing $g(0)$ to be an element of $\log f(0)$, one defines g on each subsquare in lexicographic order; on each subsquare $S_{j,k}$, g will already have been defined on at least one point of that subsquare as part of a branch of $\log f$ at that point. Because $f(S_{j,k})$ is contained in an open half-plane (with boundary line containing the origin; the diameter of $f(S_{j,k})$ is smaller than half of the distance of $f(S_{j,k})$ to the origin), there is a unique way to define a branch of $\log f$ on $f(S_{j,k})$ that agrees with the values of g already defined at points on the square; define g to equal this branch of $\log f$ on $F(S_{j,k})$. Once this is finished, the desired g will have been constructed.

NOTE: The result of this question follows directly from the (topological) fact that the complex plane is the universal cover of the punctured plane (with exp as the covering map); the proof above is very similar to the proof that the complex plane is indeed this universal cover.

Because of the highly topological nature of this argument, this question would not be out of place on the Geometry/Topology qual; although this question would be very hard for 'pure' analysts who have never seen algebraic topology, the question was rated MH instead of H or VH because the argument needed was nothing more than a standard topological argument.

LAUR (Laurent series)

Reference: Gamelin VI.1 or Ahlfors 5.1.3.

List of questions: W03 Q9 (M), F05 C3 (MH)

Frequency: 1 questions every 9 exams (extremely rare, though the knowledge of Laurent series makes the study of singularities, the next topic in the guide, much easier)

Description: The fundamental idea of a Laurent series expansion is that if f is a function defined to be analytic on an annulus of the form $\{r_1 < |z-z_0| < r_2\}$ for $r_1 < r_2$, one can write $f = f_1+f_2$ where f_1 is analytic on $\{|z-z_0| < r_2\}$ (this can be written as an ordinary power series in $z - z_0$ with radius of convergence at least r_2) and f_2 is analytic on $\{|z - z_0| > r_1\}$, i.e. on $\{|z - z_0|^{-1} < r_1^{-1}$ (this can be written as an ordinary power series in $(z - z_0)^{-1}$ with radius of convergence at least r_1^{-1}). Joining together the power series for f_1 and f_2 gives a power series for $f = f_1 + f_2$ in the form $\Sigma_n c_n(z - z_0)^n$ (for n ranging over the integers) which works to define f on the entire annulus.

While c_n can theoretically be computed by integrating $f(z - z_0)^{-1-n}$ along the boundary, for actual computations it is usually easier to find the decomposition $f = f_1 + f_2$ and use standard power series techniques to compute the Laurent series decomposition.

The best way to illustrate is with the most recent question on the topic.

C3: Find the Laurent series expansion for $f(z) = \frac{(1-z^2)}{2i(z^2-(a+1/a)z+1)}$, $|a| < 1$ valid for a neighborhood of the unit circle.

Proof: Note that the denominator factors as $2i(z - a)(z - 1/a)$; therefore, f is analytic on $|a| < |z| < |1/a|$.

One notes that the residue of f at the pole a is equal to the limit of $f(z)(z-a) = \frac{1-z^2}{2i(z-1/a)}$ as z goes to a; this expression becomes $\frac{1-a^2}{2i(a-1/a)} = \frac{-a}{2i}$ (as $(a - 1/a)a = a^2 - 1$) while

the residue of f at the pole a^{-1} is equal to the limit of $f(z)(z-a^{-1}) = \frac{1-z^2}{2i(z-a)}$ as z goes to a^{-1}; this expression becomes $\frac{1-a^{-2}}{2i(a^{-1}-a)} = \frac{-1}{2ia}$ (as $(a^{-1} - a)a^{-1} = a^{-2} - 1$).

Therefore, $f = \frac{-a}{2i}\frac{1}{z-a} + \frac{-a^{-1}}{2i}\frac{1}{z-a^{-1}} + g$ where g is entire; letting z approach infinity shows that g approaches $-\frac{1}{2i}$ as z approaches ∞ so Liouville's theorem says $g = -\frac{1}{2i}$ everywhere. Multiplying through by $-2i$ gives

$-2if = \frac{a}{z-a} + \frac{a^{-1}}{z-a^{-1}} + 1$;

the first term is analytic on $|z| > a$ (and should provide the 'negative coefficients'), the second term is analyitc on $|z| < a^{-1}$ (and should provide the 'positive coefficients'), and the last term is a constant.

Consequently, one can write

$$\frac{a}{z-a} = \frac{az^{-1}}{1-az^{-1}} = \Sigma_{n\geq 1}(az^{-1})^n = \Sigma_{n\leq -1}a^{-n}z^{-n},$$

which converges for $|az^{-1}| < 1$, i.e. $|z| > a$.

Similarly, one can write

$$\frac{a^{-1}}{z-a^{-1}} = \frac{-1}{1-az} = \Sigma_{n\geq 0} - (az)^n = \Sigma_{n\geq 0} - (a^n)z^n,$$

which converges for $|az| < 1$, i.e. $|z| > a^{-1}$.

Adding together the three power series (and noting that the constant terms cancel out) yields

$$-2if = \Sigma_{n\leq -1} a^{-n} z^n + \Sigma_{n\geq 1} - a^n z^n;$$

dividing through by $-2i$ yields

$$f = \Sigma_{n\leq -1} .5ia^{-n} z^n + \Sigma_{n\geq 1} - .5ia^n z^n;$$

this is a Laurent series valid in the annular neighborhood $\{|a| < |z| < |a|^{-1}\}$ of the unit circle.

NOTE: The 'usual' way to try to express $f(z) = \frac{g_1}{z-a} + \frac{g_2}{z-a^{-1}}$ via a partial fractions decomposition, clearing denominators by multiplying through by $2i(z-1/a)(z-a)$, would be a computational nightmare in this case, which is why residue methods were chosen instead.

SING (Singularities)

Reference: Gamelin VI.2-5 or Ahlfors 4.3 (Ahlfors gives a much more convoluted treatment because Laurent series aren't covered until Chapter 5; see the present author's notes at http://www.math.ucla.edu/ meyerson/246w7pdf for a study guide on this topic emphasizing Ahlfors' treatment).

List of questions: W02 Q12 (ME), W03 Q12 (M), W05 Q3 (MH; this is the only complex analysis qual question ever known to use Zorn's lemma in its solution), S06 Q10 (MH), F06 Q7 (M), W07 Q9 (ME), W07 Q11 (ME), F08 Q8 (MH), F09 Q9 (MH), F09 Q12 (H)

Frequency: .556 questions per exam (fairly common)

Description: A function has an isolated singularity at z if it is undefined at z but is analytic in an 'annular neighborhood' of z (i.e. a set of the form $\{w : 0 < |w-z| < \epsilon\}$). The main idea here is that there are precisely three types of isolated singularities: removable singularities (f extends continuously to z), poles ($\frac{1}{f}$ extends continuously to z and has a zero there), and essential singularities (for each complex w there exists $\{z_n\} \to z$ with $f(z_n) \to w$). By far the easiest way to classify a singularity is to look at the Laurent series: removable singularities occur when the Laurent series has no terms of negative order, poles occur when the Laurent series has only finitely many terms of negative order, and essential singularities occur when the Laurent series has infinitely many terms of negative order. Finally, one should be familiar with the word "meromorphic", which means that the function is analytic except possibly for some poles.

The following are some qualifying exam questions that have appeared recently concerning this topic:

W07 Q9: Let $f(z)$ be analytic for $0 < |z| < 1$. Suppose there are $C > 0$ and $m \geq 1$ with $|f^{(m)}(z)| \leq \frac{C}{|z|^m}$, $0 < |z| < 1$. Show f has a removable singularity at $z = 0$.

Proof: One notes that $f^{(m)}(z)$ has either a removable singularity or a pole (of order at most m) at $z = 0$; however, writing out the Laurent series for $f(z) = \Sigma_n c_n z^n$, the mth derivative has Laurent series $\Sigma_n c_n (n * (n-1) * \ldots * (n-m+1)) z^{n-m}$ (note that the coefficient becomes zero if $0 \leq m < n$). As this Laurent series does not have any terms of order less than $-m$, the original Laurent series cannot have any terms of order less than 0 (terms of order k in f correspond to terms of order $k - m$ in the mth derivative) so f has a removable singularity.

The next two problems are about as hard as singularity-related problems get.

That being said, though, even though a lot of work is required for the next question, the steps involved are all intuitive enough that this is only a medium-hard qual problem (in fact, it was only the third-hardest problem in its section!)

W06 10. Prove that $\frac{\pi^2}{\tan^2(\pi z)} = a + \Sigma_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$ for some constant a.

Letting f denote $\frac{\pi^2}{\tan^2(\pi z)}$ and g denote $\Sigma_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$, the problem asks the test-taker to show that $f - g$ is constant, but fortunately one is not asked to actually compute this constant.

Also, f and g are periodic with period 1.

Lemma: $f - g$ can be extended to an entire function.

Proof: Note that f is meromorphic on $\mathbf{C}$ with singularities at the points $.5n$ for $n \in \mathbf{Z}$ (where the denominator is either zero or undefined). If z is congruent to .5 modulo 1, then the function in question can be written as $\frac{\pi^2 \cos^2(\pi z)}{\sin^2(\pi z)}$ from which it is clear that these singularities are removable and therefore can be ignored.

Therefore, it suffices to look at $z \in \mathbf{Z}$; by periodicity we merely need to look at $z = 0$. At this point, one realizes that as $\tan^2(\pi z)$ has a zero of order two at $z = 0$ (because it can be expressed as $\frac{\sin^2(\pi z)}{\cos^2 \pi z}$, which, when divided by z^2, approaches π^2 as z approaches zero), $\frac{\pi^2}{\tan^2(\pi z)}$ has a pole of order 2.

To compute its principal part, one notes that $\frac{z^2\pi^2}{\tan^2(\pi z)}$ approaches 1 as z approaches zero which implies that at the origin, the singularity of $\frac{\pi^2}{\tan^2(\pi z)} - \frac{1}{z^2}$ at 0 is either removable or a pole of order 1. However, as the function is even, comparing principal parts removes the latter case; therefore the singularity is removable so the principal part of $f(z)$ at $z = 0$ is $\frac{1}{z^2}$. By periodicity, the principal part of $f(z)$ at $z = n$ is $\frac{1}{(z-n)}^2$.

For g, the quadratic nature of the summands gives that if z is not an integer, the series defining g converges uniformly in a neighborhood of z (and therefore g is analytic at z by Morera's theorem).

Further, if n is an integer, the series (with the term corresponding to n) removed still converges in a neighborhood of n. Therefore, the only poles are the 'obvious' ones: at $z = n$ there is a pole of order 2 with principal part $\frac{1}{(z-n)^2}$.

This implies that f and g are meromorphic with the same poles; therefore, all the singularities of $f - g$ are removable so $f - g$ can indeed be extended to an entire function, proving the lemma.

By Liouville's theorem; it now suffices to show $f - g$ is bounded (remember that the main goal is to show $f - g$ is constant). Because $f - g$ is periodic with period 1, one can suppose that $\text{Re } z \in [0, 1]$. Therefore, the strip can be divided into two separate areas; where $\text{Im } z$ is large one estimates the size of f and g directly and where $\text{Im } z$ is small one uses compactness.

$|\text{ Im } z| \geq 1$: Note that $\tan(z) = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$. If $\text{Im } z > 1$ then $|e^{-iz}| > 2|e^{iz}|$ so $|\tan z| > \frac{1}{3}$ and $|f(z)| < 3\pi^2$; for $\text{Im } z < -1$ the same bounds hold except that $-iz$ and iz need to be switched.

For $g(z)$ note that $|z - n|^2 \geq (|n| - 1)^2 + 1$ from which basic Calculus II methods give a uniform bound.

$-1 \leq \text{Im } z < 1$: The region in question is compact (because $\text{Re } z$ and $\text{Im } z$ are bounded) so f and g, being continuous on compact sets, are bounded.

Therefore, $f - g$ can be extended to an analytic function which is bounded (because the complex plane is the union of two sets such that $f - g$ is bounded on each of those sets) and therefore constant by Liouville, completing the problem.

The last problem in this section is theoretically harder, but fairly painless computationally.

F09 Q12: Let f be a non-constant meromorphic function in the complex plane. Assume that if f has a pole at the point $z \in \mathbf{C}$, then z is of the form $n\pi$ with an integer $n \in \mathbf{Z}$. Assume that for all non-real z we have the estimate

$$|f(z)| \leq (1 + |\mathrm{Im}(z)|^{-1})e^{-|\mathrm{Im}(z)|}$$

Prove that for every integer $n \in \mathbf{Z}$, f has a pole at the point πn.

We show that f is a constant multiple of the cosecant function. First, observe that the estimate implies that the poles of f on the z axis must be simple poles (also, with residue having norm at most one); at a pole $n\pi$, $f(n\pi + ri) = O(r^{-1})$. Now, note that $|\sin z| \leq e^{|\mathrm{Im}(z)|}$ (as this inequality applies to e^{iz} and e^{-iz}; setting $g(z) = f(z)\sin z$, g is entire with $|g(z)| \leq 1 + |\mathrm{Im}(z)|^{-1}$ off the real axis. Consequently, $|g| \leq 2$ outside the strip $|\mathrm{Im}(z)| < 1$. Inside the strip, consider the rectangle with vertices $(\pm n\pi, \pm 1)$. g is bounded in norm by 2 on the horizontal edges; on the vertical edges, note that $|\sin z| = O(\mathbf{Im\ z})$ (both $\sin z$ and $\mathbf{Im\ z}$ are 2π-periodic and $\frac{sinz}{z}$ is entire, so this bound is independent of n). Therefore, by the Maximum Modulus Principle, g is bounded in norm on this rectangle (independent of n) and therefore on the strip. From here, we conclude g is bounded, so (by Liouville) g is constant as desired.

RES (Residue Theory)

Reference: Gamelin Chapter VII or Ahlfors 4.5.1 and 4.5.3 (the present author has always believed that the best way to prepare for this section is to carefully work through every problem from Chapter VII of Gamelin's book)

List of questions: F01 C2 (ME), W02 Q11 (M), S02 Q12 (ME), F02 Q14 (M), W03 Q10 (M; note that the correct answer is actually half of the stated value), F03 Q10 (M), W04 CA5 (M), F04 Q8 (MH), W05 Q5 (MH), F05 C4 (M), S06 Q8 (M), S06 Q11 (VH), F06 Q9 (M), W07 Q7 (M), S08 Q9 (H), F08 Q12 (M)

Frequency: .889 questions per exam (common; this type of question used to appear like clockwork exactly once per exam, except that in S06 it appeared twice and in F07 it was skipped - and it also did not appear at all in 2009)

Description: The fundamental idea behind residue theory (the final truly 'undergraduate' topic in this guide) is to use information about some analytic function in the interior of a region to calculate its integral along the boundary of a region. The main theorem (the residue theorem) states that if U is a region whose boundary is a simple closed curve γ (oriented counterclockwise) and f is analytic along γ and meromorphic in U then $\int_\gamma f(z)dz = \Sigma_p 2\pi i Res(f;p)$ where the sum is over all the poles of f, and $Res(f,p)$ is the $(z-p)^{-1}$ coefficient in the Laurent series of f about p (note: this is only $\lim_{z\to p} f(z)(z-p)$ if f has a simple pole; the general formula is $\lim_{z\to p}(f(z)(z-p)^n)^{(n-1)}/(n-1)!$ if the pole is of order n. This is because starting with a Laurent series term of $a_{-1}(z-p)^{-1}$, multiplying by $(z-p)^n$ to make the function analytic yields $a_{-1}(z-p)^{(n-1)}$ and taking $n-1$ derivatives yields a constant term of $a_{-1}(n-1)!$.)

There are four standard contours which are most frequently used in evaluating integrals with respect to this technique:

I) The 'semicircle' contour: starting at the origin, go along the real axis to R, then counterclockwise along the circle $|z| = R$ to $-R$, and back along the real axis to 0. (Use for functions which are analytic and $O(|z|^{-2})$ along the real axis, or $O(|z|^{-1})$ along the real axis with an oscillatory term; see Gamelin VII.2 for the first case and VII.6-7 for the second case.)

VARIATION 1: The 'slice of pizza' contour. Starting at the origin, go along the real axis to R, then counterclockwise along the circle $|z| = R$ for θ radians until hitting $Re^{i\theta}$, then return to 0 in a straight line. (Use when the function's behavior on $\{Arg z = \theta\}$ can easily be related to the real line, or is computable in its own right. A classic example, as in F02 Q14 or S06 Q8, is the case $f(z) = e^{iz^2} = \cos z^2 + i \sin z^2$; if $\theta = \frac{\pi}{2}$, $f(re^{i\theta} = e^{-r^2})$. Note that in this case, the correct contour looks exactly like a slize of pizza!)

VARIATION 2: The 'depressed disc' contour. Travel as in the regular disc contour except that for each pole p on the real axis, stop at $p-\epsilon$ and travel clockwise along the circle $|z-p| = \epsilon$ until hitting $p+\epsilon$ before continuing. As one might expect, this is used in the case where f has poles on the real axis; see Gamelin VII.5 for details.

II) The 'unit circle contour': Integrate along the unit circle. (Use if f is a rational function of sin and cos and the region of integration is an interval from 0 to π or 0 to 2π; in this case, the substitution $f(z) = e^{iz}$, along with the proper expressions of sin and cos in terms of e^{iz}, are useful; see Gamelin VII.3). It should be noted that this contour appeared precisely once in the first eleven quals of the modern era (F01 through S06), and three times in the five quals since.

III) The 'keyhole' contour: Start infinitessimally above the point ϵ (denoted $\epsilon + 0i$ by Gamelin); move to the right, staying infinitessimally above the real axis until $R + 0i$. Travel counterclockwise along the circle $|z| = R$ until infinitessimally below the point R (denoted $R - 0i$ by Gamelin); move to the left, staying infinitessimally below the real axis until $\epsilon - 0i$. Finish by moving clockwise along the circle $|z| = \epsilon$ until $\epsilon + 0i$ (use for functions which have logarithm or power function behavior which requires a branch cut; this allows one to place the branch cut at the positive real axis, while the function defined on "$\epsilon + 0i$" is what the function 'should' be along the positive real axis. See Gamelin VII.4 for details.

IV) The 'dogbone' contour: Start infinitessimally above ϵ (i.e. $\epsilon + 0i$), move to the right, staying infinitessimally below the real axis until $1 - \epsilon + 0i$.. Travel clockwise along the circle $|z-1| = \epsilon$ until infinitessimally below $1-\epsilon$, at $1-\epsilon-0i$. Move to the left, staying infinitessimally below the real axis, until $\epsilon - 0i$; finish by traveling clockwise along $|z| = \epsilon$ until $\epsilon + 0i$. (Use for functions which are defined and integrated along $[0, 1]$, but can be extended to have a branch cut at $[0, 1]$; for example: $\frac{x^4}{\sqrt{x(1-x)}}$).

See Gamelin VII.8 for details; although this contour has never appeared on a qualifying exam in the modern era (and therefore there are no practice problems in the topic outside of Gamelin's book), those wishing to skip this topic should pay heed to the story of the unit circle contour, which went from being one of the least popular contours to the single most popular contour in recent years.

Below are three recent qualifying exam questions in this topic; the first is a unit circle contour, the second is a keyhole contour, and the last is the hardest question asked on an analysis qual in the modern era.

F08 Q12: Evaluate $\int_0^\pi \frac{d\theta}{a^2+\sin^2\theta}$ if $a > 0$ is real.

Proof: Because $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, $\sin^2\theta = -\frac{1}{4}(e^{2i\theta}-2+e^{-2i\theta})$. This expansion in terms of $e^{2i\theta}$ and an integrand from 0 to π (instead of the usual 0 to 2π) suggests the substitution $z = e^{2i\theta}$ which will make the integral cover one lap around the unit circle (oriented counterclockwise).

Here, $dz = 2izd\theta$ by the chain rule; therefore, the integral becomes

$$\int_C \frac{dz}{2\pi iz(a^2 - .25(z - 2 + z^{-1})},$$

where C is the unit circle oriented counterclockwise, or

$$\frac{1}{2i}\int_C \frac{-4dz}{z^2 - (2 + 4a^2)z + 1}dz.$$

By the residue theorem, this is equal to π times the sum of the residues of the poles of $\frac{-4}{z^2-(2+4a^2)z+1}$ in the unit circle.

One notes that the poles are the points where $z^2 - (2 + 4a^2)z + 1 = 0$; the quadratic formula yields $z = \frac{2+4a^2 \pm \sqrt{16a^4+16a^2}}{2}$. The larger pole is a real number greater than one and therefore not inside the unit circle; as the roots have product 1, one is left with the smaller root,

$$\frac{2 + 4a^2 - \sqrt{16a^4 + 16a^2}}{2} = 1 + 2a^2 - 2a\sqrt{a^2 + 1},$$

inside the unit circle.

The residue is equal to $\frac{-4}{(z^2-(2+4a^2)z+1)'}$ evaluated at this root, or $\frac{-4}{2z-2-4a^2}$, which becomes $\frac{1}{a\sqrt{a^2+1}}$, making the integral $\frac{\pi}{a\sqrt{a^2+1}}$.

S08 Q9: Show using the residue theorem that

$$\int_0^\infty \frac{\log^2 x}{1 + x^2} = \frac{\pi^3}{8}.$$

Proof: As this is a logarithm question, the keyhole contour is used. However, the function to integrate is not $\frac{\log^2 z}{1+z^2}$ but rather $\frac{\log^3 z}{1+z^2}$ (where the logarithm has the positive real axis as its branch cut; therefore, $\log(r + 0i) = \ln r$ and $\log(r + 0i) = \ln r + 2\pi i$ for r positive real.

The integrals along the circular portions of the contour vanish in the limit (because logarithmic singularities are absolutely integrable while the function is $O(\log^3 |z| * |z|^{-2})$ for large z); along the part just above the real axis one gets $\int_0^\infty \frac{\log^3 x}{1+x^2} dx$ and just below this axis one gets $-\int_0^\infty \frac{(\log x+2\pi i)^3}{1+x^2} dx$. Therefore, this integral will yield

$\int_0^\infty \frac{g(x)}{1+x^2} dx$ where $g(x) = \log^3 x - (\log^3 x + 6\pi i \log^2 x - 12\pi^2 \log x - 8\pi^3 i) = -6\pi i \log^2 x + 12\pi^2 \log x + 8\pi^3 i$. (Note that integrating $\log^3 x$ was necessary to yield a $\log^2$ term; if $\frac{\log^2 x}{1+x^2}$ was used, the highest order term remaining would be a multiple of $\log x$.)

Now, investigating the integrand $\frac{\log^3 z}{1+z^2}$ yields poles at $\pm i$ (both inside the keyhole contour); at i the residue is $\frac{\log^3 i}{2i} = \frac{((\pi i)/2)^3}{2i} = -\frac{\pi^3}{16}$ and at $-i$ the residue is $\frac{\log^3 -i}{-2i} = \frac{((3\pi i)/2)^3}{-2i} = \frac{27\pi^3}{16}$ from which the Residue Theorem says that the integral along the keyhole contour is $\frac{13\pi^4 i}{4}$. Taking imaginary parts of $\int_0^\infty \frac{g(x)}{1+x^2} dx$ yields that

$$\int_0^\infty \frac{-6\pi \log^2 x + 8\pi^3}{1 + x^2} dx = \frac{13\pi^4}{4};$$

from the well-known result that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$,

$$-6\pi \int_0^\infty \frac{\log^2 x}{1 + x^2} = \frac{13\pi^4}{4} - 4\pi^4 = -\frac{3\pi^4}{4};$$

dividing by -6π yields $\int_0^\infty \frac{\log^2 x}{1+x^2} = \frac{\pi^3}{8}$ as desired.

S06 Q11. Let $\epsilon = \frac{1}{100}$ and let U be an ϵ-neighborhood of the spiral $\{\theta e^{i\theta} : 0 \leq \theta \leq 4\pi\}$ and O be an ϵ-neighborhood of the spiral $\{2\theta e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Let $f : U \to \mathbf{C}$ and $g : O \to \mathbf{C}$ be the corresponding analytic continuations of

$$z \to \log(\frac{\cos(z)}{1-z^2})$$

from the ϵ-neighborhood of the origin on the real axis such that $f(0) = 0 = g(0)$. Find the imaginary part of $f(4\pi) - g(4\pi)$.

Proof: By the Fundamental Theorem of Calculus, $\int f'(z)dz$ along the spiral used to define U (call this curve S_U) is equal to $f(4\pi) - f(0) = f(4\pi)$ while $\int g'(z)$ along the spiral used to define O (called S_O) is equal to $g(4\pi) - g(0) = g(4\pi)$.

Because branches of the logarithm function only differ by constants (which vanish when derivatives are taken), it follows that if h is a branch of $\log(\frac{\cos(z)}{1-z^2})$ defined in a neighborhood of z, there exist branches L_1, L_2, and L_3 of the logarithm function such that near z, $h(z) = L_1(\cos z) - L_2(1-z) - L_3(1+z)$; therefore,

$$h'(z) = -\tan z + \frac{1}{1-z} - \frac{1}{1+z}$$

for each z. This equation works for both f and g; therefore, letting $\gamma = S_U - S_O$ (a simple closed curve based at zero; one follows S_U forwards and then S_O backwards),

$$f(4\pi) - g(4\pi) = \int_\gamma -\tan z + \frac{1}{1-z} - \frac{1}{1+z}dz.$$

When drawing γ, note that γ, being a positively oriented closed curve, is the boundary of some region V in the complex plane. Therefore, the residue theorem says that it suffices to find the residues of all the poles of the integrand in V and multiply by $2\pi i$. One can see that all of the poles of the integrand are real; the poles of $-\tan z$ all are $z = k\pi$ for k a half-integer (and the residue is 1, as can be found by a direct computation), $\frac{1}{1-z}$ has a pole at $z = 1$ with residue -1, and $-\frac{1}{1+z}$ has a pole at $z = -1$, again with residue -1.

The question is to find which portion of the real axis is in the interior of V. To do this, observe S_U crosses the real axis for θ an integral multiple of π, i.e. at the points 0, $-\pi$, 2π, -3π, and 4π. Similarly, S_O crosses the real axis for θ an integral multiple of π, i.e. at 0, -2π, and 4π.

Clearly $(-\infty, -3\pi)$ is outside of V; -3π is on the boundary (in the image of S_U) and $(-3\pi, -2\pi)$ is in V; -2π is on the boundary (in the image of S_O) and $(-2\pi, -\pi)$ is outside of V. Next, $-\pi$ is on the boundary (in the image of S_U), and $(-\pi, 0)$ is inside of V. Although 0 is in the boundary, it is in the image of both S_U and S_O, so $(0, 2\pi)$ is inside of V. 2π is on the boundary (in the image of S_U again), $(2\pi, 4\pi)$ is outside V, 4π is on the boundary (in the image of both S_U and S_O), and $(4\pi, \infty)$ is outside of V.

Note that except for the cusp points (0, and 4π), the intersection points of the images of S_U and S_O with $\mathbf{R}$ are precisely the transition points between 'inside' and 'outside'.

In other words, $V \cap \mathbf{R} = (-3\pi, -2\pi) \cup (-\pi, 0) \cup (0, 2\pi)$.

This set contains both of the poles of the integrand coming from the rational terms (-1 and 1, both with residue -1) and the following four poles of $-\tan z$: -2.5π, $-.5\pi$, $.5\pi$, and 1.5π (all with residue 1). The sum of all the residues of the poles in V is equal to 2, which implies that the integral in question is $2 * 2\pi i = 4\pi i$. Therefore, $f(4\pi) - g(4\pi) = 4\pi i$.

NOTE: When this question appeared on the author's qual, no one had any idea how to solve it. The author is still proud to have been the one to discover the solution... six weeks after the fact.

ARG (Argument principle)

Reference: Gamelin VIII.1-4 or Ahlfors 4.5.2

List of questions: F03 Q11 (M), W04 CA6 (M), F04 Q9 (MH), F05 C1 (ME), F06 Q11 (M), W07 Q8 (MH), W08 Q13 (M), S09 Q11 (MH)

Frequency: 1 question every 2.25 exams (fairly uncommon; these questions tend to be neither too easy nor too hard)

Description: In this 'transitional' topic (i.e. it is sometimes considered undergraduate level and sometimes considered graduate level), the main idea is to look at a meromorphic function f which is analytic on the boundary δU of some region U. Then, the Argument Principle states that $\frac{1}{2\pi i}\int_{\delta U}\frac{f'(z)dz}{f(z)}$ is equal to the difference between the number of poles of the function and the number of zeroes of the function (the integrand can be written as $d\log f(z)$ or also $dArgf(z)$).

Most qualifying problems (there are three exceptions) only require Rouche's theorem (which states that if f, g are analytic on U and $|g| < |f|$ on its boundary then $f+g$ and f have the same number of zeroes up to multiplicity), which is a consequence of the argument principle. Occasionally (as in S09 Q11), the topological version of the argument principle (which says that if γ is a curve which does not pass through z_0, $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}$ is equal to the winding number of γ with respect to γ) is needed; in this case, it should be noted that winding numbers are multiplicative (if γ_1, γ_2 are curves parametrized by the same interval which do not pass through zero, the winding number of the pointwise product $\gamma_1\gamma_2$ with respect to zero is the sum of the winding numbers of γ_1 and γ_2) and the winding number of a "Jordan curve" (e.g. the injective analytic image of the unit circle) always lies in the set $\{-1, 0, 1\}$.

Two qualifying exam questions are given below: one of which can be done with Rouche's theorem and the other of which needs the full strength of the argument principle.

S08 Q13: Suppose $D \subset\subset \Omega \subset \mathbf{C}$, f_n are analytic on Γ and converge uniformly to f. If f is nonzero on the unit circle, show that there exists N such that whenever $n \geq N$, f_n and f have the same number of zeroes in the unit disc D.

Proof: One notes that 'same number of zeroes' has to refer to multiplicity; to see why, take $f_n = x^2 + 1/n$ (which has two zeroes in the unit circle for $n > 1$) and $f = x^2$ (which has a single zero of multiplicity two).

With this caveat, this problem is a relatively straightforward consequence of Rouche's theorem and compactness: as δD is compact, there exists $\epsilon > 0$ with $|f| > \epsilon$ on δD. By the definintion of uniform convergence there exists N such that for $n > N$,$||f - f_n|| < \epsilon$ (this notation means that the supremum of $|f(z) - f_n(z)|$ is less than ϵ). Rouche's theorem then tells us (because $|f - f_n| = |f_n - f|$ at each point) that f and $f + (f_n - f) = f_n$ have the same number of zeroes (up to multiplicity) in the unit disc for each such n, precisely as desired.

NOTE: One could solve this problem directly by the argument principle; the hardest step here is to show that if f_n converges to f uniformly on δD and f is nonzero, then $\frac{f_n'}{f_n}$ converges uniformly to $\frac{f'}{f}$ on δD. The present author suggests that this latter method serve as a challenge to the reader.

W07 Q8: Determine the number of zeroes of the polynomial $p(z) = z^4 + z^3 + 4z^2 + 2z + 3$ in the right half-plane.

Proof: This is actually a question that requires the full argument principle; to proceed, let Γ_R be the contour that starts at $-Ri$, travels clockwise around the circle $z = R$ to Ri and finishes by going down the z axis until it hits $-Ri$.

The interior of Γ_R is those points in the right half plane with magnitude less than R; therefore, by the argument principle, the integral of $\frac{p'(z)}{p(z)}$ will yield $2\pi i$ times the number of zeroes in the right half plane wth magnitude less than R (letting R go to infinity will yield the number of zeroes in the right half plane).

Over the semicircular portion of the contour, $p(z) = z^4 + O(|z|^3)$ while $p'(z) = 4z^3 + O(|z|^2)$; from this, $p'(z)/p(z) = 4z^{-1} + O(|z|^{-2})$ from which integrating over Γ_R yields $4\pi i + O(R^{-1})$, which approaches $4\pi i$ as R goes to infinity.

Over the vertical line (i.e. $z = it$ for t decreasing from R to $-R$), one notes that the real part of p is $t^4 - 4t^2 + 3 = (t^2 - 3)(t^2 - 1)$ while its imaginary part is $-t^3 + 2t = -t(t^2 - 2)$. Therefore, p stays in the fourth quadrant (positive real part; negative imaginary part; the argument of p approaches that of the positive real axis as t approaches $+\infty$) until crossing over into the third quadrant at $t = \sqrt{3}$ (when the real part becomes negative), into the second quadrant at $t = \sqrt{2}$ (the imaginary part becomes positive), the first quadrant at $t = 1$ (the real part switches sign), the fourth quadrant at $t = 0$ (the imaginary part switches sign; note that a complete circle has been traversed clockwise by the image), the third quadrant at $t = -1$ (the real part switches sign), the second quadrant at $t = -\sqrt{2}$, and the first quadrant at $t = -\sqrt{3}$, where it stays, approaching the positive real axis as t approaches $-\infty$.

One can formalize this process by defining new branches of the logarithm each time an axis is crossed; in any event, as R approaches ∞, the resulting change in argument approaches the exact same change in argument as one would get from two clockwise revolutions around the circle, or -4π, so the integral over the vertical line approaches $-4\pi i$ as R goes to infinity.

Therefore, the limit of $\int_{\Gamma_R} \frac{p'(z)dz}{p(z)}$ as R goes to infinity is $4\pi i - 4\pi i = 0$, yielding the counterintuitive result that the polynomial has NO zeroes in the right half-plane!

NOTE: This question appears in Gamelin's book as VIII.1.5. Although the current edition of Gamelin's book states (incorrectly) that the answer is two, his errata page gives the correct answer of zero.

DISC (Analysis on the unit disc)

Reference: Gamelin IX.1-2 or Ahlfors 4.3.4

List of questions: F02 Q12 (ME), F06 Q8 (MH), S08 Q11 (M), F08 Q11 (ME; look at $\overline{f(z^*)}$ where $*$ is the map sending z to $\overline{z^{-1}}$), F09 Q11 (M)

Frequency: 1 question every 3.6 exams (rare, though this topic has become much more common in recent years, having appeared in three of the last four exams)

Description: This topic is the first true 'graduate level' topic (i.e. the material is unlikely to appear in an undergraduate complex analysis course). The main idea is Schwarz's lemma, which says that if $f : D \to D$ is analytic (D is the unit disc) and $f(0) = 0$ then $|f(z)| \leq z$ (further, if equality holds outside the origin, f is a multiple of the identity map). This can be used to classify conformal self-maps (i.e. complex diffeomorphisms) of the unit disc; they are of the form $\omega\frac{z-a}{1-\bar{a}z}$ where $|\omega| = 1$ and $|a| < 1$.

The most representative problems of this topic is listed below:

S08 Q11: Let $f : D \to D$ be a holomorphic map with two unequal fixed points $a, b \in D$. Show $f(z) = z$ for all $z \in D$. (The problem gives Schwarz's lemma as a hint).

Proof: Letting ϕ be the conformal map sending z to $\frac{z-a}{1-\bar{a}z}$ and $g = \phi \circ f \circ \phi^{-1}$, g sends D to itself and fixes two unequal points: $0 = \phi(a)$ and $\phi(b)$. By the conditions for equality in Schwarz's lemma (as $g(\phi(b)) = \phi(b)$), g is a scalar multiple of the identity (which must be the identity itself to fix a nonzero point); as $f = \phi^{-1} \circ g \circ \phi$, f is also the identity map.

CM (Conformal mappings)

Reference: Gamelin XI.1 or Ahlfors 3.2-4 (excluding 3.2.4, 3.3.2-5, and 3.4.3)

List of questions: F01 C1 (M), W02 Q9 (MH), S02 Q9 (MH), S02 Q10 (MH), F02 Q11 (ME), F03 Q8 (MH), F03 Q9 (M), W04 CA4 (H),F04 Q10 (M), W05 Q4 (VH), F05 C2 (M), F06 Q10 (MH), W07 Q10 (MH), F07 Q12 (M)

Frequency: .778 questions per exam (common; although there are a few hard questions, this topic tends not to be too difficult - it is worth noting that the majority of these questions refer to one specific idea, the mapping from the 'slit unit disc' $D \backslash [0,1)$ to the unit disc. It should also be noted that no questions have been asked from this topic since 2007.)

Description: The goal in conformal mappings is to find a 'conformal mapping' (i.e. an analytic diffeomorphism) that transforms a given region into another target region; although the Riemann mapping theorem (see Gamelin XI.2,6 or Ahlfors 6.1) says that such a map always exists between bounded simply connected regions, the goal of this section is always to construct such maps explicitly.

The most important thing to keep in mind is that all of these maps will be based on (i.e. compositions of) the following three basic maps:

I) Fractional linear transformations (i.e. $\frac{az+b}{cz+d}$ with a, b, c, d complex and $ad - bc \neq 0$; this includes the trivial cases where, say, $c = 0$ and $d = 1$)

II) Exponentials or logarithms

III) Power functions (which can be written in terms of the first two; the proof is an exercise which the present author suggests the reader should know how to do)

For a clear exposition on each of these objects, the reader should consult Gamelin II.7 for fractional linear transformations, Gamelin I.5-6 for exponentials and logarithms, and Gamelin I.7 for power functions.

Finally, it should be noted that the real and imaginary part of conformal maps are (often bounded) harmonic functions; sometimes, conformal map questions will be disguised in the form 'Find a bounded harmonic function on this domain'.

The following is a representative sample of the types of conformal mapping questions that appear on the qual.

F07 Q12: Find a conformal map to the unit disc of the half disc $\{z : |z| < 1, Rez > 0\}$.

Proof: This is one of the eight times that a variant of the question "map the slit disc to the unit disc" has been asked; this one is easier than usual because the first step (take the slit disc, i.e. the unit disc with the positive real axis removed, and apply the square root map to yield the upper half disc) has already been done (to get to the upper half disc from this problem statement, you would instead multiply by i).

At this point, the fractional linear transformation $\frac{z-1}{z+1}$ maps this half disc to the second quadrant of the complex plane (the reader is invited to follow along with pictures).

Then, squaring sends this map to the bottom half-plane, from which $\frac{z+i}{z-i}$ will send this plane to the unit disc as desired.

NOTE: This type of question has many variants, ranging from innocuous to positively evil.

Variation 1: The reader might be presented with a slit half-plane instead of a slit disc (i.e. the upper half-plane with the portion of the imaginary axis where the imaginary part is greater than one is removed); in this case, one must begin with a fractional linear transformation (here, of the form $\frac{z-i}{z+i}$) to return to the case of the slit unit disc.

Variation 2: The slit might be a 'nonstandard' slit (for example, the F06 qualifying exam had the slit removed as $[.5, 1]$ instead of $[0, 1]$); the remedy, as before, is a fractional linear transformation to move the slit to the standard slit ($\frac{z-.5}{1-.5z}$ works here; note that in both of these variations, the map moved the starting point of the slit to the origin).

Variation 3: The unit disc might have TWO slits removed (as was the case in W05 Q4, where the slits removed were $[-1, a]$ and $[b, 1]$ where $a < 0 < b$); here, one needs to use a fractional linear transformation as above to move b to zero and then proceed as in Variation 2... while carefully accounting for what happens to point a. At this point, the doubly-slitted disc will be moved to a singly slitted disc (the initial slit joining -1 to a will become a straight line segment along the IMAGINARY axis, with $-i$ as an endpoint) and one can repeat the steps for Variation 2 with this new segment.

W04 CA4: Find a bounded harmonic function ϕ on the region of the plane $x^2 \geq y^2+1$ such that as (x, y) approaches the boundary $x^2-y^2 = 1$, ϕ approaches 0 for $y > 0$ and 1 for $y < 0$.

Proof: One notes that $(x+iy)^2$ has real part $x^2 - y^2$; therefore, z^2 will send the region to the portion of the complex plane where Re $z > 1$ (and using $z^2 - 1$ would send this to the right half-plane); the boundary segments are sent to the y axis (as $(x + iy)^2$ has imaginary part $2xy$, points above the x axis are sent above the x axis and points below the axis are sent below the axis).

Now, define a branch of the logarithm with branch cut at the negative real axis; now $\log(z^2 - 1)$ is a harmonic function whose imaginary part in this region lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$ (with the upper portion of the boundary being sent to $\frac{\pi}{2}$ and the lower portion being sent to $-\frac{\pi}{2}$).

Therefore, the desired map is $\frac{\pi}{2} - Im(\log(z^2 - 1))\pi$.

F03 Q8 (This question was asked twice, in S02 and F03; it should be noted that this map has never appeared on a qual in the five years since).

Let $I = [-1, 1]$ and U be its complement in the complex plane.

(a) Show that there exists a non-constant bounded analytic function on U.

Proof: The conformal map $\frac{z-1}{z+1}$ sends U to the complex plane with the negative real axis removed, after which a branch of the square root function sends this new image to the right half plane and $\frac{z-1}{z+1}$ sends what is left to the unit circle. Therefore, the composition of these three is a non-constant bounded analytic

(b) Prove that if f is bounded and analytic on U and extends continuously to $\mathbf{C}$ then f is constant.

Proof: This is just Liouville's theorem (the extension of f is bounded on I because I is compact).

NORM (Normal families of functions)

Reference: Gamelin XI.5 (XII.1-2 are overkill) or Ahlfors 5.5 (excluding all discussion on the spherical metric)

List of questions: W02 Q10 (MH), S02 Q13 (M), F04 Q11 (H), F06 Q12 (MH), F08 Q10 (ME), S09 Q10 (M)

Frequency: 1 question every 2.83 exams (fairly uncommon; although hard questions have been known to come from this topic in the past, both of the most recent ones were actually fairly easy)

Description: One of the standard resuls of real analysis is the Arzela-Ascoli theorem; it states that if K is a compact metric space then a family of real-valued continuous functions on K is compact (i.e. every sequence has a uniformly convergent subsequence) if and only if it is complete, pointwise bounded, and equicontinuous (this last condition means that for each $\epsilon > 0$ and each $x \in K$ there exists δ such that for each f in the family of functions, $|f(x) - f(y)| < \epsilon$ for $d(x, y) < \epsilon$. In fact, K does not need to be compact (but in this case, 'uniformly convergent subsequence' is replaced by 'uniformly convergent on compact sets'); further, the proof works just as well if the family of functions is complex-valued. It should be noted that Arzela-Ascoli is on the basic exam syllabus and appeared as question S06 Q4 on the qual (real analysis section; category BAS).

Normally, equicontinuity is the hardest condition to check; however, in the case of families of analytic functions on a domain U there is an easier way to proceed. In this case, a pointwise bounded family of functions is equicontinuous if and only if it is uniformly bounded on each compact set (see, for example, Cauchy's integral formula for $f'(z)$). An outline of the proof of this revised version of the Arzela-Ascoli theorem (replacing 'equicontinuous' with 'uniformly bounded on compact sets'), which is due to Montel, appears as Question 10 of the Fall 2008 qualifying exam (one of the rare cases where the test-taker is asked to prove a major theorem instead of applying it).

It should be pointed out that in this case, the standard way to refer to a compact family of analytic functions is as a 'normal' family (and uniform convergence on compact sets is referred to as 'normal' convergence); hence the name of this section. It should also be noted (the following makes S09 Q10 easy) that L^2 convergence implies normal convergence (because, for example, the value of an analytic function at a point is equal to its average over a disc centered at that point) but not conversely (look at $\{(n+1)z^n\}$ on the unit disc).

On the next page is the hardest question that was ever asked about normal families of functions.

F04 Q11: Let $f(z)$ be a bounded analytic function on the open right half-plane such that $f(x) \to 0$ as $x \to 0$ along the positive real axis. Suppose $0 < \theta_0 < \pi/2$. Prove that $f(z) \to 0$ as $z \to 0$, uniformly in the sector $|argz| < |\theta_0|$.

Proof: Fixing such a sector, one defines $f_n(z) = f(\frac{z}{n})$.

Applying Montel's theorem to $\{f_n\}$ (uniformly bounded by the bound of f), a subsequence of f_n converges uniformly on compact sets (this is called converging normally) of the sector to some analytic function g. Since $f_n \to 0$ uniformly on the real axis, $g = 0$ on the real axis and therefore in the sector.

One finishes by noting that for each $\epsilon > 0$ there exists N such that $|f_N| < \epsilon$ in the region of this sector where $|z| < 1$; therefore, in this sector, if $|z| < N^{-1}$, $|f(z)| < \epsilon$ giving the desired convergence.

HAR (Harnack's inequality)

Reference: Gamelin XV.3 or Ahlfors 6.3 (it should be noted that this is beyond the 'standard' portion of the syllabus)

List of questions: F02 Q13 (M), W05 Q2 (M), W07 Q12 (M), F08 Q9 (MH)

Frequency: 1 question every 4.25 exams (rare; note that F02 Q13, W05 Q2, and W07 Q12 are nearly identical)

Description: Harnack's inequality concerns positive harmonic functions on the unit disc and reads as follows: if f is a nonnegative harmonic function on the unit disc, then for each z in the unit disc

$$\frac{1-|z|}{1+|z|}f(0) \leq f(z) \leq \frac{1+|z|}{1-|z|}f(0);$$

the proof (which has never been tested) follows from looking closely at the Poisson kernel.

If g is a positive harmonic function on a connected domain U, one can look at a point $z_0 \in U$ and let δ be its distance to the boundary. By applying Harnack's inequality to $g(z_0+\delta w)$ and noting that $\frac{1+|z|}{1-|z|} = 3$ for $z = \frac{1}{2}$, one concludes that if $|z - z_0| < .5\delta$ then $\frac{1}{3}g(z_0) \leq g(z) \leq 3g(z_0)$. Standard topological arguments show that g is bounded on any compact subset of U (with this bound merely depending on the choice of subset and the value of g at one fixed point in the subset); the reader should now be able to do F02 Q13, W05 Q2, and W07 Q12.

The only other type of Harnack's inequality question asked on the qual is given below.

F08 Q9. Let D be the open unit disc and H be the upper half plane.

(a) Explicitly describe all conformal maps g from H to D that obey $g(i) = 0$.

Proof: (It should be noted that this part by itself would be a CM question.) The map $\frac{z-i}{z+i}$ fits the description; therefore, the maps in question are maps of the form $g \circ h$ where g is a conformal self-map of the unit disc fixing the origin and h is $\frac{z-i}{z+i}$; therefore, g must be multiplication by a complex number with norm 1.

In other words, the maps are the ones sending z to $e^{i\theta}\frac{z-i}{z+i}$ for some fixed real θ.

(b) Suppose $f : D \to H$ has $f(0) = i$, f holomorphic. Show Im $f(x) \geq \frac{1-x}{1+x}$ for all $x \in (0, 1)$.

Proof: Im f is a positive harmonic function on D (as each point of $f(D)$ has positive imaginary part, and components of analytic functions are harmonic); the inequality is simply the left-hand side of Harnack's inequality for z positive real.

MISC (Miscellaneous questions)

Reference: Provided after each question.

List of questions: S06 Q9 (H), F07 Q9 (H), S09 Q7 (VH), S09 Q8 (MH), S09 Q9 (H), S09 Q12 (H), F09 Q7 (MH)

Frequency: 1 question every 2.58 exams (fairly uncommon, but it should be noted that all seven examples are recent and the majority of them come from the notorious S09 qualifying exam)

Description: Of the 99 questions asked on the complex analysis section of the analysis qualifying exams, seven of them do not fit into the broad categories previously listed. In determining whether or not to create a new category for a broad method (or combination of methods), the following rule of thumb was used: if a topic has had two or more questions asked (such as Laurent series), it is important enough to warrant its own category, while if questions related to the topic have been asked only once, it is placed in the 'miscellaneous' section. (This is a slight modification from the same rule of thumb from the real analysis section because complex analysis topics tend to be more specific and focused; for example, HAR refers to a single inequality.) It should be noted that technically, the two harmonic analysis questions appearing here should be moved to their own category because of this rule; however, the present author has chosen not to do so at the moment (in part because the natural three-letter abbreviation is already taken).

While (usually) not impossible, these questions tend to be fairly difficult; often these questions arise because the author of the qualifying exam is interested in a particular non-standard topic. In these cases, as it is not necessary to answer every question (in fact, the instructions explicitly tell the student to leave two to three questions blank), it is often intended that this will be the question that the students skip.

As in the real analysis section, the best way to deal with these questions is to list them and give hints or a brief outline on how to solve them (along with a reference for the corresponding area of mathematics).

S06 Q9: Show that any analytic function $f : \mathbf{C} \to \mathbf{C}$ that obeys $|f(z)| = 1$ on the real line can be written as $f(z) = e^{g(z)}$ for some analytic function $g(z)$ (the problem states 'Prove an analogue of the Schwarz reflection principle' as a hint).

Hints: Begin by noting that it suffices to show that $f \neq 0$ on the complex plane (when this is done, one can let g be a primitive of $\frac{f'}{f}$ with a constant added to make $e^{g(0)} = f(0)$).

To do this, look at $(f(\bar{z}))^*$, which is meromorphic and agrees with f on the real line (recall $z^* = \overline{z^{-1}}$) and therefore agrees with f; as f has no poles, $(f(\bar{z}))^*$ has no zeroes and therefore so does f.

Note: Although this problem uses the Schwarz reflection principle (with the simple reflections $\bar{z}$ and $*$), like two other problems on the qualifying exam, this is not the reason that it appears in the MISC section. The reason is actually that it combines the Schwarz reflection principle with analytic continuation in a nontrivial way.

Reference: For analytic continuation by integrals see Gamelin VIII.6-8 or Ahlfors 4.4; for the Schwarz reflection principle (which in its easiest form says that f is analytic on the upper half plane and real valued on the real line, f can be extended analytically below the real line as $\overline{f(\bar{z})}$), see Gamelin X.3 or Ahlfors 4.6.5.

F07 Q9: Suppose $h : D - \{(0,0)\} \to \mathbf{R}$ is harmonic where D is the unit disc.

(a) Show there is exactly one real number A such that $h - A\ln|z|$ is the real part of a holomorphic function on $D - \{(0,0\}$. (The hint given is to consider the harmonic conjugate constructed by contour integration.)

(b) Use part (a) to show that if h is bounded then h extends to be a harmonic function on D.

Hints: (a) Uniqueness is trivial (otherwise, $\ln|z|$ would extend to be the real part of such a function and it has the wrong type of singularity to do so).

For existence, write $f = h_x - ih_y$; then f is analytic and has a well-defined primitive if and only if the integral of f around a simple closed curve (say $|z| = \frac{1}{2}$) is equal to zero. Because the integral of

$$z^{-1} = \frac{x - iy}{x^2 + y^2} = (\ln|z|)_x - i(\ln|z|)_y)$$

along this curve is equal to $2\pi i$, this can be forced by subtracting the appropriate multiple $A\ln|z|$ of $\ln|z|$ from f. (The integral is pure imaginary because $h_x dx + h_y dy = dh$, so A is real).

Assuming this is the case, there is a well-defined primitive on the punctured disc whose real part differs from that of h by a constant; by adding this constant one creates an analytic function whose real part is h.

(b) If A were nonzero then any analytic function whose real part was $h - A\ln|z|$ would have the wrong type of singularity at the origin, so $A = 0$, forcing the associated singularity to be removable (as the real part is bounded) so the analytic function from (a) extends to the origin and therefore so does h.

Note: This is one of three problems (the others are a miscellaneous one from a recent real analysis qual and S09 Q9) to study harmonic functions in detail; the reason that this is not an INT question is that the basic complex integration methods to find an analytic conjugate only work in star-shaped regions.

Reference: The key step is in Ahlfors 4.6.1 (the creation of $u_x - iu_y$); the study of singularities is in Gamelin VI.2.

S09 Q7: Let μ be a finite positive Borel measure on $\mathbf{C}$.

a) Prove $F(z) := \int_{\mathbf{C}} \frac{d\mu(w)}{z-w}$ exists for almost every $z \in \mathbf{C}$ and the integral of F is finite over every compact subset K of the plane.

b) Use (a) to show that for almost every horizontal line L (i.e. the set of ordinates, which are y-coordinates, of horizontal lines which fail form a nullset), the line integral of F over every compact K is finite.

c) Show that for almost all open squares S (the squares are parametrized by (z_1, z_2) which are their lower left and upper right vertices; the sense is with respect to Lebesgue measure on $\mathbf{C}^2$), $\mu(S) = \frac{1}{2\pi i}\int_{\delta S} F(z)dz$.

Hint: a) It suffices to show that if $K = B(0,r)$ then $\int_K \int_{\mathbf{C}} |\frac{1}{z-w}| d\mu(w)|dz| < \infty$. One does this by switching the order of integration, substituting $z = w + z'$, and using polar coordinates to evaluate the inner integral (note that z^{-1} is locally integrable in $\mathbf{R}^2$, even though it isn't in $\mathbf{R}$).

b) It suffices to consider only K of the form $[-N, N]$ (indexed by abscissas, which are x-coordinates) for N integral; because the integral of $|F|$ over $[-N, N]$ x $[-M, M]$ is finite by the preceding part, the integral over $[-N, N]$ is finite for almost all L with ordinate in $[-M, M]$. The conclusion follows from basic measure theory.

c) The hint says to use b) and the analogous result for vertical lines; by this part, it suffices to consider those squares S such that $|F|$ is integrable over the boundary of S (and by finiteness of μ, we may also suppose $\mu(\delta S) = 0$). The conclusion follows by integrating F over δS (using the integral definition), using Fubini to switch the order of integration, and then using Cauchy's integral formula to get $2\pi i 1_S$ for the inner integral (except possibly on δS, which is a μ-nullset anyway).

Note: The sole complex analysis content of this entire (long) question is in the last part of c), which is just a trivial application of Cauchy's integral formula. If not for c), this question would instead be classified as RAID and rated H (the main thing that bumps the question up to VH is its length. The object developed in the question is called the Cauchy integral, which is NOT to be confused with the Cauchy integral formula!

Reference: The only 'standard' reference the present author could find for the Cauchy integral is in Section VII.4 of Stein's "Harmonic Analysis".

S09 Q8: Let f be an entire non-constant function with $f(1-z) = 1 - f(z)$ for each complex z. Show $f(\mathbf{C}) = \mathbf{C}$.

Proof: Picard's theorem states that if f is entire, it is either constant or has image omitting at most one value. Now, if $t \notin f(\mathbf{C})$ then either $t = .5$ (a contradiction as $f(.5) = .5$) or not (then $1-t$ is not in the image of f; if $f(z) = 1-t$ then $f(1-z) = t$, which gives two distinct values not in the image of f and therefore contradicting Picard's theorem) proving the desired result.

Note: The question would be fairly easy except for the use of Picard's theorem (which is not listed in the official syllabus and goes beyond the 'standard' sections of Ahlfors and Gamelin that are usually covered in the qual). That being said, the same can be said for Harnack's inequality (which itself has appeared on a number of qualifying exam questions). As Picard's theorem is a lot like Harnack's inequality in that it is a lot easier to use than to prove, the present author believes that the best strategy for Picard's theorem is a lot like that for Harnack's inequality (focus on knowing how to use it).

Reference: Gamelin XII.1-2 (XII.1 consists of lemmata in spherical geometry used to prove the theorem, which is found in XII.2) or Ahlfors 8.3.

S09 Q9: Let $f(z)$ be entire with $f(0) \neq 0$. Let $\{a_n\}$ be the zeroes of f counted by multiplicity.

a) Let $R > 0$ with f nonzero on $|z| = R$. Show that the average value of $\log f$ on $|z| = R$ (i.e. $\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta = \log|f(0)| + \Sigma_{a_n<R}\log(\frac{R}{|a_n|})$.

b) Assume $|f(z)| \leq Ce^{|z|^\lambda}$ where C, λ are positive constants. Show that if $\epsilon > 0$, $\Sigma(\frac{1}{|a_n|})^{\lambda+\epsilon}$ is finite.

Hints: a) Write $f(z) = g(z) * \Pi_n(z - a_n)$ where $g \neq 0$ on $|z| \leq R$ and the product is taken over all zeroes in $|z| < R$. The RHS becomes $\log|g(0)| + NR$ where N is the number of zeroes in the product. Now, use that the average value of $\log g(z)$ over $|z| = R$ is $\log g(0)$ (by the mean value property of harmonic functions) and that the function sending w to the average value of $\log|z - w|$ over $|z| = R$ is harmonic and radial on $|z| < R$, which means it is constant and equal to its value at $w = 0$, i.e. $\log R$.

b) The hint asks the test-taker to estimate the number of zeroes with $|a_n| < R$ by using (a) for the circle of radius 2R; as the LHS gives a value which is at most $(2R)^\lambda$ up to some additive constant and each zero with $|a_n| < R$ counts in the RHS as at least $\log 2$, the number of zeroes is at most $AR^\lambda + B$ for some constants A, B. Now, let Z_0 be the desired sum over all a_n with norm < 1 (this is finite because zeroes of nonconstant functions are isolated) and Z_n (for $n > 0$) be the desired sum over all a_n with norm in $[.5 * 2^n, 2^n)$ and observe that the Z_n decay geometrically.

Note: This is the second question (both asked in recent years) about harmonic functions in detail which appeared in the complex analysis qual. One of the reasons that these two questions are still considered miscellaneous instead of forming their own category is that the relevant material cannot be found in Gamelin (only in Ahlfors).

Reference: Ahlfors 4.6.2.

S09 Q12: Let Q be the closed unit square in the complex plane $\mathbf{C}$ and let R be the closed rectangle in $\mathbf{C}$ with vertices $\{0, 2, i, 2+i\}$. Prove that there does NOT exist a surjective homeomorphism $f : Q \to R$ which is conformal on the interior and maps corners to corners.

Hint: As f is a homeomorphism of boundaries as well, we may suppose f maps CORRESPONDING corners to CORRESPONDING corners (i.e. lower left to lower left, etc). Now, estimate the integral of $|f'|$ (which is taken to be the conjugate-analytic derivative if f is conjugtae analytic) over Q by two different ways: from above (Cauchy-Schwarz gives an upper bound of $\sqrt{\int_Q 1 * \int_Q |f'|^2} = \sqrt{2}$) and from below (as f sends each horizontal line segment of length 1 in Q to something joining $x = 0$ to $x = 2$, the integral of $|f'|$ over each such segment is at least two, which therefore serves as a lower bound) to get a contradiction because $2 > \sqrt{2}$.

Note: The focus of this question, quasiconformal mappings, is actually very closely related to the author's research.

Reference: This argument cannot be found in either Ahlfors' or Gamelin's book. Instead, it is a simplified version of Section I.B of Ahlfors's book "Lectures on quasiconformal mappings". The reader can read Sections I.A-D of the book to see the proper place of this question in mathematics (in terms of quasiconformal maps and extremal length); for a more modern perspective, the reader can consult Chapter 7 of Juha Heinonen's excellent book "Lectures on analysis on metric spaces" (which was the present author's introduction to his current areas of research). However, the present author does NOT recommend that the reader look at Heinonen's book in depth until AFTER having passed the analysis qual!

F09 Q7: a) Define unitary operator on a complex Hilbert space.

b) Let S be a unitary operator on a complex Hilbert space. Using your definition, prove that for every complex number $|\lambda| < 1$ the operator $S - \lambda I$ (where I is the identity operator) is invertible.

c) Fix v in the Hilbert space; for all $\lambda \in \mathbf{C}$ with norm less than one, we define

$$h(\lambda) =< (S + \lambda I)(S - \lambda I)^{-1} v, v > .$$

Show that Re h is a positive harmonic function (without invoking the spectral theorem).

Hints: a) S is unitary iff $< Sv, w >=< v, S^{-1} w >$ for all v, w in the Hilbert space.

b) $< Sv, Sv >=< v, v >$ so $||S|| = 1$ in operator norm; write $S - \lambda I = S(I - \lambda S^{-1})$ and use the standard power series expansion for $(I - \lambda S^{-1})^{-1}$.

c) For 'harmonic', show h is analytic by using the power series expansion from the previous part. For 'positive', fix λ, let $w = (S - \lambda)^{-1} w$ and note $h(\lambda) =< (S + \lambda I)w, (S - \lambda I)w >$, which is easy to expand.

Note: This is the second straight Question 7 from a complex analysis qual which had very little complex analysis content. The sole complex analysis portion of this question is the usage of a power series expansion in c) to show analyticity. Otherwise, this question would be a standard real analysis question in the HIL section (for Hilbert spaces) as nearly all of the question is standard Hilbert space theory.

Reference: The standard references in the real analysis section for Hilbert space theory (Folland 5.5 or Stein-Shakarchi Chapter 4) are the appropriate ones here.