Week 7: Isolated Singularities

References: Ahlfors 4.3, Gamelin VI.2-5

The topic for this is isolated singularities: we wish to know what happens when an otherwise analytic function fails to be defined at a single point.

Definition: A function f has an isolated singularity at a point $z \in \mathbf{Z}$ if there is a neighborhood U of z such that f is defined and analytic on $U \backslash \{z\}$, but f is not defined at z.

Nonexample: The function $\log z$ does not have an isolated singularity at the origin because any branch of the logarithm function is undefined on some curve starting at the origin. Every neighborhood of the origin must contain some segment of that curve.

The important fact about isolated singularities is that there are only three types of them:

I) f has a removable singularity at z_0. This means that f is bounded near z_0; whenever this happens, f can be extended to z_0 such that f is continuous and analytic at z_0.

(Example: $f = \frac{z^2}{z}$; although this function is technically undefined at the origin, the singularity there is removable as we can clearly define $f(0) = 0$, which will make f equal to the identity function z.)

II) For some positive integer n, f has a pole of order n at z_0. This means that $\frac{1}{f}$ is defined sufficiently close to z_0 and, defining $\frac{1}{f}(z_0) = 0$, $\frac{1}{f}$ has a zero of order n at z_0. In other words, the limit of $(z - z_0)^n f(z)$ as z approaches z_0 exists and is nonzero.

(Example: $f = \frac{1}{z^2} + \frac{7}{z^3}$ has a pole of order 3 at the origin. Its reciprocal, $\frac{z^3}{z+7}$, is defined at the origin and clearly has a zero of order 3 there.)

III) f has an essential singularity at z_0. This is the weirdest of the three cases; for every complex number w (including ∞) there exists a sequence $\{z_k\}$ converging to z_0 such that $f(z_k)$ converges to w. You can show the existence of an essential singularity, though, by showing that there exists a subsequence for two distinct w (for a removable singularity $f(z_k)$ would always converge to the same finite limit; for a pole it would always converge to ∞ in the Riemann sphere).

(Example: $f = e^{-\frac{1}{z^2}}$ has an essential singularity at the origin. As z approaches 0 along the real line, $f(z)$ approaches 0; as z approaches 0 along the purely imaginary line, $f(z)$ approaches ∞.)

Just as last week, we shall work out two problems from qualifying exams; one fairly easy and one a bit harder.

W07 9: Let $f(z)$ be analytic for $0 < |z| < 1$ and suppose $C > 0, m \geq 1$, and $|f^{(m)}(z)| \leq \frac{C}{|z|^m}$ for all z in the domain. Show f has a removable singularity at the origin.

Before beginning, it is worthwhile to note the following fact about derivatives and isolated singularities.

I) If f has a removable singularity at z_0 then so does f' (the derivative of an analytic function is analytic by Cauchy's integral theorem).

II) If f has a pole at z_0 of order n then f' has a pole at z_0 of order $n+1$ (the easiest way to see this is to compare principal parts).

III) If f has an essential singularity at z_0 then so does f' (picking $\epsilon, N > 0$ with N an integer, $\epsilon < .5$ and z_1 with $0 < |z_1 - z_0| < \epsilon$ and $|f(z_1)| > 1$ there exists w with $0 < |w - z_0| < |z_1 - z_0|$ such that $|f(w)| > |f(z_1)||w - z_0|^{-N}$ because otherwise $f(z-z_0)^N$ has a removable singularity at z_0; by drawing an appropriate path joining z to w one finds a point w_1 with $|w_1 - z_0| < \epsilon$ where $|f'(w_1)| > C|w_1 - z_0|^{-N}$).

NOTE: A much easier proof of III) is possible if you know how to work with Laurent series.

Armed with this classification, we note that under the circumstances of the problem, $f^{(m)}z^m$ has a removable singularity at 0 so $f^{(m)}$ has either a removable singularity or a pole of order $\leq m$ at 0. However, if the isolated singularity of f at $z = 0$ was a pole (whose order is strictly positive), $f^{(m)}$ would therefore have a pole of order strictly greater than m at $z = 0$, and if it were an essential singularity, $f^{(m)}$ would have one as well. Both contradict that $f^{(m)}$ has either a removable singularity at the origin or a pole of order m or less, so f has a removable singularity at the origin.

The next problem is about as hard as singularity-related problems get. That being said, though, even though a lot of work is required, the steps involved are all intuitive enough that this is only a medium-hard qual problem (in other words, it's significantly easier than W04 CA2, which we did last week.)

W06 10. Prove that $\frac{\pi^2}{\tan^2(\pi z)} = a + \Sigma_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$ for some constant a.

Letting f denote $\frac{\pi^2}{\tan^2(\pi z)}$ and g denote $\Sigma_{n \in \mathbf{Z}} \frac{1}{(z-n)^2}$, the problem asks us to show that $f - g$ is constant, but fortunately one is not asked to actually compute this constant.

Also, we can quickly observe that both f and g are periodic with period 1.

Our first goal is to show that $f - g$ can be extended to an entire function. One begins by noting that clearly f is meromorphic on $\mathbf{C}$ with singularities at the points $.5n$ for $n \in \mathbf{Z}$ (where the denominator is either zero or undefined). If z is congruent to .5 modulo 1, then the function in question can be written as $\frac{\pi^2 \cos^2(\pi z)}{\sin^2(\pi z)}$ from which it is clear that these singularities are removable and therefore can be ignored.

Therefore, it suffices to look at $z \in \mathbf{Z}$; by periodicity we merely need to look at $z = 0$. At this point, one realizes that as $\tan^2(\pi z)$ has a zero of order two at $z = 0$ (because it can be expressed as $\frac{\sin^2(\pi z)}{\cos^2 \pi z}$, which, when divided by z^2, approaches π^2 as z approaches zero), $\frac{\pi^2}{\tan^2(\pi z)}$ has a pole of order 2.

To compute its principal part, one notes that $\frac{z^2\pi^2}{\tan^2(\pi z)}$ approaches 1 as z approaches zero which implies that at the origin, the singularity of

$\frac{\pi^2}{\tan^2(\pi z)} - \frac{1}{z^2}$ at 0 is either removable or a pole of order 1. However, as the function is even, comparing principal parts removes the latter case; therefore the singularity is removable so the principal part of $f(z)$ at $z = 0$ is $\frac{1}{z^2}$. By periodicity, the principal part of $f(z)$ at $z = n$ is $\frac{1}{(z-n)}^2$.

For g, the quadratic nature of the summands (which allows us to compare the terms with $\frac{1}{n^2}$) tells us that if z is not an integer, the series defining g converges uniformly in a neighborhood of z (and therefore g is analytic at z by Morera's theorem).

Further, if n is an integer, the series (with the term corresponding to n) removed still converges in a neighborhood of n, so we conclude that the only poles are the 'obvious' ones: at $z = n$ we have a pole of order 2 with principal part $\frac{1}{(z-n)^2}$.

This implies that f and g are meromorphic with the same poles; therefore, all the singularities of $f - g$ are removable so $f - g$ can indeed be extended to an entire function.

Now that we know $f - g$ is entire, we want to show that it is constant. Our main tool, just like last week, will be Liouville's theorem; therefore we only need to show $f - g$ is bounded. Because $f - g$ is periodic with period 1, we can also suppose that Re $z \in [0, 1]$. Therefore, we can break this strip up into two separate areas; where Im z is large we can estimate the size of f and g directly and where Im z is small we use compactness.

$|$ Im $z| \geq 1$: We note that $\tan(z) = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$. If Im $z > 1$ then $|e^{-iz}| > 2|e^{iz}|$ so $|\tan z| > \frac{1}{3}$ and $|f(z)| < 3\pi^2$; for Im $z < -1$ the same bounds hold except that $-iz$ and iz need to be switched.

For $g(z)$ we note that $|z - n|^2 \geq (|n| - 1)^2 + 1$ from which basic Calculus II methods give a uniform bound.

$-1 \leq$ Im $z < 1$: The region in question is compact (because Re z and Im z are bounded) so f and g, being continuous on compact sets, are bounded.

Therefore, $f - g$ can be extended to an analytic function which is bounded (because the complex plane is the union of two sets such that $f - g$ is bounded on each of those sets) and therefore constant by Liouville, completing the problem!

NOTE: If one wants to actually evaluate a, the easiest way to do so would be to evaluate f and g at points where computation is fairly easy. One can consider, for example, a zero of f (we already determined that $f(.5) = 0$).

Now, $g(.5) = \Sigma_{n\in\mathbf{Z}} \frac{1}{(.5-n)^2} = 2\Sigma_{n\in\mathbf{Z}_{\geq 0}+.5} \frac{1}{n^2}$.

To evaluate this, one notes that

$$\Sigma_{n\in\mathbf{Z}_{\geq 0}+.5} \frac{1}{n^2} = 4\Sigma_{n\in 2\mathbf{Z}_{\geq 0}+1} \frac{1}{n^2}$$

$$= 4(\Sigma_{n\in\mathbf{N}} \frac{1}{n^2} - \Sigma_{n\in 2\mathbf{N}} \frac{1}{n^2})$$

$$= 4(\Sigma_{n\in\mathbf{N}} \frac{1}{n^2} - \frac{1}{4}\Sigma_{n\in\mathbf{N}} \frac{1}{n^2})$$

$$= 3\Sigma_{n\in\mathbf{N}} \frac{1}{n^2} = \frac{\pi^2}{2}$$

(because the sum of the reciprocals of the squares is known to equal $\frac{\pi^2}{6}$)

so $g(.5) = \pi^2$ and $a = -\pi^2$.