Problems graded: 20.16 (10 points), 28.3 (15 points); the others were graded for completeness and worth five points each (though 28.7, which was longer, was worth ten) and scores were then doubled to make the total value 100.

20.12. (a) (Graph should start out just below the origin for large negative x; gradually decreasing as x increases until $x = 0$, where $f(x) = -.25$. From here on out, f should be rapidly decreasing until $x = 1$, where f is undefined, but the limit from the left is $-\infty$. The limit from the right is ∞; from here, the function should swoop down to a minimum value for x somewhere between 1 and 2; technically, the coordinates are $(4/3, 6.75)$. Then the function rapidly increases to another vertical asymptote at $x = 2$; past $x = 2$ the function decreases, staying positive for $x > 2$ although the function should approach the x axis as x gets large).

(b) $\lim_{x\to 2^+} f(x)$: Consider $x = 2 + h$ for h small and positive. $f(x) = (1+h)^{-1} * h^{-2}$; h^{-2} approaches ∞ as h approaches zero from the right and $1 + h$ approaches 1; this gives that $f(x)$ approaches ∞ as h approaches 0 from the right by the limit laws from Section 9.

$\lim_{x\to 2^-} f(x)$: Consider $x = 2 + h$ for x small and negative. $f(x) = (1+h)^{-1} * h^{-2}$; h^{-2} approaches ∞ as h approaches zero from the left and $1 + h$ approaches 1; this gives that $f(x)$ approaches ∞ as h approaches 0 from the left by the limit laws from Section 9.

$\lim_{x\to 1^+} f(x)$: Consider $x = 1 + h$ for h small and positive. $f(x) = h^{-1} * (1+h)^{-2}$; $(1+h)^{-2}$ approaches 1 as h approaches zero from the right and h^{-1} approaches ∞; this gives that $f(x)$ approaches ∞ as h approaches 0 from the right by the limit laws from Section 9.

$\lim_{x\to 1^-} f(x)$: Consider $x = 1 + h$ for h small and negative. $f(x) = h^{-1} * (1+h)^{-2}$; $(1+h)^{-2}$ approaches 1 as h approaches zero from the left and h^{-1} approaches $-\infty$; this gives that $f(x)$ approaches $-\infty$ as h approaches 0 from the right by the limit laws from Section 9.

(c) By Theorem 20.10, we know from the preceding part that since the right hand limit and the left hand limit of $f(x)$ agree (both are ∞) as x approaches two, this is indeed $\lim_{x\to 2} f(x)$. However, since these limits are different (one is ∞ and the other is $-\infty$) for x approaching one, $\lim_{x\to 1} f(x)$ does not exist.

20.16. (a) Suppose $L_1 > L_2$; let $\epsilon = .5(L_1 - L_2)$. For $\epsilon > 0$, by Corollary 20.8 there exists $\delta_1 > 0$ such that for $x \in (a, a + \delta_1)$, $f_1(x) > L_1 - \epsilon = .5(L_1 + L_2)$. Further, there exists $\delta_2 > 0$ such that for $x \in (a, a + \delta_2$, $f_2(x) < L_2 + \epsilon = .5(L_1 + L_2)$. Fixing δ to be the minimum of δ_1 and δ_2 it follows that for $x \in (a, a + \delta), f_1(x) > .5(L_1 + L_2) > f_2$ producing a contradiction so $L_1 \leq L_2$.

(b) Even if $f_1 < f_2$ on (a, b) we cannot conclude that $L_1 < L_2$; for example consider $a = 0, b = 1, f_1(x) = x, f_2(x) = 2x$. In this case, $L_1 = L_2 = 0$ because both f_1 and f_2 extend continuously to 0 where they have the value 0.

20.17. By Corollary 20.8 it suffices to show that for an arbitrary fixed $\epsilon > 0$ there exists δ such that for $x \in (a, a + \delta)$, $f(x) \in (L - \epsilon, L + \epsilon)$.

However, fixing such an ϵ, this same corollary gives us a $\delta_1 > 0$ such that for $x \in (a, a + \delta_1)$, $f_1(x) > L - \epsilon$ and a $\delta_3 > 0$ such that for $x \in (a, a + \delta_3$, $f_3(x) < L + \epsilon$. Setting δ to be the minimum of δ_1 and δ_3 we conclude that for $x \in (a, a + \delta)$,

$$L - \epsilon < f_1(x) \leq f_2(x) \leq f_3(x) < L + \epsilon$$

as desired.

28.1. We begin by making the following observation: if f and g are functions which agree on an open interval I and f is differentiable on I then so is g (fixing $x \in I$ there exists $\delta > 0$ such that for $|h| < \delta, x + h \in I$, so $\frac{g(x+h)-g(x)}{h} = \frac{f(x+h)-f(x)}{h}$).

(a) We note that $e^{|x|} = e^x$ on $(0, \infty)$, which is assumed to be differentiable there with derivative e^x. Further, $e^{|x|} = e^{-x}$ on $(-\infty, 0)$, which is differentiable there with derivative $-e^{-x}$ by the Chain Rule.

The only point remaining to account for is at $x = 0$. However, using $f(x)$ to denote $e^{|x|}$, we have that the right-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to zero for $x = 0$) is the limit of $\frac{e^{x+h}-e^x}{h}$ as h goes to zero, which is the derivative of e^x at 0, or 1. However, the left-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h goes to zero for $x = 0$) is the limit of $\frac{e^{-(x+h)}-e^{-x}}{h}$ as h goes to zero, which is the derivative of e^{-x} at 0, or -1. As these two limits do not agree, the difference quotient does not have a single limit as h approaches zero so f is NOT differentiable at 0.

(c) If k is an even integer, $|\sin x| = \sin x$ on $(k\pi, (k+1)\pi)$, which is assumed to be differentiable with derivative $\cos x$. If k is an odd integer, $|\sin x| = -\sin x$ on $(k\pi, (k+1)\pi)$, which is differentiable with derivative $-\cos x$ by Theorem 28.3.(i).

The only points left to be accounted for are $x = k\pi$ with k an integer.

k is an even integer: Setting $f(x) = |\sin x|$, the right-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero is equal to the same limit with f replaced by $\sin x$; this is the derivative of the sine function at $x = k\pi$, or $\cos k\pi = 1$. By contrast, the left-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero is equal to the same limit with f replaced by $-\sin x$; this is the derivative of the additive inverse of the sine function at $x = k\pi$, or $-\cos k\pi = -1$. As these two limits do not agree, the difference quotient does not have a single limit as h approaches zero so f is NOT differentiable at $k\pi$.

k is an odd integer: Setting $f(x) = |\sin x|$, the right-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero is equal to the same limit with f replaced by $-\sin x$; this is the derivative of the additive inverse of the sine function at $x = k\pi$, or $-\cos k\pi = 1$. By contrast, the left-hand limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero is equal to the same limit with f replaced by $\sin x$; this is the derivative of the sine function at $x = k\pi$, or $\cos k\pi = -1$. As these two limits do not agree, the difference quotient does not have a single limit as h approaches zero so f is NOT differentiable at $k\pi$.

CONCLUSION: The points of non-differentiability for $|\sin x|$ are $k\pi$ for $k \in \mathbf{Z}$.

(e) As $x^2-1=(x-1)(x+1)$ which is negative for $x \in (-1,1)$ and nonnegative elsewhere, we have that $|x^2-1|$ agrees with x^2-1 (which is differentiable with derivative $2x$ by Example 3 of this section and Theorem 28.3.(ii)) on $(-\infty,-1)$ and $(1,\infty)$ and with $1-x^2$ (which is differentiable with derivative $-2x$ by Theorem 28.3.(i)) on $(-1,1)$; the only points left to check are $x = \pm 1$. Here we use $f(x)$ to denote $|x^2-1|$.

$x = -1$: We note that for $h > 0$, the limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero from above is equal to the same limit with f replaced by $1-x^2$ so this right-hand limit is equal to $(1-x^2)'$ evaluated at $x=-1$, or $-2*-1=2$. However, for $h<0$, the limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero from below is equal to the same limit with f replaced by x^2-1 so this left-hand limit is equal to $(x^2-1)'$ evaluated at $x=-1$, or $-2*1=-2$. As these two limits do not agree, the difference quotient does not have a single limit as h approaches zero so f is NOT differentiable at -1.

$x = 1$: We note that for $h > 0$, the limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero from above is equal to the same limit with f replaced by x^2-1 so this right-hand limit is equal to $(x^2-1)'$ evaluated at $x=1$, or $2*1=2$. However, for $h<0$, the limit of $\frac{f(x+h)-f(x)}{h}$ as h approaches zero from below is equal to the same limit with f replaced by $1-x^2$ so this left-hand limit is equal to $(1-x^2)'$ evaluated at $x=1$, or $-2*1=-2$. As these two limits do not agree, the difference quotient does not have a single limit as h approaches zero so f is NOT differentiable at 1.

CONCLUSION: The points of non-differentiability here are ± 1.

28.3. (a) Fix $x > 0$. We have that $h(x)$ is the limit, as t goes to zero (we can assume $|t| < x$), of $\frac{h(x+t)-h(x)}{t} = \frac{\sqrt{x+t}-\sqrt{x}}{t}$.

Multiplying numerator and denominator by $\sqrt{x+t}+\sqrt{x}$ gives us $\frac{t}{t(\sqrt{x+t}+\sqrt{x})} = \frac{1}{\sqrt{x+t}+\sqrt{x}}$. As the denominator approaches $2\sqrt{x}$ as t approaches zero, we indeed have that $h'(x) = \frac{1}{2}x^{-1/2}$.

(b) Fix $x \neq 0$. We have that $h(x)$ is the limit, as t goes to zero (we can assume $|t| < |x|$), of $\frac{h(x+t)-h(x)}{t} = \frac{(x+t)^{1/3}-x^{1/3}}{t}$.

Multiplying numerator and denominator by $(x+t)^{2/3}+(x+t)^{1/3}x^{1/3}+x^{2/3}$ (here: $(a-b)(a^2+ab+b^2) = a^3-b^3$) gives us $\frac{t}{t((x+t)^{2/3}+(x+t)^{1/3}x^{1/3}+x^{2/3})} = \frac{1}{(x+t)^{2/3}+(x+t)^{1/3}x^{1/3}+x^{2/3}}$. As the denominator approaches $3x^{2/3}$ as t approaches zero, we indeed have that $h'(x) = \frac{1}{3}x^{-2/3}$.

(c) At $x = 0$, for nonzero t the difference quotient becomes $\frac{t^{1/3}}{t} = t^{-2/3}$ which approaches ∞ as t approaches zero; as the limit must be finite for the derivative to exist, f is NOT differentiable at $x = 0$.

28.7. (a) (Graph should look like the x-axis for $x \leq 0$ and like the right half of $y = x^2$ for $x \geq 0$; this shape is sometimes referred to as a 'smirk').

(b) We note that $\frac{f(x+h)-f(x)}{h}$ is equal to 0 for $h < 0$ and $\frac{h^2}{h} = h$ for $h > 0$ at $x = 0$; as this difference quotient has magnitude less than ϵ whenever $|h| < \epsilon$, it approaches zero as h does so $f'(0) = 0$.

(c) (Graph should look like the x-axis for $x \leq 0$ and like a line through the origin with slope two for $x \geq 0$).

(d) f' is continuous on $\mathbf{R}$; by our observation at the beginning of this section, $f' = 0$ on $(-\infty, 0)$ and $f' = 2x$ on $(0, \infty)$ so the only point to check is $x = 0$. However, $f'(0) = 0$ and if $0 < |x| < .5\epsilon$, $|f'(x)| < \epsilon$ so $f'(x)$ approaches zero as x does giving the desired continuity. However, $f'(x)$ is NOT differentiable at zero; the difference quotient $\frac{f'(x+h)-f'(x)}{h}$ (for $x = 0$) is equal to 2 for $h > 0$ and 0 for $h < 0$ so it cannot have a limit as h approaches zero.