Problems graded: p129 6, p148 2, p161 3

p129 3. If e^z had a removable singularity at ∞ it would be bounded in a neighborhood of ∞ (in a neighborhood of 0) and if e^z had a pole at ∞ there would exist some k with $z^{-k}e^z$ bounded in a neighborhood of ∞. However, basic calculus tells us that as z approaches ∞ along the positive real axis, $z^{-k}e^z$ approaches ∞ so the singularity is essential.

Similarly, as $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, which grows faster than any polynomial along the negative imaginary axis (at $-ti$, the value is $\frac{e^t-e^{-t}}{2i}$), and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, which also grows faster than any polynomial along the negative imaginary axis, these functions have essential singularities at ∞ as well.

4. If f is meromorphic in the extended complex plane (with poles $z_1, \ldots, z_n$) we let f_k be the singular part of f at z_k and note that $f - f_1 - \ldots - f_n$ has only removable singularities in the entire complex plane (including at infinity). Being bounded and analytic, this new function is constant so $f = (f - f_1 - \ldots - f_n) + f_1 + \ldots + f_n$ is a sum of rational functions and therefore rational.

5. Because Im z is equal to Re iz (and multiplying by i or -1 does not change the nature of a singularity), WLOG we suppose Re $f(z)$ is bounded from above and let A be the location of the singularity. By translation we may suppose that near A, Re $f(z) \leq 0$. Therefore, defining $g(z) = \frac{f(z)+1}{f(z)-1}$, g is bounded near A (in fact, $|g| < 1$) and therefore has an removable singularity at A. Let B be the limit of $g(z)$ as z goes to A.

However, we note that $g = 1 + \frac{2}{f-1}$ so, solving, $f = \frac{2}{g-1} + 1$ which is defined near A (except possibly at A) because $|g| < 1$ near A when it is defined. This implies that $f(z)$ approaches $\frac{2}{B-1} + 1$ as z goes to A and therefore is itself removable.

6. If A is a removable singularity of $f(z)$ then it is a removable singularity of $e^{f(z)}$ (as z approaches A, $e^{f(z)}$ approaches e raised to the limit of $f(z)$).

We next note that if Re $f(z) \leq C|z-A|^{-1}$ for z near A and some constant C then $f(z)(z-A)$ has a removable singularity at A by the preceding problem. If f does not have a removable singularity then we can write $f(z) = g(z) + \frac{w}{z-A}$ near A where g is analytic and w is a complex constant. In this case, we note that $e^{f(z)} = e^{g(z)}e^{\frac{w}{z-A}}$; the continuity of G implies that e^f and $e^{\frac{w}{z-A}}$ have the same type of singularity at A. However, approaching A along the line $A+tw$, for t positive real and going to zero, gives the same growth estimates as in Problem 3 implying that the singularity is essential.

The only other possibility is when Re $f(z) > |z-A|^{-1}$ for some z sufficiently close to A (in other words, for each $\delta > 0$ there is z within δ of A with Re $f(z) > |z-A|^{-1}$). As $|e^z|$ depends only on Re z, the same growth estimates again tell us that the singularity is essential.

Therefore, an isolated singularity of f must either be removable or essential as a singularity of exp f.

p133 1. We note that if w is complex then $z^2 + z - w$ has two zeroes which sum up to -1. Therefore, we conclude that if $z_1, z_2 \in \mathbf{C}$ then $z_1^2 + z_1 = z_2^2 + z_2$ iff $z_1 + z_2 = -1$. Therefore, the largest disc about the origin about which $z^2 + z$ is one to one is the largest disc centered at 0 which does not contain any two points summing up to -1. $B(0;.5)$ satisfies this (any two points in this disc have real part greater than $-.5$, but any larger ball contains $-.5\pm\epsilon$ for ϵ sufficently small).

2. Because $e^z=e^{z'}$ iff $z'-z=2k\pi i$ for some integer i, e^z is one to one iff there exist no two points which differ by some nonzero multiple of $2\pi i$. Therefore, our desired disc is $B(0;\pi)$ (no two points in that disc can differ by any nonzero multiple of $2\pi i$ by the triangle inequality; however, any larger disc contains $\pm\pi i$).

4. Because $f(z)=f(0)+f'(0)z+O(z^2)$, $f(z^n)=f(0)+f'(0)z^n+O(z^{2n})$. This implies that $f(z^n)-f(0)=z^n(f'(0)+O(z^n)$. Therefore, $h(z):=\frac{f(z^n)-f(0)}{z^n}$, which has a removable singularity at the origin (where the limiting value is $f'(0)$), has image near $z=0$ contained in some ball which does not contain 0. Therefore, a branch of $\log h$ can be defined in a neighborhood of the origin. Letting $g_1=\frac{1}{n}\log h$ (using the appropriate branch), one notes that $(g_1z)^n=f(z^n)-f(0)$ near 0 so setting $g(z)=g_1(z)*z$, $f(z^n)=f(0)+g(z)^n$ in a neighborhood of 0.

p148 2. The connectivity of the region obtained by removing m points (say $z_1,\dots,z_m$) from a simply connected U is $m+1$ because its complement has $m+1$ components in the Riemann sphere: $\{z_1\},\dots,\{z_m\}$, and the complement of U.

Let δ be the minimal distance from any z_i to either another z_j or the boundary of U. Then, we can set γ_i to be the curve defined as follows: $z_i+.1\delta e^{it}$ for $0\le t\le 2\pi$. We note that $n(\gamma_i,z_j)=\delta_{i,j}$ (the δ used is the Kronecker delta) and $n(\gamma_i,z)=0$ for z in the complement of U (as γ_i is a cycle in U, which is simply connected).

These cycles (which are clearly linearly independent) form a homology basis in our new region because if γ is another cycle in our region, we can let n_i be $n(\gamma,z_i)$ for each i and note that setting $\sigma=\gamma-\Sigma_i n_i\gamma_i$, $n(\sigma,z_i)=n_i-n_i=0$ for each i and $n(\sigma,z)=0$ for each z in the complement of U as σ is a cycle in U. Therefore, $\sigma\sim 0$ so $\gamma\sim\Sigma_i n_i\gamma_i$.

3. Let $\{U_\alpha\}$ be the set of regions determined by a closed curve γ (where the α range over some large index set A). We note that the complement of U_α in the complex plane is connected because the component which contains γ must also contain U_β for each $\beta\ne\alpha$ (as the closure of U_β intersects γ and the U_β themselves are connected). If U_α is bounded, then the point at infinity is in the closure of the complement of U_α in the Riemann sphere so this complement has one component and U_α is simply connected. If U_α is unbounded then as γ is bounded, there exists R such that U_α contains all points in the complex plane of magnitude at least R. This implies that the point at infinity is in a different component of the complement of U_α from the complement of U_α in the complex plane, so U_α is doubly connected.

4. If U is simply connected and does not contain the origin, then whenever γ is a cycle in U we have $n(\gamma,0)=0$. This implies that the integral of $\frac{1}{z}$ is independent of path in U so $\frac{1}{z}$ has a primitive f in U. Setting $g=e^f$ in U, we note that $g'=e^ff'=\frac{g}{z}$ in U. From this differential equation we note that $\frac{g}{z}$ is

constant on U as its derivative is $\frac{g}{z^2} - \frac{g}{z^2} = 0$. Therefore, there is some nonzero constant C such that for all $z \in U$, $e^{f(z)} = Cz$. Setting z_0 to be a number with $e^{z_0} = C^{-1}$ and h to be equal to $f + z_0$, we have $e^{h(z)} = z$, i.e. h is an analytic branch of the logarithm defined on U.

Because branches of z^α and z^z are defined as $e^{\alpha \log z}$ and $e^{z \log z}$ respectively, given branches of the logarithm, $e^{\alpha h(z)}$ and $e^{zh(z)}$ are branches of those two functions defined in our region.

5. One tempting way to proceed would be, as in the previous problem, to find a branch of $\log(1-z^2)$ on such a region U. However, this does not work because its derivative, $\frac{1}{z-1} + \frac{1}{z+1}$ (note $\log(1-z^2) = \log(1-z) + \log(1+z) + C$ for some constant C, which is a multiple of $2\pi i$), is not necessarily homologous to zero;

the integral of this function along some curve $\gamma \in U$ is $n(\gamma, 1) + n(\gamma, -1)$ and we only know that these two numbers are equal (as 1 and -1 are in the same component of the complement).

On the other hand, we know that $n(\gamma, 1) - n(\gamma, -1)$, the integral of $\frac{1}{z-1} - \frac{1}{z+1}$, is equal to zero for any γ, so $\frac{1}{z-1} - \frac{1}{z+1}$ has a primitive f in U. Setting $g = e^f \frac{z+1}{z-1}$, we note that g is constant on U. as its derivative is equal to

$$(e^f)' \frac{z+1}{z-1} + e^f \frac{(z+1)'}{z-1} + e^f (z+1) (\frac{1}{z-1})'$$

$$= g(\frac{1}{z-1} - \frac{1}{z+1} + \frac{1}{z+1} - \frac{1}{z-1}) = 0$$

on U.

Letting z_0 be a number such that e^{-z_0} is equal to this constant value of g, we therefore have that setting $h(z) = f(z) + z_0$, $e^{h(z)} = \frac{z-1}{z+1}$. This means that if we set $\phi(z) = i(z+1)e^{.5h(z)}$ then $\phi(z)^2 = -(z+1)^2 \frac{z-1}{z+1} = -(z+1)(z-1) = 1 - z^2$ on U so ϕ is indeed a single-valued analytic branch of $\sqrt{1-z^2}$.

To find the integral of $\frac{dz}{\sqrt{1-z^2}}$ over a closed curve in the region we note that there are two possibilities; the component containing ± 1 may be either bounded or unbounded. If the component were unbounded then $\sqrt{1-z^2}$ could have been defined on the entire complement of that component and the integral would be an integral of an analytic function over a closed curve in a simply connected region, therefore equal to zero.

If the component is bounded we can again suppose that the function was defined on all of the complement of the component. Therefore, there exists $M > 0$ such that for all $|z| > M$, our branch $\phi(z)$ of $\sqrt{1-z^2}$ is defined on z. If $R > M$ then the cycle $Re^{i\theta}$ (for $0 < \theta < 2\pi$) forms a single-element homology basis for this region; therefore, the integral of $\frac{dz}{\phi(z)}$ over any closed curve in the region will be an integer multiple of the integral over this curve. To evaluate it we suppose R is sufficiently large and make the substitution $u = \frac{1}{z}$, so $z = \frac{1}{u}$ and $dz = \frac{-1}{u^2} du$.

Then we are integrating the function $\frac{-(u^{-2})}{\sqrt{1-u^{-2}}}$ with respect to u over $R^{-1}e^{-i\theta}$ (for $0 < \theta < 2\pi$).

This function can be written as $\frac{-1}{u\sqrt{u^2-1}}$ (note that all branches of the square root function are defined in terms of ϕ). Therefore, letting γ_R be the circle centered at the origin of radius R^{-1} oriented clockwise, the integral we seek is $\int_{\gamma_R} \frac{du}{u\sqrt{u^2-1}}$. By Cauchy's integral formula, this integral is $2\pi i * W$ where W is the value of the square root of negative one with respect to our branch of the square root (which can either be i or $-i$). This implies that the integral with respect to this curve is either 2π or -2π (both of which are integral multiples of each other) so in this case, the integral must be $2\pi k$ for some integer k and each such value can be attained.

p154 2. We note that on the circle $|z| = 1$, $|-6z| = 6$, while $|z^4 + 3| \leq 4$ so Rouche's theorem says that $z^4 - 6z + 3$ has the same number of roots as $-6z$ on $\{|z| < 1\}$; in other words: one.

On the circle $|z| = 2$, $|z^4| = 16$ while $|-6z+3| \leq 6 * 2 + 3 = 15$ so Rouche's theorem says that $z^4 - 6z + 3$ has the same number of roots (up to multiplicity) as z^4 on $\{|z| < 2\}$; in other words: four (and this is all the possible roots as the polynomial has degree four). Clearly $|z^4 - 6z + 3| \geq 1$ on both $|z| = 1$ and $|z| = 2$ so up to multiplicity, $z^4 - 6z + 3$ has three roots with modulus between 1 and 2. These are all simple roots because the derivative of the polynomial, $4z^3 - 6$, has its zeroes at the three cube roots of 1.5. If z is a root of both $z^4 - 6z + 3$ and $4z^3 - 6$, then $z^4 = 1.5z$ which implies that $0 = z^4 - 6z + 3 = -4.5z + 3 = 0$ so $z = -\frac{2}{3}$ which is not a root of either expression, so all roots are simple and exactly three roots have modulus between one and two.

3. We begin by looking at the behavior of $f(z) := z^4 + 8z^3 + 3z^2 + 8z + 3$ on the negative axis (where $z = it$). Then, the expression can be written as $(t^4 - 3t^2 + 3) + (-8t^3 + 8t)i$. This expression is real exactly when $8t^3 - 8t = 0$, i.e. for $t = 0$ (when the expression is 3), 1 (when the expression is 1), and -1 (when the expression is -1). In other words, on the imaginary axis, our polynomial never hits the nonpositive real axis. This means that no zeroes of our polynomial lie on the imaginary axis; further, a well-defined branch of the logarithm can be defined on this axis.

Next, we note that f has at most four roots (being of degree 4); let M be the maximum modulus of a root. Therefore, if $R > M$ we can define the contour γ_R as follows; $\gamma_{R,1}$ is the semicircle $|z| = R$ to the right of the imaginary axis oriented counterclockwise, $\gamma_{R,2}$ is the line from Ri to $-Ri$ oriented downwards, and $\gamma = \gamma_{R,1} + \gamma_{R,2}$; all roots of f in the right half plane lie in the interior of γ_R.

Therefore, the argument principle tells us that the number of roots of f in the right half plane is equal to $\frac{1}{2\pi i} \int_{\gamma_R} \frac{f'(z)dz}{f(z)}$.

To evaluate this integral, we begin by looking at $\gamma_{R,1}$.

On this curve, $\frac{f'(z)}{f(z)} = \frac{4z^3+24z^2+6z+8}{z^4+8z^3+3z^2+8z+3}$.

Suppose $|z| > 100$. Then, $|z^4 + 8z^3 + 3z^2 + 8z + 3| = |z^4||1 - (8z^{-1} + 3z^{-2} + 8z^{-3} + 3)| > .5|z^4|$

so

$$|\frac{4z^3+24z^2+6z+8}{z^4+8z^3+3z^2+8z+3}-\frac{4z^3}{24z^2+6z+8}|<\frac{|24z^2+6z+8|}{.5|z|^4}<100|z|^{-2}.$$

Further,

$$|\frac{4z^3}{z^4+8z^3+3z^2+8z+3}-\frac{4z^3}{z^4}|=|\frac{4z^3(8z^3+3z^2+8z+3)}{(z^4+8z^3+3z^2+8z+3)(z^4)}|$$

$$<\frac{32|z|^6+12|z|^5+32|z|^4+12|z|^3}{.5|z|^8}<200|z|^{-2}$$

so by the triangle inequality,

$$|\frac{f'(z)}{f(z)}-\frac{4}{z}|=|\frac{4z^3+24z^2+6z+8}{z^4+8z^3+3z^2+8z+3}-\frac{4z^3}{z^4}|<250|z|^{-2}.$$

As the integral of $\frac{4}{z}$ with respect to z over $\Gamma_{R,1}$ is $4\pi i$, this implies that $\int_{\Gamma_{R,1}}\frac{f'(z)dz}{f(z)}=4\pi i+O(R^{-1})$. (The error term $O(R^{-1})$ is bounded above in magnitude by $250\pi R^{-1}$).

Along $\gamma_{R,2}$ we note that as f omits the nonpositive real axis, there exists a branch L of the logarithm function defined in a neighborhood of the image of f on $\gamma_{R,2}$ (this is the standard branch of the logarithm defined away from the nonpositive real axis) and $(L(f))'=\frac{f'}{f}$); therefore, $\int_{\gamma_{R,2}}\frac{f'(z)dz}{f(z)}=L(f(-Ri))-L(f(Ri))$.

We note that

$$L(f(-Ri))=L(R^4-3R^2+3+i(8R^3-8R))$$

$$=L(R^4)+L(1-3R^{-2}+3R^{-4}+i(8R^{-1}-8R^{-3})=L(R^4)+O(R^{-1})$$

while

$$L(f(Ri))=L(R^4-3R^2+3+i(-8R^3+8R))$$

$$=L(R^4)+L(1-3R^{-2}+3R^{-4}+i(-8R^{-1}+8R^{-3})=L(R^4)+O(R^{-1})$$

so $\int_{\gamma_{R,2}}\frac{f'(z)dz}{f(z)}=O(R^{-1})$.

As $\gamma_R=\gamma_{R,1}+\gamma_{R,2}$, $\int_{\gamma_R}\frac{f'(z)dz}{f(z)}=4\pi i+O(R^{-1})$. This integral is independent of $R>M$ (by the argument principle again) so the integral is indeed equal to $4\pi i$ and the number of zeroes up to multiplicity is $\frac{4\pi i}{2\pi i}=2$. To show that there are exactly two zeroes (and not just one zero with multiplicity zero) we note that f has no zeroes on the nonnegative real axis (as all coefficients, including constant term, are positive) and, because f is real on the real axis, $f(z)=\overline{f(\overline{z})}$ everywhere. In other words, z is a zero iff $\overline{z}$ is, so this means that as f has two zeroes (up to multiplicity) in the right half plane, none of which are on the real axis, f has one zero in the first quadrant and one zero in the fourth quadrant (which are clearly different) or, in other words, two zeroes in the right half plane.

p161 2. The right-angled triangle γ_{X_1,X_2} (for X_1, X_2 positive) shall be defined as follows:

the segment $\gamma_{X_1,X_2,1}$ joins $-X_1$ to X_2, the segment $\gamma_{X_1,X_2,2}$ joins X_2 to $(-.5X_1 + .5X_2) + i(.5X_1 + .5X_2)$, and the segment $\gamma_{X_1,X_2,3}$ joins $(-.5X_1 + .5X_2) + i(.5X_1 + .5X_2)$ to $-X_1$. (Orientation is taken to be from the first point listed to the second).

By Cauchy's integral formula, in order to get a limit along $\gamma_{X_1,X_2,1}$ it suffices to show that as X_1 and X_2 tend independently to zero, the integrals along $\gamma_{X_1,X_2,2}$ and $\gamma_{X_1,X_2,3}$ each tend to zero as well. (All the zeroes of the rational function in the denominator in the upper half plane are contained in $\{|z| < M\}$ for some $M > 0$; we note that if S is less than the minimum of $.1X_1$ and $.1X_2$ then our triangle contains the rectangle with vertices $-X_1 + S$, $X_2 - S$, $-X_1 + S + Si$, $X_2 - S + Si$, so for X_1, X_2 sufficiently large, all zeroes are already claimed in the interior of the triangle.

To compute the integral, we parametrize the curve by imaginary component.
Therefore,

$$|\int_{\Gamma_{X_1,X_2,2}} R(z)e^{iz}dz| \leq \int_0^{.5X_1+.5X^2} |R(X_2 + (i-1))t|2e^{-t}dt$$

(because $|dz| = \sqrt{2}dt \leq 2dt) \leq C\int_0^\infty \frac{e^{-t}}{|X_2+(i-1)t|}dt$ (as $|zR(z)|$ is bounded, say by C, for $|z|$ sufficiently large since we assume a pole of order at least one; this is guaranteed when X_1 and X_2 are sufficiently large) $\leq \frac{2C}{X_2}\int_0^\infty \frac{e^{-t}}{d}t$ ($|X_2 + (i-1)t| \geq .5|X_2|$ for t real) $= \frac{2C}{X_2}$.

Similarly,
Therefore,

$$|\int_{\Gamma_{X_1,X_2,3}} R(z)e^{iz}dz| \leq \int_0^{.5X_1+.5X^2} |R(-X_1 + (i+1))t|2e^{-t}dt$$

$\leq C\int_0^\infty \frac{e^{-t}}{|-X_1+(i+1)t|}dt \leq \frac{2C}{X_1}\int_0^\infty \frac{e^{-t}}{d}t$ ($|-X_1 + (i+1)t| \geq .5|X_1|$ for t real) $= \frac{2C}{X_1}$ so the total integral over $\Gamma_{X_1,X_2,2} + \Gamma_{X_1,X_2,3}$ is $O(\frac{1}{X_1} + \frac{1}{X_2}$ just as in the case of the square.

3. (Note: each subpart was given either a check for 'nearly correct or completely correct', a minus for 'some progress', or an X. From a total of ten possible points, a half point was taken off for each minus sign and a full point was taken off for each X, making the lowest possible score 1. Each score was then rounded up to the nearest integer.)

a) We first solve for a positive real and note that as both sides extend analytically as functions of a, the formula works in general.

By symmetry ($\sin^2 x$ is symmetric about $\frac{\pi}{2}$ and has period π) we have

$$\int_0^{\pi/2} \frac{dx}{a + \sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{dx}{a + \sin^2 x}$$

$$= \frac{1}{4}\int_C \frac{dz}{zi(a-(z-(z^{-1})^2)/4)}$$

(where C is the unit circle; we substitute $z = e^{ix}$ so $\frac{dz}{zi} = dx$ and $\sin x = \frac{z-z^{-1}}{2i}$

$$= \int_C \frac{z^2 dz}{zi(4az^2-(z^2-1)^2)}$$

$$= -i\int_C \frac{zdz}{4az^2-(z^2-1)^2}.$$

To find the poles of the integrand inside the unit circle we set $u = z^2$ and note a pole occurs when $4au = (u-1)^2$; in other words: $u^2-(2+4a)u+1 = 0$. Solving, $u = 1+2a\pm 2\sqrt{a^2+a}$. As these two roots have product 1 and $1+2a+2\sqrt{a^2+a}$ is clearly outside the unit circle, $1+2a-2\sqrt{a^2+a}$ lies inside the unit circle.

The poles, therefore, are the two square roots of $1+2a-2\sqrt{a^2+a}$ (and they are clearly simple).

To find the residue of $\frac{z}{4az^2-(z^2-1)^2}$ at a pole P we note that the value is the limit of $\frac{z(z-P)}{4az^2-z^4+2z^2-1}$ as z approaches P, which we evaluate by using L'Hopital's rule to get $\frac{P}{8aP-4P^3+4P} = \frac{-1}{4(P^2-2a-1)}$. For each pole, $P^2 = 1+2a-2\sqrt{a^2+a}$; therefore, the residue is $\frac{-1}{-2\sqrt{a^2+a}} = \frac{1}{8\sqrt{a^2+a}}$. Because there are two poles, the residue theorem tells us

$$= \int_C \frac{zdz}{4az^2-(z^2-1)^2)} = \frac{\pi i}{2\sqrt{a^2+a}}$$

so

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \frac{\pi}{2\sqrt{a^2+a}}.$$

This formula can be extended to all a with norm greater than one by extending $\sqrt{a^2+a}$ analytically; one can do this, for example, by finding a branch for $\log z/(z+1)$.

b) As the integrand is an even function, the integral is one half of the integral over the real line. Computing the integral of $\frac{z^2}{z^4+5z^2+6}$ with respect to z falls exactly under the case of example 2 (the numerator has degree 2 and the denominator has degree 4) so we find the poles in the upper half plane.

Again, substituting $u = x^2$, the poles satisfy $u^2+5u+6 = 0$, i.e. $u = -2$ or $u = -3$. This means that the poles in the upper half plane (which are clearly simple) are $i\sqrt{2}$ and $i\sqrt{3}$.

To compute the residue at a pole P, we again find the limit of $\frac{z^2(z-P)}{z^4+5z^2+6}$ as z approaches P; L'Hopital's Rule gives us $\frac{P^2}{4P^3+10P} = \frac{P}{4P^2+10}$, which is $\frac{i\sqrt{2}}{2}$ for $i\sqrt{2}$ and $\frac{i\sqrt{3}}{-2}$ for $i\sqrt{3}$. This means that by the formula in Example 2 (integral is $2\pi i$ times the sum of the residues), the integral over the entire real line is $2\pi(-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2})$ so the integral from 0 to ∞ is $\pi\frac{\sqrt{3}-\sqrt{2}}{2}$.

c) Once again, Example 2 applies directly (numerator degree 2; denominator degree 4) with respect to the rational function $\frac{z^2-z+2}{z^4+10z^2+9}$ so we find the potential poles in the upper half plane.

We factor the denominator of the integrand: $x^4+10x^2+9=(x^2+1)(x^2+9)$ from which it is clear that the poles are $\pm i$ and $\pm 3i$, of which i and $3i$ are in the upper half plane. The residue of the integrand at a potential pole P (as before, the poles are clearly simple) is the limit, as z approaches P, of $\frac{(z^2-z+2)(z-P)}{z^4+10z^2+9}$, which, by L'Hopital's Rule, is $\frac{P^2-P+2}{4P^3+20P}$. For $P=i$ this works out to $\frac{-i+1}{16i}$ while for $P=3i$ it works out to $\frac{-3i-7}{-48i}$. Therefore, the sum of the two residues is

$$\frac{-i+1}{16i}+\frac{-3i-7}{-48i}=\frac{3i-3}{16i}+\frac{-3i-7}{-48i}=\frac{-10}{-48i}$$

so the integral becomes $\frac{10}{48i} * 2\pi i=\frac{5\pi}{12}$.

d) Once again, the integrand is even so the desired integral is $.5\int_{-\infty}^{\infty}\frac{z^2dz}{(z^2+a^2)^3}$.

Clearly the only pole above the y axis is at $|a|i$; unfortunately, it is a triple pole. (Note: the integral clearly diverges for $a=0$ so we ignore this case.) To calculate the residue of f at a triple pole z_0, one needs to evaluate the limit of $\frac{1}{2}(f(z)(z-z_0)^3)''$ as z goes to z_0.

In this case, this expression is $\frac{1}{2}(\frac{z^2}{(z+|a|i)^3})''$.

The first derivative of $\frac{z^2}{(z+|a|i)^3}$ is $\frac{2z}{(z+|a|i)^3}-\frac{3z^2}{(z+|a|i)^4}$; the second derivative is

$$\frac{2}{(z+|a|i)^3}-\frac{6z}{(z+|a|i)^4}-\frac{6z}{(z+|a|i)^4}+\frac{12z^2}{(z+|a|i)^5},$$

which, when evaluated at $|a|i$, gives $\frac{2i}{8|a|^3}-\frac{6|a|i}{16|a|^4}-\frac{6|a|i}{16|a|^4}+\frac{12|a|^2i}{32|a|^5}$, or $\frac{-i}{8|a|^3}$. This implies that the residue of f at the pole is $\frac{-i}{16|a|^3}$ so the Residue Theorem gives an integral of $\frac{\pi}{8|a|^3}$. However, this is the integral over the entire real line; our desired integral, from 0 to ∞, will be half that, or $\frac{\pi}{16|a|^3}$.

e) As the integrand is even, we have $\int_0^{\infty}\frac{\cos x}{x^2+a^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}dx$ and are in the situation of Example 3. Letting A equal $\int_{-\infty}^{\infty}\frac{e^{iz}}{z^2+a^2}dz$, our integral will be half of the real part of A. (Once again, the integral diverges if $a=0$ so we ignore this case.)

To find A we note that the (simple) poles are at ai and $-ai$, of which $|a|i$ is the only pole in the upper half plane. The residue of the integrand at $|a|i$ is equal to the limit of $\frac{e^{iz}(z-|a|i)}{z^2+a^2}=\frac{e^{iz}}{z+|a|i}$ as z approaches $|a|i$, or $\frac{e^{-|a|}}{2|a|i}$. Therefore, $A=\frac{\pi e^{-|a|}}{|a|}$ so the integral we initially sought, $\int_0^{\infty}\frac{\cos x}{x^2+a^2}dx=\frac{\pi e^{-|a|}}{2|a|}$.

f) Once again, the integrand is even so we have

$$\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx=.5\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}dx$$

and are again in the situation of Example 3. Letting $B=\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz$, our initial integral is equal to half the imaginary part of B.

To find A we note that the (simple) poles are at ai and $-ai$, of which $|a|i$ is the only pole in the upper half plane. The residue of the integrand at $|a|i$ is equal to the limit of $\frac{ze^{iz}(z-|a|i)}{z^2+a^2} = \frac{ze^{iz}}{z+|a|i}$ as z approaches $|a|i$, or $\frac{e^{-|a|}}{2}$. Therefore, $A = \pi e^{-|a|}i$ so the integral we initially sought, $\int_0^\infty \frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-|a|}}{2}$.

g) We are now exactly in the case of example 4 with $\alpha = \frac{1}{3}$ and $R(z) = \frac{1}{1+z^2}$. Therefore,

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}dx = \int_0^\infty \frac{2t^{5/3}}{1+t^4}dt$$

$$= (1-e^{2\pi i/3})^{-1}\int_{-\infty}^{\infty} \frac{2z^{5/3}}{1+z^4}dz.$$

(The branch of $z^{2/3}$ chosen is the branch whose argument lies between $-\frac{\pi}{3}$ and π; the negative imaginary axis is deleted here.)

Letting $Z = \int_{-\infty}^{\infty} \frac{2z^{5/3}}{1+z^4}dz$, our desired integral is $\frac{Z}{1-(-.5+.5i\sqrt{3})} = \frac{Z}{1.5-.5i\sqrt{3}}$. To evaluate Z we note that the poles of the integrand are $e^{k\pi i/4}$ for $k = 1, 3, 5, 7$ of which only $k = 1, 3$ correspond to points in the upper half plane (and all poles are simple).

Therefore, the residue at some pole P is the limit of $\frac{2z^{5/3}(z-P)}{1+z^4}$ as z approaches P, or $\frac{2P^{5/3}}{4P^3}$ (by L'Hopital's Rule), which is $.5P^{-4/3}$. For $k = 1$ this becomes $.5e^{-\pi i/3}$; for $k = 3$ this is simply $-.5$. Therefore, the sum of the residues is $.5(.5 - .5i\sqrt{3} - 1)$; therefore, $Z = \pi(.5\sqrt{3} - .5i)$ and the integral we seek, $\int_{0\infty} \frac{x^{1/3}}{1+x^2}dx$, becomes $\frac{\pi}{\sqrt{3}}$.

h) To evaluate $\int_0^\infty \frac{\log x}{1+x^2}$ we can again get away with using the contour in Figure 4-13.

Formally, for $R, \epsilon > 0$ (we think of R large and ϵ small) we define $\gamma_{R,\epsilon}$ as equal to the sum of the following four curves:

$\gamma_{R,\epsilon,1}$ goes clockwise along the circle centered at the origin of radius ϵ from $-\epsilon$ to ϵ.

$\gamma_{R,\epsilon,2}$ goes horizontally from ϵ to R,

$\gamma_{R,\epsilon,3}$ goes counterclockwise along the circle centered at the origin of radius R from R to $-R$, and

$\gamma_{R,\epsilon,4}$ goes horizontally from $-R$ to R.

This is a simple closed curve and we seek to integrate $\frac{\log z}{1+z^2}dz$ over $\gamma_{R,\epsilon}$. (The branch of the logarithm is defined to avoid the negative real axis and take arguments between $-\frac{\pi}{2}$ and $3\frac{pi}{2}$.

The integrand has only two simple poles: at $\pm i$ (and only i is in the interior of our contour). At i the residue is equal to the limit of $\frac{\log z}{1+z^2}(z-i) = \frac{\log z}{z+i}$ as z approaches i, or $\frac{\pi/2i}{2i} = \frac{\pi}{4}$; this implies that $\int_{\gamma_{R,\epsilon}} \frac{\log z}{1+z^2}dz = \frac{pi^2}{2}i$.

However, we also have

$$\int_{\gamma_{R,\epsilon}} \frac{\log z}{1+z^2}dz = \int_{\gamma_{R,\epsilon,1}} \frac{\log z}{1+z^2}dz + \int_{\gamma_{R,\epsilon,2}} \frac{\log z}{1+z^2}dz$$

$$\int_{\gamma_{R,\epsilon,3}} \frac{\log z}{1+z^2}dz \int_{\gamma_{R,\epsilon,4}} \frac{\log z}{1+z^2}dz.$$

The integral over $\gamma_{R,\epsilon,1}$ approaches zero as ϵ approaches zero (the integrand has magnitude approximately $\log \epsilon$ and we are integrating over a length of $\pi\epsilon$) while the integral over $\gamma_{R,\epsilon,3}$ approaches zero as R approaches ∞ (the integrand has magnitude approximately $R^{-2}\log R$ and we are integrating over an area of length πR).

The integral over $\gamma_{R,\epsilon,2}$ is simply $\int_\epsilon^R \frac{\log x}{1+x^2}dx$ while the integral over $\gamma_{R,\epsilon,4}$ is

$$\int_{-R}^{-\epsilon} \frac{\log x}{1+x^2}dx = \int_R^\epsilon \frac{\log -x}{1+x^2} - dx$$

$$= \int_\epsilon^R \frac{\log x + \pi i}{1+x^2}$$

so, letting R approach ∞ and ϵ approach 0, $\int_0^\infty \frac{2\log x + \pi i}{1+x^2} = \frac{\pi^2}{2}i$.

Taking real parts tells us $\int_0^\infty \frac{2\log x}{1+x^2} = 0$ from which it is clear that $\int_0^\infty \frac{\log x}{1+x^2} = 0$.

i) We note that $\int_\epsilon^R \log(1+x^2)\frac{dx}{x^{1+\alpha}}$ is equal to $\log(1+x^2) * -\alpha^{-1}x^{-\alpha}$ evaluated at R minus the same expression evaluated at ϵ, plus $\int_\epsilon^R \frac{2x}{\alpha x^\alpha (1+x^2)}dx$ (integrating by parts with $dv = \frac{dx}{x^{1+\alpha}}$ and $u = \log(1+x^2)$); letting ϵ go to zero and R go to infinity, the boundary terms vanish (the term at R because $\alpha > 0$; the term at ϵ because $\alpha < 2$ and $\log(1+x^2) = O(x^2)$) and we get that

$$\int_0^\infty \log(1+x^2)\frac{dx}{x^{1+\alpha}} = \int_0^\infty \frac{2x dx}{\alpha x^\alpha (1+x^2)}.$$

We use the exact same contour $\gamma_{R,\epsilon}$ as in the preceding part; also, our branch of x^α is taken to be defined off the negative imaginary axis and return arguments between $-\frac{\pi\alpha}{2}$ and $\frac{3\pi\alpha}{2}$.

As before, the integrand has only two simple poles: at $\pm i$ (and only i is in the interior of our contour). At i the residue is equal to the limit of $\frac{2z}{\alpha z^\alpha (1+z^2)}(z-i) = \frac{2z}{z^\alpha (z+i)}$ as z approaches i, or $\frac{1}{\alpha i^\alpha}$, which is simply $\alpha^{-1}e^{-\pi i\alpha/2}$ so the integral over $\gamma_{R,\epsilon}$ is $2\pi i\alpha^{-1}e^{-\pi i\alpha/2}$.

However, we also have

$$\int_{\gamma_{R,\epsilon}} \frac{2z}{\alpha(1+z^2)z^\alpha}dz = \int_{\gamma_{R,\epsilon,1}} \frac{2z}{\alpha(1+z^2)z^\alpha}dz + \int_{\gamma_{R,\epsilon,2}} \frac{2z}{\alpha(1+z^2)z^\alpha}dz$$

$$\int_{\gamma_{R,\epsilon,3}} \frac{2z}{\alpha(1+z^2)z^\alpha}dz \int_{\gamma_{R,\epsilon,4}} \frac{2z}{\alpha(1+z^2)z^\alpha}dz.$$

The integral over $\gamma_{R,\epsilon,1}$ approaches zero as ϵ approaches zero (the integrand has magnitude approximately $2\epsilon^\beta$ for some $\beta > -1$ and we are integrating over a length of $\pi\epsilon$) while the integral over $\gamma_{R,\epsilon,3}$ approaches zero as R approaches ∞

(the integrand has magnitude approximately $R^{-1}R^{-\alpha}$ and we are integrating over an area of length πR).

The integral over $\gamma_{R,\epsilon,2}$ is simply $= \int_\epsilon^R \frac{2xdx}{\alpha x^\alpha(1+x^2)}$ while the integral over $\gamma_{R,\epsilon,4}$ is

$$= \int_{-R}^{-\epsilon} \frac{2xdx}{\alpha x^\alpha(1+x^2)} = \int_R^\epsilon \frac{2(-x) - dx}{\alpha(-x)^\alpha(1+x^2)}$$

$$= -e^{-\pi i\alpha} \int_\epsilon^R \frac{2xdx}{\alpha(x)^\alpha(1+x^2)}$$

so, letting R approach ∞ and ϵ approach 0,

$$= (1 - e^{-\pi i\alpha}) \int_0^\infty \frac{2xdx}{\alpha x^\alpha(1+x^2)} = \alpha^{-1}e^{-\pi i\alpha/2}.$$

Solving, the integral we initially sought,
$\int_\epsilon^R \log(1+x^2)\frac{dx}{x^{1+\alpha}}$, is $\frac{2\pi i}{\alpha e^{\pi i\alpha/2} - e^{-\pi i\alpha}/2} = \frac{2\pi i}{2i\alpha \sin(\pi\alpha/2)} = \frac{\pi}{\alpha\sin(\pi\alpha/2)}$.

4. This problem is exactly Problem 3 (with even the same hint, albeit in less detail) on page 120, which was assigned in the previous homework; see the previous assignment for solutions.

5. We note that the integral of $\frac{f(z)dxdy}{(1-\bar{z}\zeta)^2}$ over the unit disc is equal to

$$\int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})rdrd\theta}{(1-\zeta re^{-i\theta})^2}.$$

To evaluate this, we look at the inner integral,

$$\int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})rd\theta}{(1-\zeta re^{-i\theta})^2},$$

and substitute $z = re^{i\theta}$ (so $d\theta = \frac{dz}{iz}$) to get

$$\int_C \frac{f(z)rdz}{iz(1-\zeta r^2 z^{-1})^2},$$

where C is the unit circle.

This simplifies to $\int_C \frac{f(z)rz}{i(z-\zeta r^2)^2 dz}$. To evaluate this, we note that the integrand has a pole of order 2 at ζr^2 (which is inside the unit circle) and no other singularities; the residue corresponding to this pole is $r/i(zf(z))'$ evaluated at $z = \zeta r^2$.

Therefore, defining the function $g(z) = zf(z)$, the inner integral is $2\pi g'(\zeta r^2)$. Therefore,

$$\int_{r=0}^1 \int_{\theta=0}^{2\pi} \frac{f(re^{i\theta})rdrd\theta}{(1-\zeta re^{-i\theta})^2}$$

$= \int_{r=0}^1 \pi g'(\zeta r^2)2rdr$.

If $\zeta = 0$ we are integrating $2r\pi g'(0)$ with respect to r from 0 to 1 and get $\pi g'(0) = \pi f(0)$; we temporarily assume this is not the case.

Substituting $z = \zeta r^2$, $dz = \zeta 2r dr$ and the integral becomes $\pi \int_L \zeta^{-1} g'(z) dz$ where L is the straight line joining 0 to ζ; this integral is $\pi\zeta^{-1}(g(\zeta) - g(0)) = \pi\zeta^{-1}(\zeta f(\zeta) - 0) = \pi f(\zeta)$.

In any event, we have that the integral of $\frac{f(z)dxdy}{(1-\bar{z}\zeta)^2}$ over the unit disc is $\pi f(\zeta)$; therefore,

$$f(\zeta) = \frac{1}{\pi} \int \int_{|z|<1} \frac{f(z)dxdy}{(1 - \bar{z}\zeta)^2}$$

just as desired.