1i. (Having the correct system of equations was worth ten points.)

For this problem, we use the same convention as the lecture notes in stating that $n+1=1$ because $n=0$ (formally, the nodes are given modulo $n$).

Letting $x_j(t)$ be the probability that the walker is in the $j$th node at time $t$, we note the following:

If the walker is in the $j$th node at time $t$, the walker has probability $1-3h$ of staying there until time $t+h$.

If the walker is in the $j-1$th node at time $t$, the walker has probability $2h$ of moving to the $j$th node by time $t+h$.

If the walker is in the $j+1$th node at time $t$, the walker has probability $h$ of moving to the $j$th node by time $t+h$.

If the walker is at any other node at time $t$, the walker has no chance of moving to the $j$th node by time $t+h$.

Therefore, we compute

$$x_j(t+h) = (1-3h)x_j(t) + 2hx_{j-1}(t) + hx_{j+1}(t);$$

this tells us

$$\frac{x_j(t+h) - x_j}{h} = -3x_j(t) + 2x_{j-1}(t) + x_{j+1}(t)$$

and taking the limit as $h$ goes to zero gives us our desired equation

$$x_j' = -3x_j + 2x_{j-1} + x_{j+1}.$$

ii. (This part was also worth ten points.)

Letting $x(t) = (x_1(t), \ldots, x_n(t))$, our differential equation becomes $x' = Ax$ where $A$ is the matrix with a line of $-3$'s on the diagonal, a line of 2's directly below the diagonal (where the top row is directly below the diagonal), a line of 1's directly above the diagonal, and 0's everywhere else.

Analogously to the lecture notes, we guess that the eigenvectors are of the form

$$(1, e^{\frac{2\pi ki}{n}}, \ldots, e^{\frac{2\pi k(n-1)i}{n}})$$

for $k = 0, \ldots, n-1$.

Indeed, when multiplied by $A$, the $(j+1)$th entry, $e^{\frac{2\pi kji}{n}}$, becomes

$$\begin{aligned} &2e^{\frac{2\pi k(j-1)i}{n}} - 3e^{\frac{2\pi kji}{n}} + e^{\frac{2\pi k(j+1)i}{n}} \\ &= e^{\frac{2\pi kji}{n}}(-3 + 2e^{\frac{-2\pi ki}{n}} + e^{\frac{2\pi ki}{n}}) \end{aligned}$$

giving us a family of $n$ linearly independent eigenvectors (indexed by $k = 0, \ldots, n-1$) where the $k$th one has the eigenvalue

$$-3 + 2(\cos(-\frac{2\pi k}{n}) + i\sin(-\frac{2\pi k}{n})) + (\cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n}))$$

$$= -3 + 3\cos(\frac{2\pi k}{n}) - i\sin(\frac{2\pi k}{n})$$

(because the cosine function is even whereas the sine function is odd). We note that all of these eigenvectors have eigenvalues with strictly negative real part except when $k = 0$ (and $\cos \frac{2\pi k}{n} = 1$), whose eigenvalue is zero.

Consequently, all solutions of our differential equation are of the form $x(t) = C(1, \ldots, 1) + y(t)$ where $C$ is a constant and $y$ goes to zero as $t$ goes to infinity.

However, as $x_1(t) + x_2(t) + \ldots + x_n(t) = 1$ for all $t$ (this is the probability that the walker will be at *some* node at time $t$), we have that $nC = 1$, i.e. that $C = \frac{1}{n}$.

In other words, regardless of initial conditions, $x(t)$ will approach $(\frac{1}{n}, \ldots, \frac{1}{n})$ as $t$ approaches infinity, i.e. the probability the wanderer is at node $j$ approaches $\frac{1}{n}$ for each $j$ as $t$ approaches infinity (and, just as in the example in class, each node is equally likely).

2i. (Having the correct system was worth ten points.)

Using the notation of the previous problem, our aim shall be to calculate $x_j(t+h)$ in terms of $x_j(t)$; we distinguish the two cases $j \neq 5$ (edge nodes) and $j = 5$ (corner node).

$j \neq 5$: If we are at node $j$ at time $t$ we have a $1-h$ chance of remaining there at time $t+h$; if we are at node 5 at time $t$ we have a $h/4$ chance of moving to node $j$ at time $t$ and otherwise we have no chance of moving to node $j$ by time $t$. Therefore,

$$x_j(t+h) = (1-h)x_j(t) + \frac{h}{4}x_5(t)$$

which, upon subtracting $x_j(t)$ from both sides, dividing both sides by $h$, and taking the limit as $h$ goes to zero, tells us $x'_j = -x_j + \frac{1}{4}x_5$.

$j = 5$: If we are at node $k$ at time $t$ for $k \neq 5$ we have a $h$ chance of moving there by time $t+h$ and if we are at node 5 we have a $1-h$ chance of staying there. This tells us

$$x_5(t) = h(x_1(t) + x_2(t) + x_3(t) + x_4(t)) + (1-h)x_5(t);$$

subtracting $x_5(t)$ from both sides, dividing both sides by $h$, and taking the limit as $h$ goes to zero, tells us $x'_5 = x_1 + x_2 + x_3 + x_4 - x_5$.

ii. (This part was also worth ten points.)

Writing $x(t) = (x_1(t), \ldots, x_5(t))$ gives us the differential equation $x' = Ax$ where $A$ is the matrix equal to $-1$ on the diagonal, .25 on the rightmost edge away from the diagonal, 1 on the bottom edge away from the diagonal, and 0 everywhere else.

To search for eigenvectors, we begin by noting that $A$ looks like $-I$ except for the bottom edge and rightmost row so we look for eigenvectors with eigenvalue $-1$. Writing $v = (v_1, \ldots, v_5)$ we want to solve $(A+I)v = 0$; however, $(A+I)v = (v_5, v_5, v_5, v_5, v_1+v_2+v_3+v_4-v_5)$. This gives us a three-dimensional eigenspace consisting of those vectors where $v_1 + v_2 + v_3 + v_4 = 0 = v_5$.

The symmetry of the problem suggests that for other eigenvectors we want to look at the case where $v_1 = v_2 = v_3 = v_4$. Writing a vector of this form as $(v, v, v, v, w)$, we note that $A(v, v, v, v, w) = (-v + .25w, -v + .25w, -v + .25w, -v + .25w, 4v - .w)$. In other words, we want to find $\lambda$ such that $\lambda v = -v + .25w$ and $\lambda w = 4v - w$. From the first equation, $v(\lambda + 1) = .25w$; from the second, $w(\lambda + 1) = 4v$. Therefore, $v(\lambda + 1)^2 = .25w(\lambda + 1) = .25 * 4v = v$ so $\lambda + 1$ is equal to either 1 or $-1$. In the first case, $\lambda = 0$ so $v = .25w$ (i.e. the eigenvectors are multiples of $(1, 1, 1, 1, 4)$) and in the second case, $\lambda = -2$ so $v = -.25w$ (i.e. the eigenvectors are multiples of $(1, 1, 1, 1, -4)$). As we have found five linearly independent eigenvectors, we have found all possible eigenspaces.

As the only eigenspace whose real part is not strictly negative is the one corresponding to $(1, 1, 1, 1, 4)$, all solutions of our differential equation are of the form $x(t) = C(1, 1, 1, 1, 4) + y(t)$ where $C$ is a constant and $y$ goes to zero as $t$ goes to infinity.

However, as $x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) = 1$ for all $t$ (this is the probability that the walker will be at *some* node at time $t$), we have that $C(1 + 1 + 1 + 1 + 4) = 1$, i.e. that $C = \frac{1}{8}$.

In other words, regardless of initial conditions, $x(t)$ will approach

$$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$$

as $t$ approaches infinity, i.e. the probability the wanderer is at node $j$ approaches $\frac{1}{8}$ for $j \neq 5$ and $\frac{1}{2}$ for $j = 5$ as $t$ approaches infinity.