The following problems were graded in detail (to be worth ten points each): p9 3, p15 2, p20 1.

p6 1. We use the following map from the matrics to the complex numbers: a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

is sent to the complex number $\alpha+\beta i$. This map is clearly bijective and sends identities to identities (the zero matrix is the additive identity and the identity matrix is the multiplicative identity) so it suffices to show that the map preserves addition and multiplication. As matrix addition is done componentwise, the addition part is clear; it suffices to verify multiplication; however, if $\alpha, \beta, \gamma, \delta$ are real numbers then

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha\gamma - \beta\delta & \alpha\delta + \beta\gamma \\ -(\alpha\delta + \beta\gamma) & \alpha\gamma - \beta\delta \end{pmatrix}$$

while $(\alpha + \beta i)(\gamma + \delta i) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)i$ showing that multiplication is preserved.

p9 3. If $|a| = 1$ then the numerator of our fraction has norm $|\bar{a}||a - b| = |1 - \bar{a}b|$ (as $|a| = |\bar{a}| = 1$ and $a\bar{a} = 1$); if $|b| = 1$ the numerator has norm $|b||\overline{a - b}| = |b\bar{a} - 1| = |1 - \bar{a}b|$; in either case the numerator and denominator have the same norm. Therefore, unless the denominator is zero (which can only happen if $\bar{a}b = 1$, which requires $|a| = |b| = 1$ and $a = b$) we conclude $|\frac{a-b}{1-\bar{a}b}| = 1$.

4. Writing $a = a_1 + a_2 i$, $b = b_1 + b_2 i$, $z = x + yi$, $c = c_1 + c_2 i$, the equation is equivalent to

$$\begin{pmatrix} a_1 + b_1 & -a_2 + b_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

which has a unique solution in x and y iff the first matrix has nonzero determinant; however, this determinant is $(a_1^2 - b_1^2) - (b_2^2 - a_2^2) = |a|^2 - |b|^2$; therefore, we have a unique solution iff $|a| \neq |b|$.

In this case, we note that if $c = 0$ then 0 is the solution; we assume this is not the case. Letting $a' = a/c, b' = b/c$ we note that $|a'| \neq |b'|$. Solving our original system is the same as solving $a'z + b'\bar{z} = 1$. To do this, we consider $w = \bar{a'} - b'$ and note that $a'w + b'\bar{w} = |a'|^2 - |b'|^2$ (the other terms cancel) so setting $z = \frac{\bar{a'} - b'}{|a'|^2 - |b'|^2}$ gives us our desired solution.

5. We note that

$$|\Sigma_{i=1}^n a_i b_i|^2 = \Sigma_{i=1}^n |a_i|^2 |b_i|^2 + \Sigma_{1\le i<j\le n} a_i b_i \overline{a_j b_j} + a_j b_j \overline{a_i b_i}.$$

However, for $1 \le i < j \le n$,

$$|a_i \bar{b_j} - a_j \bar{b_i}|^2 = (a_i \bar{b_j} - a_j \bar{b_i})(\bar{a_i} b_j - \bar{a_j} b_i) = |a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - a_i b_i \overline{a_j b_j} - a_j b_j \overline{a_i b_i}$$

from whence we conclude

$$a_i b_i \overline{a_j b_j} + a_j b_j \overline{a_i b_i} = |a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - |a_i \bar{b_j} - a_j \bar{b_i}|^2$$

so plugging in gives us

$$|\Sigma_{i=1}^n a_i b_i|^2 = \Sigma_{i=1}^n |a_i|^2 \Sigma_{i=1}^n |b_i|^2 - \Sigma_{1\le i<j\le n} |a_i \bar{b_j} - a_j \bar{b_i}|^2.$$

p11 1. Writing $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ (with the a_j, b_j real), the numerator becomes $a_1 - b_1 + (a_2 - b_2)i$, whose absolute value, when squared, becomes $a_1^2 + b_1^2 + a_2^2 + b_2^2 - 2(a_1 b_1 + a_2 b_2) = |a|^2 + |b|^2 - 2(a_1 a_2 + b_1 b_2)$ and the denominator is $1 - (a_1 b_1 + a_2 b_2) + (a_2 b_1 - a_1 b_2)i$, whose norm, when squared, is $1 + a_1^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2 - 2(a_1 b_1 + a_2 b_2) = 1 + |a|^2|b|^2 - 2(a_1 b_1 + a_2 b_2)$. However, if x and y are elements of $[0, 1)$ then $1 + xy - (x + y) = (1 - x)(1 - y) > 0$ which implies that the denominator has greater norm than the numerator, i.e. that our fraction has norm less than 1.

4. For "only if" the triangle inequality gives us $|z-a|+|z+a| = |a-z| + |a+z| \geq 2|a|$ which can only equal $2|c|$ for $a \geq c$; for "if" one supposes $|a| \leq |c|$ notes that if $z=0$ then $|z-a|+|z+a| = 2|a| \leq 2|c|$ but as z approaches infinity along the real line, clearly $|z-a|+|z+a|$ does as well. Continuity of absolute value therefore says that there must exist some real z with $|z-a|+|z+a| = 2|c|$ in this case.

To find the minimal and maximal values of $|z|$ (where $|a| \geq |c|$) we begin by considering the case where both a and c are real; write $z = x+yi$ with x, y real.

Squaring both sides, $|z-a|+|z+a| = 2|c|$ if and only if

$$(x^2-2ax+a^2+y^2)+(x^2+2ax+a^2+y^2)+2\sqrt{(x^2-2ax+a^2+y^2)(x^2+2ax+a^2+y^2)} = 4c^2;$$

regrouping gives us

$$x^2+y^2+a^2-2c^2 = -\sqrt{(x^2+a^2+y^2)^2-4a^2x^2}.$$

The square of the left hand side is $(x^2+a^2+y^2)^2 - 4c^2(x^2+y^2+a^2)+4c^4$ so we conclude

$$4c^2(x^2+y^2+a^2)-4c^4 = 4a^2x^2,$$

or, regrouping,

$$x^2(c^2-a^2)+y^2c^2 = c^2(c^2-a^2)$$

which is the equation of the ellipse

$$\frac{x^2}{c^2}+\frac{y^2}{c^2-a^2} = 1.$$

Because $x^2 \leq c^2, y^2 \leq c^2-a^2$, $x^2+y^2+a^2-2c^2 \leq 0$ which implies that the second squaring did not give us any extraneous root and every point on the ellipse satisfies the initial condition. From this equation we conclude the minimal value of $|z|$ is $\sqrt{|c|^2-|a|^2}$ and the maximal value is $|c|$.

This holds in general (even for a, c not real) because if a is not real it can be written as $a'w$ for a' real and $w = 1$ while $|z-a|+|z+a| = |z\bar{w}-a'|+|z\bar{w}+a'|$ where $z\bar{w}$ has the same norm as z.

p15 2. Adding z to all three vertices (which clearly keeps equilateral triangles equilateral) changes the LHS from $a_1^2 + a_2^2 + a_3^2$ into $a_1^2 + a_2^2 + a_3^2 + 2z(a_1 + a_2 + a_3) + 3z^2$ and the RHS from $a_1a_2 + a_2a_3 + a_3a_1$ to $a_1a_2 + a_2a_3 + a_3a_1 + z(a_2 + a_1 + a_3 + a_2 + a_1 + a_3) + 3z^2$, adding the same quantity to each vertex.

Therefore, we can suppose (by adding $-a_1$ to all three vertices) that $a_1 = 0$. Similarly, multiplying all three vertices by a nonzero complex number t keeps equilateral triangles equilateral and multiplies both sides by t^2 so as long as $a_2 \neq 0$ (in which case the identity holds iff $a_3 = 0$ and the triangle is trivially equilateral) we can divide all three vertices by a_2 and therefore we merely need to consider the case where $a_1 = 0, a_2 = 1$.

Then, the triangle is equilateral iff $|a_3| = |a_3 - 1| = 1$; writing $a_3 = x + yi$, this means $x^2 + y^2 = x^2 - 2x + y^2 + 1 = 1$. Subtracting the first expression from the second tells us $x = .5$ which implies that $y = \pm\frac{\sqrt{3}}{2}$; direct computation tells us that if $a_3 = .5 \pm \frac{\sqrt{3}}{2}i$ then $|a_3| = |a_3 - 1| = 1$.

Now, our identity becomes $a_3^2 + 1 = a_3$, whose roots are indeed $.5 \pm \frac{\sqrt{3}}{2}i$ by the quadratic formula, so we conclude that the triangle is equilateral iff $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

4. Let C refer to the center of this circle and R denote its radius. We begin by looking at the triangle with vertices a_1, C, and a_2. It is icosceles (with vertex angle at C) because $|a_1 - C| = R = |a_2 - C|$. Drawing the altitude from C to the side joining a_1 and a_2 (which is also a median by the Pythagorean Theorem) we note that $R \sin\theta = .5|a_1 - a_2|$ where θ is the measure of the angle from a_1 to C to the midpoint of a_1 and a_2. As the median is an angle bisector (notice, for example, that a_1 is the reflection of a_2 in this segment) and inscribed angles have half the measure of central angles intercepting the same arc (the geometric proof for this can be directly translated into analytic language), θ is the angle of our original triangle at a_3.

Also, using the formula that the area of a triangle is half the product of two sides and the sine of their included angle (which is just $\frac{1}{2}bh$), we conclude that letting A denote the area of the triangle in question, $R\frac{2A}{|a_1-a_3||a_2-a_3|} = .5|a_1-a_2|$ from which we conclude $R = \frac{|a_1-a_2||a_1-a_3||a_2-a_3|}{4A}$.

There are many symmetric expressions for the circumcenter C

(Wikipedia alone gives three) of which the simplest uses trilinear coordinates.

If a triangle has vertices a_1, a_2, a_3 then we define its trilinear coordinates as follows. For each point p in the plane we let p_1 be the distance from p to the side joining a_2 to a_3 (distance is considered to be positive if p is on the same side of the segment joining a_2 and a_3 as the rest of the triangle and negative otherwise) and similarly for p_2 (the side joining a_3 to a_1) and p_3 (the side joining a_1 to a_2); the trilinear coordinates of p are now $(p_1 : p_2 : p_3)$. (Under this notation, $(p_1 : p_2 : p_3) = (\lambda p_1 : \lambda p_2 : \lambda p_3)$ for each λ; this is no problem because the relative distances to a_1, a_2, a_3 uniquely determine p).

By direct construction, the circumcenter is $R \cos A_3$ from the segment joining a_1 and a_2 (where A_3 denotes the angle at a_3) and similarly for the other angles, so the circumcenter is the point with trilinear coordinates $(A_1 : A_2 : A_3)$.

p16 4. Because h is not a multiple of n, $\omega^h \neq 1$ so $1 - \omega^h = 0$ which implies that

$$1 + \omega^h + \ldots + \omega^{(n-1)h} = \frac{1 - \omega^{nh}}{1 - \omega^h} = 0$$

as desired.

5. Case 1: h/n is a half-integer, i.e. $\omega^h = -1$.

In this case, n is even and the sum is alternating; half the terms are 1's and the other half are -1 making the net result zero.

Case 2: h/n is not a half-integer.

In this case, the formula for a geometric series tells us that our sum will be

$$\frac{1 + (-1)^{n-1}\omega^{nh}}{1 + \omega^h} = \frac{1 + (-1)^{n-1}}{1 + \omega^h}.$$

p17 2. Ellipse: $|z - a_1| + |z - a_2| = r$ (the definition of an ellipse is, given two foci and a number r, the set of all points whose sum of distances from the foci is r).

Hyperbola: $|z - a_1| - |z - a_2| = r$ (the definition of a hyperbola is, given two foci and a number r, the set of all points whose difference of distances from the foci is r).

Parabola (rotating, if necessary, to place the directrix at the x-axis, noting that the parabola is the set of points equidistant from a focus to a directrix): $|z - a| = .5i(\bar{z} - z)$.

3. Letting A be one vertex of a parallelogram, B and C the adjacent vertices, and D the opposite vertex (and letting these letters simultaneously denote complex numbers), we note $B - A = D - C$. Therefore, the midpoint of the diagonal AD, $(A + D)/2$, is equal to $(B + C)/2$, the midpoint of the diagonal BC, or in other words the diagonals bisect each other.

For the case of a rhombus, by translation we can suppose that one vertex is the origin; therefore, we can write the four vertices in complex form as 0, w, z, $w + z$ with $|w| = |z|$. The vertices intersect at $(w + z)/2$; to show that they intersect at a right angle (i.e. that the angle from 0 to $(w + z)/2$ to w is right) we note that

$$\begin{gathered} |(w+z)/2|^2 + |w - (w+z)/2|^2 \\ = 1/4(|w|^2 + w\bar{z} + z\bar{w} + |z|^2) + 1/4(|w|^2 - w\bar{z} - z\bar{w} + |z|^2) \\ = .5(|w|^2 + |z|^2) = |w|^2 \end{gathered}$$

which implies, by the Pythagorean Theorem (which states that a triangle is right IF AND ONLY IF the sum-of-squares relation holds), that the angle in question is right.

5. By rotation it suffices to suppose a is positive real (and switching the roles of the two points we can suppose $a > 1$); letting C be the center of a circle joining a^{-1} to a and X be an intersection point of this circle with the unit circle, we shall show that the angle 0XC is right.

We begin by noting that C is equidistant to a^{-1} and a; therefore, $C = .5(a + a^{-1}) + bi$ for some number b (and the radius of the circle will be the square root of $(.5(a - a^{-1}))^2 + b^2$).

The distance from X to 0 is 1 and the distance from X to C is the square root of $(.5(a - a^{-1}))^2 + b^2$, which when added to 1, becomes $(.5(a + a^{-1}))^2 + b^2$, the distance from 0 to C, so by the Pythagorean Theorem, the triangle OXC is right, precisely as desired.

p20 1. We begin by noting that the origin and the point at infinity, which correspond to the south and north pole respectively, 'trivially' satisfy the condition so we assume that z is neither of those points.

Therefore, there exists a unique z' which corresponds to a diametrically opposite point on the Riemann sphere and a unique point z^* such that $z\bar{z}^* = -1$ (choose $z^* = -\bar{z}^{-1}$); it suffices to show that $z*$ and z correspond to diametrically opposite points on the unit sphere.

Letting $z = x + yi$, we note that z corresponds to

$$(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}).$$

Because $z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$, we have that $z^* = -\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$. Therefore, z^* corresponds to

$$(-\frac{\frac{2x}{x^2+y^2}}{1+(x^2+y^2)^{-1}}, -\frac{\frac{2y}{x^2+y^2}}{1+(x^2+y^2)^{-1}}, \frac{1-(x^2+y^2)^{-1}}{1+(x^2+y^2)^{-1}},$$

which is equal to

$$(-\frac{2x}{1+x^2+y^2}, -\frac{2y}{1+x^2+y^2}, -\frac{1-x^2-y^2}{1+x^2+y^2}),$$

on the unit sphere as desired.

2. Letting (a, b, c) be one of the vertices of the cube, the others are of the form $(\pm a, \pm b, \pm c)$ with every possible combination of signs. Because the side length is $2a = 2b = 2c$ (WLOG $a, b, c \geq 0$), we have $a = b = c = \frac{1}{\sqrt{3}}$ by the Pythagorean Theorem. Therefore, the four top vertices, with height $\frac{1}{\sqrt{3}}$, are multiplied by $\frac{1}{1-\frac{1}{\sqrt{3}}} = \sqrt{3}(\sqrt{3}+1)/2 = \frac{3+\sqrt{3}}{2}$ and therefore correspond to $\pm\frac{\sqrt{3}+1}{2} \pm \frac{\sqrt{3}+1}{2}i$ (for every possible combination of signs) while the bottom four vertices, with height $-\frac{1}{\sqrt{3}}$, are multipled by $\frac{1}{1+\frac{1}{\sqrt{3}}} = \frac{3-\sqrt{3}}{2}$ and therefore correspond to $\pm\frac{\sqrt{3}-1}{2} \pm \frac{\sqrt{3}-1}{2}$ (for every possible combination of signs).

4. Let P be the plane through N spanned as an affine set by the sides NZ and NZ' (so P contains the triangles NZZ', Nzz'). Because P is two-dimensional and not parallel to the xy-coordinate plane (otherwise the only point on the sphere it could touch is N), P can be oriented so that along the 'horizontal' direction, the z-coordinate is unchanged while along the 'vertical' direction, the z-coordinate changes (increasing as we go up).

We note that the intersection of P with the unit sphere is a circle C, which is oriented so that N is its top point.

Consequently, letting L be the horizontal line tangent to the bottom point of C (call this point B), we observe that because z and z' have the same vertical component, letting Y, Y' be the intersection of NZ, NZ' respectively with L we have NZZ' and NYY' are similar triangles.

The lines in question divide the circle into four arcs: NZ (of measure α_1), ZB (of measure α_2), BZ' (of measure α_3), and $Z'N$ (of measure α_4) with $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = \pi$.

We note that angle NYY' has measure $.5(\alpha_3 + \alpha_4 - \alpha_2) = .5\alpha_1$, which is the same as the measure of angle $Nz'z$. As triangles $Nz'z$ and NYY' have the same angle at N, we conclude that triangle $Nz'z$ is similar to triangle NYY', which is similar to triangle NZZ' (note the vertices for Nzz' are switched in relation to Ahlfors).

Therefore, $\frac{|Z'-Z|}{|z-z'|} = \frac{|N-Z'|}{|N-z|}$.

Clearly $|N - z| = \sqrt{1 + |z|^2}$; for $N - Z'$ we note that because Z' has height $\frac{1-|z'|^2}{1+|z'|^2}$ and distance $\frac{2|z'|}{1+|z'|^2}$ from the vertical axis,

$$|N - Z'|^2 = \frac{(2|z'|^2)^2 + (2|z'|)^2}{(1 + |z'|^2)^2} = \frac{4}{1 + |z'|^2}.$$

Solving, $|Z' - Z| = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$ as desired.

5. To find the radius, we use (28) to find the diameter. Letting M be the point on the circle with magnitude $a + R$, we let N be the point on the circle whose stereographic projection is furthest away from the stereographic projection of M. However, if N is the point on the circle with magnitude $|a-R|$ (on the opposite end from a), then N maximizes the numerator for (28) at $4R$ and minimizes the denominator (by minimizing norm) at $\sqrt{(1 + (a + R)^2)(1 + (a - R)^2)}$ so the radius is $\frac{1}{2}d(M, N)$, or

$$\frac{2R}{\sqrt{(1 + (a + R)^2)(1 + (a - R)^2)}}.$$