Problems graded: 29.3 (15 points), 29.13 (10 points), 32.2 (10 points); the other problems were graded based on completion and worth five points each (except that I counted 28.8, which was longer, as two problems). To convert from 65 points to 100, scores were multiplied by 1.5; then we gave everyone 2.5 points, rounding all fractions up at the end.

Final homework grade: Students have received nine scores, each out of 100. As per the syllabus, we dropped the two lowest scores for each students and converted the remaining sum to a percentage out of 100, rounding any fractional part up ($623/700 = 89$ percent but $624/700 = 89.14$ percent gets bumped up to 90 percent).

28.8. (Note: I did this question in section on November 18.)

(a) Suppose $\epsilon > 0$. If $|x - 0| < \sqrt{\epsilon}$ then $f(x)$ is equal to either $x^2 \in [0, \epsilon)$ or 0; in any event, $|f(x) - f(0)| < \epsilon$ implying continuity at zero.

(b) x rational nonzero: Let $\epsilon = x^2$ and suppose $\delta > 0$. There exists an irrational number q in $(x - \delta, x + \delta)$ (see Exercise 4.12 from HW 2). However, while $|x - q| < \delta$, $|f(x) - f(q)| = \epsilon$; δ can be made arbitrarily small so f is NOT continuous at x.

x irrational: Let $\epsilon = .1x^2$ and suppose $\delta > 0$ (we may further suppose that $\delta < .5|x|$). There exists a rational number q in $(x - \delta, x + \delta)$; by the triangle inequality, $|q| > .5|x|$ so $f(q) = .25x^2$. Therefore, $|f(x) - f(q)| = .25x^2 > \epsilon$; as δ can be made arbitrarily small, f is not continuous at x.

As all nonzero points are either nonzero rational or irrational, f is discontinuous for all nonzero x.

(c) If $x = 0$ and $h \neq 0$, $\frac{f(x+h)-f(x)}{h} = \frac{f(h)}{h}$, which is equal to h if h is rational and 0 otherwise. In any event, the expression clearly goes to zero as h goes to zero (as it is bounded above in magnitude by $|h|$) so f is differentiable at $x = 0$ with derivative 0.

29.3. (a) By Theorem 29.3 there exists $x \in (0,2)$ with $f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{1}{2}$.

(b) By Theorem 29.3 there exists $y \in (0,1)$ with $f'(y) = \frac{f(1)-f(0)}{1-0} = 1$; there also exists $z \in (1,2)$ with $f'(z) = \frac{f(2)-f(1)}{2-1} = 0$. As $y < z$ and $\frac{1}{7}$ lies between $f'(y) = 1$ and $f'(z) = 0$, Theorem 29.8 tells us there exists $x \in (y,z) \subset (0,2)$ with $f'(x) = \frac{1}{7}$.

29.5. Suppose f is not constant, i.e. there exists $a < b$ with $f(a) \neq f(b)$. Letting $Z = \frac{f(b)-f(a)}{(b-a)^2}$ there exists a natural number n such that $Zn > 1$. We let $a_j = a + \frac{j}{n}$ for $j = 0, 1, \ldots, n$; by the triangle inequality, there exists $j \in [0, n-1]$ such that $|f(a_{j+1}) - f(a_j)| \geq n^{-1}|f(b) - f(a)|$. However, this tells us

$$|f(a_{j+1}) - f(a_j)| \geq n^{-1}|f(b) - f(a)| = Zn^{-1}(b-a)^2 = Zn^{-1}(n(a_{j+1} - a_j))^2$$

$$\geq Zn(a_{j+1} - a_j)^2 > (a_{j-1} - a_j)^2$$

producing a contradiction so f is indeed constant.

29.13. Suppose there exists $x > 0$ with $f(x) > g(x)$, i.e. $f(x) - g(x) > 0$. As $f(0) - g(0) = 0$, setting $h = f - g$ tells us that h is differentiable with $h(x) > h(0)$ so by Theorem 29.3 there exists $t \in (0,x)$ with $h'(t) > 0$, i.e. $f'(t) - g'(t) > 0$ (by Theorem 28.3) or $f'(t) > g'(t)$ contradicting that $f'(x) \leq g'(x)$ for all real x so $f(x) \leq g(x)$ for $x \geq 0$.

29.15. We note that $h(x) = (x^m)^{1/n}$; by the Chain Rule, $h'(x) = \phi(x^m) * (x^m)'$ (where $\phi(t)$ is the derivative of the function $x^{1/n}$ evaluated at t; this equals $\frac{1}{n}x^{\frac{1}{n}-1}$ by Example 2 of this section) and $(x^m)' = mx^{m-1}$ by Example 3 of the preceding section, so

$$h'(x) = \frac{1}{n}(x^m)^{\frac{1}{n}-1} * mx^{m-1} = \frac{m}{n}x^{\frac{m}{n}-m+m-1} = rx^{r-1}$$

as desired.

32.1. Fix n; we consider the partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$ where $t_k = b\frac{k}{n}$ for each k.

Consequently, $U(f, P) = \Sigma_{k=1}^{n} (\frac{kb}{n})^3 * \frac{b}{n}$ (each interval has length $\frac{b}{n}$; on the interval $[t_{i-1}, t_i]$, the maximum value of f is attained at $t_i = \frac{i}{n}$)

$= (\frac{b}{n})^4 \Sigma_{k=1}^{n} k^3 = (\frac{b}{n})^4 (\frac{n^2+n}{2})^2$ (by Exercise 1.3 from the first homework and Example 1 from section 1 to evaluate the sum of cubes)

$= \frac{1}{4} b^4 (1 + 2n^{-1} + n^{-2}$ which approaches $\frac{1}{4} b^4$ as $n \to \infty$ so $U(f) \leq \frac{1}{4} b^4$.

However, $L(f, P) = \Sigma_{k=0}^{n-1} (\frac{kb}{n})^3 * \frac{b}{n}$ (each interval has length $\frac{b}{n}$; on the interval $[t_i, t_{i+1}]$, the minimum value of f is attained at $t_i = \frac{i}{n}$)

$= (\frac{b}{n})^4 \Sigma_{k=0}^{n-1} k^3 = (\frac{b}{n})^4 (\frac{n^2-n}{2})^2$ (substituting in $n - 1$ for the formula of the sum of the first n cubes)

$= \frac{1}{4} b^4 (1 - 2n^{-1} + n^{-2})$ which approaches $\frac{1}{4} b^4$ as $n \to \infty$ so $L(f) \geq \frac{1}{4} b^4$. Therefore $U(f) \leq L(f)$; by Theorem 32.4, $U(f) \geq L(f)$ so $U(f) = L(f) = \frac{1}{4} b^4$.

32.2. (a) Upper integral:

To establish an upper bound, for a fixed n we consider the partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$ where $t_k = b\frac{k}{n}$ for each k.

Consequently, $U(f, P) = \Sigma_{k=1}^{n} \frac{kb}{n} * \frac{b}{n}$ (each interval has length $\frac{b}{n}$; on the interval $[t_{i-1}, t_i]$, the maximum value of f is attained at $t_i = \frac{i}{n}$)

$= (\frac{b}{n})^2 \Sigma_{k=1}^{n} k = (\frac{b}{n})^2 \frac{n^2+n}{2}$

$= \frac{1}{2} b^2 (1 + 2n^{-1})$ which approaches $\frac{1}{2} b^2$ as $n \to \infty$ so $U(f) \leq \frac{1}{2} b^2$.

For a lower bound we look an arbitrary partition $P = \{0 = t_0 < t_1 < \ldots < t_n < b\}$ and note that

$U(f, P) = \Sigma_{k=1}^{n} t_k (t_k - t_{k-1})$ (each interval has length $\frac{b}{n}$; the supremum of f on $[t_{i-1}, t_i]$ is t_i as one can pick rational numbers in the interval arbitrarily close to t_i)

$\geq \Sigma_{k=1}^{n} \frac{(t_k + t_{k-1})}{2} (t_k - t_{k-1})$ (as $t_k > t_{k-1}, t_k > \frac{t_k + t_{k-1}}{2}$)

$= \frac{1}{2} \Sigma_{k=1}^{n} (t_k^2 - t_{k-1}^2) = \frac{1}{2} b^2$ (this was a telescoping series); as this holds for all partitions we indeed have $U(f) = \frac{1}{2} b^2$.

Lower integral: For each partition $P = \{0 = t_0 < t_1 < \ldots < t_n < b\}$, there exists an irrational number in $[t_{i-1}, t_i]$ for each i so the minimal value of f on $[t_{i-1}, t_i]$ is 0 which implies that $L(f, P) = 0$ for each P so $L(f) = 0$.

(b) Assuming $b > 0$: as $U(f) \neq L(f)$, f is not integrable on $[0, b]$. (None of the definitions involved make sense on degenerate intervals).

32.3. (a) Upper integral:

To establish an upper bound, for a fixed n we consider the partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$ where $t_k = b\frac{k}{n}$ for each k.

Consequently, $U(f, P) = \Sigma_{k=1}^{n}(\frac{kb}{n})^2 * \frac{b}{n}$ (each interval has length $\frac{b}{n}$; on the interval $[t_{i-1}, t_i]$, the maximum value of f is attained at $t_i = \frac{i}{n}$)

$= (\frac{b}{n})^3 \Sigma_{k=1}^{n} k^2 = (\frac{b}{n})^3 \frac{n(n+1)(2n+1)}{6}$ (by Exercise 1.1 for the sum)

$= \frac{1}{6}b^3(2 + 3n^{-1} + n^{-2})$ which approaches $\frac{1}{3}b^3$ as $n \to \infty$ so $U(f) \leq \frac{1}{3}b^3$.

For a lower bound we look an arbitrary partition $P = \{0 = t_0 < t_1 < \ldots < t_n < b\}$ and note that

$U(f, P) = \Sigma_{k=1}^{n} t_k^2(t_k - t_{k-1})$ (each interval has length $\frac{b}{n}$; the supremum of f on $[t_{i-1}, t_i]$ is t_i^2 as one can pick rational numbers in the interval arbitrarily close to t_i)

$\geq \Sigma_{k=1}^{n} \frac{(t_k^2 + t_k t_{k-1} + t_{k-1}^2)}{3}(t_k - t_{k-1})$ (as $t_k > t_{k-1} > 0, t_k^2 > \frac{t_k^2 + t_k t_{k-1} + t_{k-1}^2}{3}$)

$= \frac{1}{3}\Sigma_{k=1}^{n}(t_k^3 - t_{k-1}^3) = \frac{1}{3}b^3$ (this was a telescoping series); as this holds for all partitions we indeed have $U(f) = \frac{1}{3}b^3$.

Lower integral: For each partition $P = \{0 = t_0 < t_1 < \ldots < t_n < b\}$, there exists an irrational number in $[t_{i-1}, t_i]$ for each i so the minimal value of f on $[t_{i-1}, t_i]$ is 0 which implies that $L(f, P) = 0$ for each P so $L(f) = 0$.

(b) Assuming $b > 0$: as $U(f) \neq L(f)$, f is not integrable on $[0, b]$. (None of the definitions involved make sense on degenerate intervals).