Problems graded: 14.6 (10 points), 15.6 (15 points), 17.5 (10 points), 17.13 (10 points); the other problems were graded on completion to be worth five points each

14.2. (a) For $n \geq 2$, $(n-1) \geq .5n$ so $\frac{n-1}{n^2} \geq \frac{1}{2n}$. However, $\Sigma\frac{1}{n}$ diverges to ∞ by Formula 2 of Example 1 of this section ($p = 1$); therefore, so does $\Sigma\frac{1}{2n}$. By the Comparison Test (and the fact that the convergence of a series is independent of its first term), $\Sigma(n-1)/n^2$ diverges to ∞.

(e) We note that the ratio of the $(n+1)$th term to the nth term is $\frac{(n+1)^2/(n+1)!}{n^2/n!} = \frac{((n+1)/n)^2}{n+1}$ which approaches 0 as n approaches ∞ (by the limit rules of Section 9; $\frac{n+1}{n} = 1 + n^{-1}$ which approaches 1 as n approaches ∞) so the Ratio Test tells us $\Sigma n^2/n!$ converges.

(g) We note that the ratio of the $(n+1)$th term to the nth term is $\frac{(n+1)/2^{n+1}}{n/2^n} = \frac{1+n^{-1}}{2}$ which approaches $\frac{1}{2}$ as n approaches ∞ (by the limit rules of Section 9) so the Ratio Test tells us $\Sigma n/2^n$ converges.

14.6. (a) Suppose $\Sigma|a_n|$ converges; by Theorem 14.4, it satisfies the Cauchy criterion. To show $\Sigma a_n b_n$ converges we let $\epsilon > 0$ and $B > 0$ be an upper bound for $|b_n|$; we seek to show $\Sigma a_n b_n$ satisfies the Cauchy criterion. As $\Sigma|a_n|$ satisfies this criterion, there exists M such that for $n > m > M$, $\Sigma_{k=m}^n |a_k| < \epsilon/B$. Therefore, for such m and n,

$|\Sigma_{k=m}^n a_k b_k| \leq \Sigma_{k=m}^n |a_k||b_k|$ (by the triangle inequality)

$\leq B\Sigma_{k=m}^n |a_k| < B\epsilon/B = \epsilon$ so we note that $\Sigma a_n b_n$ satisfies the Cauchy criterion and therefore converges by Theorem 14.4.

(b) Corollary 14.7 states that absolutely convergent series are convergent; letting Σa_n be an absolutely convergent series, this is a special case of part (a) where $b_n = 1$ for each n.

14.7. If Σa_n is a convergent series of nonnegative numbers and $p > 1$ the Cauchy criterion tells us that for each $\epsilon > 0$ there exists $M > 0$ such that for $n \geq m > N$, $|\Sigma_{j=m}^n a_j| < \epsilon$. Supposing $\epsilon < 1$, the a_j are nonnegative so the absolute value sign is redundant; this tells us $a_j \in [0, 1)$ for $j > N$ and therefore $a_j^p \leq a_j$ so $|\Sigma_{j=m}^n a_j^p| \leq \Sigma_{j=m}^n a_j < \epsilon$ giving the desired convergence.

14.10. We let $a_n = 2^{n(-1)^n}$; we note that the nth root of a_n is equal to 2 for even n and .5 for odd n so the limit superior of $|a_n|^{1/n}$ is 2; therefore, the series converges by the Root Test.

However, $\frac{a_{n+1}}{a_n} = 2^{2n+1}$ for n odd and $2^{-(2n+1)}$ for n even; as 2^{2n+1} diverges to infinity and $2^{-(2n+1)}$ converges to zero, we have that lim sup $|a_{n+1}/a_n| = \infty$ and lim inf $|a_{n+1}/a_n| = 0$ so the Ratio Test gives no information. (NOTE: Answers may vary.)

14.12. We choose the sequence of positive integers $\{n_k\}$ as follows.

Step 1: Let n_1 be the first positive integer with $|a_{n_1}| < 2^{-1}$ (we can do this because lim inf $|a_n| = 0$).

Step k, $k > 1$ (assuming step $k-1$ was done): Let n_k be the first positive integer greater than n_{k-1} with $|a_{n_k}| < 2^{-k}$.

When this is done, $\{a_{n_k}\}$ is a subsequence indexed by k where $\Sigma_k a_{n_k}$ converges by the comparison test (the series to be compared to is $\Sigma_k 2^{-k}$, which converges by Example 1 of this section).

15.1. (a) $\Sigma(-1)^n/n$ converges by the alternating series test (clearly $\{1/n\}$ is a decreasing series of positive numbers which converges to zero).

(b) $\Sigma(-1)^n n!/2^n$ does NOT converge by the ratio test (the ratio of the $(n+1)$th term to the nth term is $-(n+1)/2$, whose absolute value diverges to ∞).

15.5. We could not use the Comparison Test to prove Theorem 15.1 for $p > 1$ because in order to show that the series Σn^{-p} converged by this test we would need to find another series with terms that were larger in magnitude which also converged. Thus far, we had seen that the series Σn^{-1} diverged. While we could then use the Comparison Test to show the divergence of series of positive terms with larger magnitude (like Σn^{-p} for $p < 1$), the test would be useless to look at series of positive terms with smaller magnitude like Σn^{-p} for $p > 1$.

15.6. (a) Let $a_n = n^{-1}$; by Example 1 of this section, Σa_n diverges but by Example 2 of the same section, Σa_n^2 converges.

(b) If Σa_n is a convergent series of nonnegative terms, then $\Sigma |a_n| = \Sigma a_n$ converges. However, $a_n \to 0$ by Corollary 14.5, so by Exercise 14.7 (letting $b_n = a_n$), we indeed have Σa_n^2 converges.

(c) $\Sigma n^{-.5}(-1)^n$ converges by the Alternating Series Theorem; however, here $a_n = n^{-.5}(-1)^n$ and $a_n^2 = n^{-1}$ so Σa_n^2 diverges by Example 1 of this section.

17.2. (a) $(f+g)(x) = 4 + x^2$ for $x \geq 0$ (where $f(x) = 4$) and $(f+g)(x) = x^2$ for $x < 0$ (where $f(x) = 0$).

$(fg)(x) = 4x^2$ for $x \geq 0$ and $(fg)(x) < 0$ for $x = 0$.

$(f \circ g)(x) = 4$ for all x (as $g(x) = x^2 \geq 0$ for all x).

$(g \circ f)(x) = 16$ for $x \geq 0$ (where $f(x) = 4$) and $(g \circ f)(x) = 0$ for $x < 0$ (where $f(x) = 0$).

The domains of all these functions are $\mathbf{R}$.

(b) f is not continuous at 0; the sequence $\{-1/n\}$ converges to 0 but $\{f(-1/n)\}$ is the constant sequence $\{0\}$ which does not converge to $f(0) = 4$ so f is not continuous. (Note that f is continuous at all nonzero points: if $x \neq 0$ and $\epsilon > 0$, then picking $\delta = |x|$ gives that for $|y - x| < \delta$, $f(x) = f(y)$ so $|f(x) - f(y)| < \epsilon$).

g is continuous because if $\{x_n\}$ is a sequence converging to x, $\{g(x_n)\} = \{x_n^2\}$ is a sequence which converges to $x^2 = g(x)$ by the limit laws of Section 9.

$f + g$ is not continuous at 0; the sequence $\{-1/n\}$ converges to 0 but $\{(f + g)(1/n)\}$ is the sequence $\{n^{-2}\}$ which does not converge to $(f + g)(0) = 4$ (it converges to 0) so $f + g$ is not continuous.

fg is continuous at all nonzero points by Theorem 17.4 (ii); it is continuous at 0 because if $\{x_n\}$ is a sequence that converges to 0, $|fg(x_n)| = |f(x_n)||g(x_n)| \leq 4x_n^2$ (this upper bound converges to zero by the convergence laws in Section 9). Therefore, $\{fg(x_n)\}$ converges to 0 by problem 8.6 (from the second homework), establishing continuity at 0 and therefore at all points.

$g \circ f$ is not continuous at 0; the sequence $\{-1/n\}$ converges to 0 but $\{g \circ f(-1/n)\}$ is the constant sequence $\{0\}$ which does not converge to $g \circ f(0) = 16$ so f is not continuous.

$f \circ g$ is the constant function 4 which is clearly continuous (for any sequence $\{x_n\}$ converging to x, the sequence $\{f(x_n)\}$ is the constant sequence 4 which clearly converges to $4 = f(x)$).

NOTE: All of these functions can be shown to be continuous at all nonzero points.

17.5. (a) We show this by induction on m. $m = 1$ is trivial because if $\{x_n\}$ converges to x, then here $(f(x) = x^m = x)$, $\{f(x_n)\} = \{x_n\}$ which clearly converges to $x = f(x)$. Assuming the result for a given $m \geq 1$ it follows for $m+1$ because $x^{m+1} = x^m * x$ which, being the product of two continuous functions (x^m is continuous by the induction hypothesis), is continuous by Theorem 17.4.(ii).

(b) If $p(x) = a_0 + a_1 x + \ldots + a_n x^n$, then $p(x)$ is the sum of $n+1$ continuous functions (by the preceding part); n applications of Theorem 17.4.(i) show continuity. (In fact, one can use Theorem 17.4.(i) and induction to show that any finite sum of functions which are all continuous at x_0 is itself continuous at x_0).

17.8. (a) We consider two cases: $f \geq g$ or $f < g$ (and note all computations are pointwise).

If $f \geq g$ then $\min(f, g) = g$; as $|f - g| = f - g$, $\frac{1}{2}(f + g) - \frac{1}{2}|f - g| = .5f + .5g - (.5f - .5g) = g$ so we indeed have $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

If $f < g$ then $\min(f, g) = f$; as $|f - g| = g - f$, $\frac{1}{2}(f + g) - \frac{1}{2}|f - g| = .5f + .5g - (.5g - .5f) = g$ so we indeed have $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

(b) Once again, we consider two cases: $f \geq g$ or $f < g$ (and also, all computations are still pointwise).

If $f \geq g$ then $\min(f, g) = g$ and $-f \leq -g$ so $\max(-f, -g) = -g$ so $\min(f, g) = -\max(-f, -g)$.

If $f < g$ then $\min(f, g) = f$ and $-f > -g$ so $\max(-f, -g) = -f$ so $\min(f, g) = -\max(-f, -g)$.

(c) Using (a): At x_0, $\min(f, g)$ is the sum of two continuous functions: $\frac{1}{2}(f + g)$ ($f + g$ is continuous at x_0 by Theorem 17.4.(i); multiplying by $\frac{1}{2}$ leaves the function continuous by Theorem 17.3) and $-\frac{1}{2}|f - g|$ ($f - g$ is continuous at x_0 by Theorem 17.4.(i); applying the absolute value preserves this continuity by Theorem 17.3 and multiplying by $-\frac{1}{2}$ also preserves this continuity by Theorem 17.3) and is therefore continuous at this point by Theorem 17.4.(i).

Using (b): Because $-f$ and $-g$ are continuous at x_0 by Theorem 17.3, so is $\max(-f, -g)$ (by Example 5 of this section);

therefore, so is $-\max(-f, -g) = \min(f, g)$ (the continuity follows by another application of Theorem 17.3; the equality follows by part (b) of this question).

17.9. (b) (NOTE: $f(x) = \sqrt{x}$ is only defined for $x \geq x_0 = 0$.) Let $\epsilon > 0$. Choose $\delta = \epsilon^2$; if $|x - x_0| < \delta$ and x is in the domain of the function, $0 \leq x < \epsilon^2$ so $0 \leq \sqrt{x} < \epsilon$ and $|f(x) - f(x_0)| < \epsilon$; as ϵ is arbitrary we indeed have continuity of f at x_0.

(c) Let $\epsilon > 0$. Choose $\delta = \epsilon$ (here, $f(x) = x\sin(1/x)$ for $x \neq 0$, $f(0) = 0$, and $x_0 = 0$) and look at x with $|x - x_0| < \delta$. If $x = 0$ then clearly $f(x) = f(x_0)$; assume this is not the case. Then, $|f(x)| = |x||\sin(1/x)| \leq |x| < \epsilon$ so $|f(x) - f(x_0)| < \epsilon$; as ϵ is arbitrary we indeed have continuity of f at x_0.

17.10. (b) Let $x_n = (\pi(n + .5))^{-1}$ for each positive integer n; then $g(x_n) = \sin x_n^{-1} = \sin(\pi(n + .5))$ which equals -1 for odd n and 1 for even n; in any event, as $x_0 = g(x_0) = 0$, we have $|g(x_n) - g(x_0)| = 1$ for each n; although $x_n \to x_0$ (by the limit rules of section 9), $g(x_n)$ does NOT converge to $g(x_0)$.

(c) Let $x_n = n^{-1}$; $\operatorname{sgn}(x_n) = 1$ which clearly does not converge to $\operatorname{sgn}(x_0) = 0$ as $x_0 = 0$.

17.13. (a) We consider two cases: x is rational and x is irrational.

x is nozero rational: Let $\delta > 0$. There is an irrational number y in $(x - \delta, x + \delta)$ by Exercise 4.12 from the second homework. Although $|y - x| < \delta$, $|f(y) - f(x)| = |0 - 1| = 1$; therefore, setting $\epsilon = 1$ there exists no $\delta > 0$ where $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. This means by Theorem 17.2 that f is not continuous at x.

x is irrational: Let $\delta > 0$. There is a rational number y in $(x - \delta, x + \delta)$ by Exercise 4.11 from the second homework. Although $|y - x| < \delta$, $|f(y) - f(x)| = |1 - 0| = 1$; therefore, setting $\epsilon = 1$ there exists no $\delta > 0$ where $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. This means by Theorem 17.2 that f is not continuous at x.

Therefore, f is discontinuous at every real x.

(b) We consider three cases: $x = 0$, x is nonzero rational, and x is irrational.

$x = 0$: Fix $\epsilon > 0$ and suppose $\delta = \epsilon$. If $|y - x| < \delta$ then either $f(y) = 0$ or $f(y) = y$; in any event, $|f(y) - f(x)| = |f(y)| \leq |y| < \epsilon$ so we indeed have continuity at 0.

x is rational: Let $\delta > 0$. There is an irrational number y in $(x - \delta, x + \delta)$ by Exercise 4.12 from the second homework. Although $|y - x| < \delta$, $|f(y) - f(x)| = |0 - x| = |x|$; therefore, setting $\epsilon = |x|$ there exists no $\delta > 0$ where $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. This means by Theorem 17.2 that f is not continuous at x.

x is irrational: Let $\delta \in (0, .5|x|)$ (we can do this because x is nonzero). There is a rational number y in $(x - \delta, x + \delta)$ by Exercise 4.11 from the second homework. Although $|y - x| < \delta$, $|f(y) - f(x)| = |y - 0| = |y| > .5|x|$ (by the triangle inequality); therefore, setting $\epsilon = .5|x|$ there exists no $\delta > 0$ where $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. This means by Theorem 17.2 that f is not continuous at x.

Therefore, f is discontinuous at every nonzero real x but contiuous at 0.

17.14. We consider two cases: x is rational and x is irrational.

x is rational: Let $\delta > 0$. There is an irrational number y in $(x - \delta, x + \delta)$ by Exercise 4.12 from the second homework. Although $|y - x| < \delta$, $|f(y) - f(x)| = |0 - f(x)| = f(x) > 0$; therefore, setting $\epsilon = f(x)$ there exists no $\delta > 0$ where $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. This means by Theorem 17.2 that f is not continuous at x.

x is irrational: Let $\epsilon > 0$ and N be a positive integer. greater than ϵ^{-1}. There are only finitely many rational numbers in $(x - 1, x + 1)$ with denominator less than N (at most $2k + 1$ have denominator k; note that our interval contains at least one integer because the smallest integer greater than $x - 1$ is at most equal to x); let δ be the minimal distance from x to such a number. If $|y - x| < \delta$ then $|f(y) - f(x)| < N^{-1} < \epsilon$ so we indeed have continuity at x.

Therefore, we indeed know that f is continuous at each point of **R**
Q (the irrationals) and discontinuous at each point of **Q** (the rationals).