(Question 1 was worth 30 points: 10 for each part.)

1i. We note that w can be written in the form

$$\Sigma_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \hat{g}(n);$$

setting $w = 0$, we have $r = 0$ so all the terms except for the $n = 0$ term vanishes, telling us $w(0) = \hat{g}(0)$. However, $\hat{g}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta$ which tells us that $w(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta$.

ii. Suppose C is a circle completely contained in the interior of D; let c be its center and r be its radius. We define the function v on the closed unit disc as follows: $v(z) = w(zr + c)$. We note from part i) that $v(0)$ is the average of the values of v on the boundary of D; however, $v(0) = w(c)$ and the average of the values of v on the boundary of D is the average of the values of w on the boundary of C (because z is in the boundary of C if and only if $\frac{z-c}{r}$ is in the boundary of D) so the value of w at the center of the circle is the average of the values of w on the circle.

iii. Let F be a disk centered at some point c in D such that F has radius s and is completely contained in the interior of D. Then, because the points in D are precisely those of the form $re^{i\theta} + c$ for $0 \leq r \leq s$ and $-\pi \leq \theta \leq \pi$, the integral of w over the radius of the disk is

$$\int_{r=0}^{s}\int_{\theta=-\pi}^{\pi} w(re^{i\theta} + c)rdrd\theta = \int_{r=0}^{s} 2\pi w(0)rdr$$

(by part ii)) $= \pi s^2 w(0)$, so the average value of w on F is equal to this integral divided by πs^2 (the area), or $w(0)$.

(Question 2 was worth 20 points: 10 for each part.)

2i. Let $Z = \Sigma_{j=1}^{n} v_j w_j Q_{jj}$. Because $Av = \lambda_1 Qv$, the jth component of $\lambda_1 Z$ is $\lambda_1 w_j Q_{jj}$ so $\lambda_1 Z =< Av, w >$. However, as $Aw = \lambda_2 Qw$, the jth component of $\lambda_2 Z$ is $\lambda_2 w_j Q_{jj}$ so $\lambda_2 Z =< v, Aw >$. As A is self adjoint, $< Av, w >=< v, Aw >$ so $\lambda_1 Z = \lambda_2 Z$. As $\lambda_1 \neq \lambda_2, Z = 0$.

ii. Lemma: This holds for the case $r = 0$.

Proof: In this case, we integrate by parts:

$$\int_a^b (py_1')'y_2 = p(b)y_1'(b)y_2(b) - p(a)y_1'(a)y_2(a) - \int_a^b py_1'y_2'$$

$$= p(b)y_1'(b)y_2(b) - p(a)y_1'(a)y_2(a) - (y_1(b)p(b)y_2'(b) - y_1(a)p(a)y_2'(a))$$

$$+ \int_a^b y_1(py_2')'$$

(because $py_1'y_2' = y_1'py_2'$)

$$= -p(b)W(y_1, y_2)(b) + p(a)W(y_1, y_2)(a) + \int_a^b y_1(py_2')'$$

where $W(u, v)$ (the Wronskian of u and v) is the determinant of the 2 x 2 matrix whose first row consists of u and v and whose second row consists of u' and v'. The boundary conditions $c_1y_i(a) + c_2y_i'(a) = 0$ tell us that the column vectors $[y_1(a), y_1'(a)]$ and $[y_2(a), y_2'(a)]$ are linearly dependent, so $W(y_1, y_2)(a) = 0$. The boundary conditions $d_1y_i(b) + d_2y_i'(b) = 0$ tell us that the column vectors $[y_1(b), y_1'(b)]$ and $[y_2(b), y_2'(b)]$ are linearly dependent, so $W(y_1, y_2)(b) = 0$. Therefore, the boundary terms cancel, so in this case $(r = 0soL(y) = (py')')$,

$$\int_a^b L(y_1)y_2 = \int_a^b L(y_2)y_1,$$

proving the lemma.

In general, we use the lemma to compute

$$\int_a^b L(y_1)y_2 = \int_a^b ((p(y_1)')' + ry_1)y_2 = \int_a^b (p(y_1)')'y_2 + \int_a^b ry_1y_2$$

$$= \int_a^b (p(y_2)')'y_1 + \int_a^b (ry_2)y_1 = \int_a^b L(y_2)y_1$$

as desired (where the first equation on the last line followed from the lemma.)