

1. (This question was worth ten points: five for each part.)

i. We consider $y(0) = (1, 0, \dots, 0)$. One observes that for k an integer with $1 \leq k \leq n-1$,

$$\sum_{j=0}^{n-1} \cos\left(\frac{2\pi jk}{n}\right) = 0$$

as this is the real part of

$$\sum_{j=0}^{n-1} e^{i\frac{2\pi jk}{n}} = \frac{1-1}{1-e^{i\frac{2\pi k}{n}}} = 0.$$

Consequently, $y(0) = \frac{1}{n} \sum_{k=0}^{n-1} v_k$ where the j th component of v_k is $\cos\left(\frac{2\pi j(k-1)}{n}\right)$.

As the eigenvalue corresponding to v_k is $2 \cos\left(\frac{2\pi(k-1)}{n}\right) - 2$ (to be called λ_k), we observe $y(t) = \frac{1}{n} \sum_{k=0}^{n-1} e^{\lambda_k t} v_k$. As $y(t)$ is a positive linear combination of vectors such that the first component is the largest component (and for $k \neq 0$, the strictly largest), the first position will always be the most likely position to stay in.

ii. Consider the initial condition $z = (1/3, 1/3, 1/3, 0, \dots, 0)$ (every component after the third is 0). We note that $y_2'(0) = 0$ while $y_1'(0) = y_3'(0) = -1/3$. Therefore, at times slightly larger than $t = 0$, y_2 is larger than both y_1 and y_3 (and therefore the walker is most likely to be at node 2) while at times slightly smaller, y_2 is smaller than both of those nodes (so the walker is most likely to be at node 1 or 3). Clearly (by continuity of y , for t near zero, $y_j > y_k$ if $j \in \{1, 2, 3\}$ and $k \notin \{1, 2, 3\}$).

2. (This question was worth 20 points: five for each set of relations.)

Because $\sin(x + y) = \sin x \cos y + \cos x \sin y$ and $\sin(x - y) = \sin x \cos y - \cos x \sin y$, $\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))$.

This means that $\cos k\theta \sin j\theta = \frac{1}{2}(\sin(j + k)\theta + \sin(j - k)\theta)$; integrating this with respect to θ from $-\pi$ to π tells us that $\int_{-\pi}^{\pi} \cos k\theta \sin j\theta d\theta = 0$.

Now, $\cos(x + y) = \cos x \cos y - \sin x \sin y$ while $\cos(x - y) = \cos x \cos y + \sin x \sin y$. From this, we note $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$.

This means that $\sin k\theta \sin j\theta = \frac{1}{2}(\cos(k - j)\theta - \cos(k + j)\theta)$. As $k \geq 0$ and $j \geq 1$, $k + j$ is nonzero so the integral of $\cos(k + j)\theta$ with respect to θ from $-\pi$ to π is zero. Therefore, if $k \neq j$ (and therefore $k - j$ is nonzero), we conclude that $\int_{-\pi}^{\pi} \sin k\theta \sin j\theta d\theta = 0$; however, if $k = j$, $\cos(k - j)\theta = 1$ so

$$\int_0^{2\pi} \sin k\theta \sin j\theta d\theta = \int_{-\pi}^{\pi} \frac{1}{2} d\theta = \pi.$$

It also means that $\cos k\theta \cos j\theta = \frac{1}{2}(\cos(k - j)\theta + \cos(k + j)\theta)$. As before, the integral of $\cos(k + j)\theta$ with respect to θ from $-\pi$ to π is zero. Therefore, if $k \neq j$ (and therefore $k - j$ is nonzero), we conclude that $\int_{-\pi}^{\pi} \cos k\theta \cos j\theta d\theta = 0$; however, if $k = j$, $\cos(k - j)\theta = 1$ so

$$\int_0^{2\pi} \cos k\theta \cos j\theta d\theta = \int_{-\pi}^{\pi} \frac{1}{2} d\theta = \pi.$$

For the final set of identities, we note that if $k \neq j$, the antiderivative of $e^{ik\theta} e^{-ij\theta}$ is equal to $\frac{1}{i(k-j)} e^{i(k-j)\theta}$, which attains the same value at both π and $-\pi$ ($\frac{1}{i(k-j)}$ if $k - j$ is even and $\frac{-1}{i(k-j)}$ if $k - j$ is odd) so $\int_{-\pi}^{\pi} e^{ik\theta} e^{-ij\theta} d\theta = 0$. However, if $k = j$, $e^{ik\theta} e^{-ij\theta} = 1$, which means that

$$\int_{-\pi}^{\pi} e^{ik\theta} e^{-ij\theta} d\theta = \int_{-\pi}^{\pi} 1 d\theta = 2\pi.$$

3. For this part, we use inner product notation: $\int f, g$ denotes the integral of fg with respect to θ for θ ranging from $-\pi$ to π . We note that $\langle f, f \rangle = \|f\|_2^2$ and the standard rules of real inner products (i.e. symmetry, which says that $\langle f, g \rangle = \langle g, f \rangle$, and bilinearity hold).

Therefore,

$$\begin{aligned} \|f - g\|_2^2 &= \langle f - g, f - g \rangle = \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\ &= \|f\|_2^2 + \|g\|_2^2 - 2\langle f, g \rangle \end{aligned}$$

so $\|f - g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$ iff and only if $\langle f, g \rangle = 0$, i.e. if and only if f is orthogonal to g .