Supplemental Notes for Week 8 and 9

The main topics covered over the past week and a half were the argument prinicple and the Schwarz reflection principle.

I) Argument principle

(Ahlfors section 4.5.4; Gamelin sections VIII.1-4)

Suppose $U \subset\subset V$ are open (this notation means that the closure of U is contained in V) and $\delta U = \gamma$ is a simple closed curve. If f is meromorphic in V and analytic (and nonzero) on γ then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$ is equal to the difference of the number of zeroes in U and the number of poles in V, counting each one up to multiplicity.

(This is known as the argument principle because the integrand can be thought of as $d(Argz)$; although the argument of z may not be well-defined on the entire curve γ, it is defined locally at each point, so the differential $d(Argz)$ makes sense.)

On most qualifying exam problems, all you need is the following simple consequence.

Theorem (Rouche): Suppose U, V, γ are as in the argument principle. Let f, g be analytic on V and suppose $|g| < |f|$ on γ (this rules out any zeroes of f on this set). Counting multiplicity, $f+g$ and f have the same number of zeroes in U.

(This follows directly from the argument principle: one notes that $d(Arg(f+g)) = d(Arg(f)) + d(Arg(1+g/f))$; as g/f has strictly positive real part on the domain, one can define the argument of $1+g/f$ smoothly on γ.)

S08 problem 13: Suppose $D \subset\subset \Omega \subset \mathbf{C}$, f_n are analytic on Γ and converge uniformly to f. If f is nonzero on the unit circle, show that there exists N such that whenever $n \geq N$, f_n and f have the same number of zeroes in the unit disc D.

One notes that 'same number of zeroes' has to refer to multiplicity; to see why, take $f_n = x^2 + 1/n$ (which has two zeroes in the unit circle for $n > 1$) and $f = x^2$ (which has a single zero of multiplicity two).

With this caveat, this problem is a relatively straightforward consequence of Rouche's theorem and compactness: as δD is compact, there exists $\epsilon > 0$ with $|f| > \epsilon$ on δD. By the definintion of uniform convergence there exists N such that for $n > N$, $||f - f_n|| < \epsilon$ (this notation means that the supremum of $|f(z) - f_n(z)|$ is less than ϵ). Rouche's theorem then tells us (because $|f - f_n| = |f_n - f|$ at each point) that f and $f + (f_n - f) = f_n$ have the same number of zeroes (up to multiplicity) in the unit disc for each such n, precisely as desired.

(NOTE: One could solve this problem directly by the argument principle; the hardest step here is to show that if f_n converges to f uniformly on δD and f is nonzero, then $\frac{f_n'}{f_n}$ converges uniformly to $\frac{f'}{f}$ on δD. The current writer suggests that the latter serve as a challenge to the reader.)

II) Schwarz Reflection Principle (Ahlfors 4.6.5; Gamelin X.3)

The principle works as follows. Suppose γ and η are circles in the complex plane (this allows for the case of lines) and f is a meromorphic function defined on one side of γ. If f extends continuously to γ by sending the points on γ to η then one can extend f meromorphically to the entire complex plane; further, this extension is unique.

We do this by letting ϕ_γ, ϕ_η be the standard reflections across γ and η (i.e. the reflections that are conformally equivalent to complex conjugation). On the 'other side' of f, we extend f by $\phi_\eta \circ f \circ \phi_\gamma$.

(NOTE: Technicalities involving the point at infinity are nearly invariably best dealt with as they occur).

There has been exactly one problem in recent qual history (not counting the 'strictly subharmonic' one) that requires the Schwarz Reflection Principle; it is the following.

S06 9: Show that an entire function f such that $|f(z)| = 1$ whenever z is real can be written as $f(z) = e^{g(z)}$ for some analytic function $g(z)$.

(The problem gave the hint 'Prove an analogue of the Schwarz reflection principle').

One can easily stare at this hard problem (which I left blank when I took that qual) for a while without having any idea how this exponential relates to the reflection principle.

Consequently, we begin by looking for conditions for f to equal e^g.

I) $f(z) \neq 0$ is required because the exponential function is never zero.

II) Differentiate both sides: $f' = g'e^g$; dividing by f (we can do this because $f \neq 0$) tells us $\frac{f'}{f} = g'$.

Now, once we have I) and II) for some function g (i.e. g can be thought of as a branch of the argument), what do we know about e^g?

The best way to proceed is to take advantage of I to look at $\frac{e^g}{f}$; its derivative will be $\frac{g'e^g}{f} - \frac{f'e^g}{f^2} = \frac{g'e^g}{f} - \frac{g'e^g}{f} = 0$ (using II). This means that $\frac{e^g}{f} = C$ for some constant C (which is clearly nonzero). Letting z_0 be a number such that $e^{z_0} = C$, we note that $e^{g-z_0} = fC * C^{-1} = f$, so the existence of a 'logarithm' for f is given whenever there exists a function g satisfying I) and II).

However, because f is defined on the entire complex plane, if f is nonzero there exists a function g satisfying II) if and only if the integral of $\frac{f'}{f}$ over any simple closed curve is nonzero (this is the fundamental theorem of calculus for complex integrals).

Therefore, to solve the problem, it suffices to show

I) $f \neq 0$ on the complex plane

II') $\int_\gamma \frac{f'(z)}{f(z)} dz = 0$ whenever γ is a simple closed curve.

As II') is a trivial consequence of I) (by either Cauchy's theorem or the argument principle), we merely need to show that f is nonzero.

We do this by using the reflections 'bar' (complex conjugation) and 'star' ($z* = \overline{z^{-1}}$; this is reflection across the unit circle).

One notes that $(f(\bar{z}))*$ is a meromorphic function of $\mathbf{C}$ which equals f on the real line by hypothesis (the real line is fixed by conjugation; the unit circle is fixed by $*$)

so (removing all the poles of this new function, if necessary; then extending to them later by continuity) we have that $f(z) = (f(\bar{z}))*$ for all complex z. This implies that f has no zeroes because if f had a zero at z then f would have to have a pole at $\bar{z}$ contradicting that f is entire.

By everything said earlier, this indeed gives an analytic g with $f = e^g$.

(NOTE: This was not even the hardest question on that particular qualifying exam; that one will be done in Week 10 either in section or as a handout.)