Problems graded: 11.9 (10 points), 12.4 (10 points), 12.12 (15 points)

The other problems were graded on completion to be worth five points each (12.3, which was longer, was worth 15); as this made the assignment worth 80 points, all scores were multiplied by 1.25 and rounded up to the next integer to make the assignment scores out of 100.

11.6. Referring to our initial sequence as $\{x_n\}$, we can refer to a subsequence as $\{x_{n_k}\}$ (indexed by k) where $\sigma : \mathbf{N} \to \mathbf{N}$ is a selection function and $\sigma(k) = n_k$ for each $n \in \mathbf{N}$. We can denote $y_k = x_{n_k}$ for each positive integer k to simplify notation; our subsequence is $\{y_k\}$. The object we care about is a subsequence of a subsequence, i.e. a subsequence of $\{y_k\}$. However, it can be written as $\{y_{k_j}\}$ (indexed by j) where $\tau : \mathbf{N} \to \mathbf{N}$ is a selection function and $\tau(j) = k_j$ for each $j \in \mathbf{N}$. To show that this new subsequence is a subsequence of the original sequence, we consider the function $\sigma \circ \tau : \mathbf{N} \to \mathbf{N}$; being the composition of two selection functions, it is itself a selection function. We note that

$$x_{\sigma\circ\tau(j)} = x_{\sigma(k_j)} = x_{n_{k_j}} = y_{k_j}$$

which is exactly the jth term of our desired sequence; therefore, a subsequence of a subsequence is indeed a subsequence of the originial sequence.

NOTE: The order in which the selection functions are performed is the REVERSE of the order in which the subsequences are applied; when this type of order-reversing behavior occurs in mathematics, it is called "contravariant".

11.7. We select our subsequence by the following steps.

Step 1: Let n_1 be the first positive integer such that $r_{n_1} > 1$ (we can do this because the rationals are unbounded from above).

Step $k, k > 1$ (assuming step $k-1$ is complete): Let n_k be the first positive integer greater than n_{k-1} such that r_{n_k} is greater than the maximum of max r_i and k (where the maximum is over $i \leq n_{k-1}$); once again we can do this as the rationals are unbounded from above.

Once we have completed this procedure we have a subsequence $\{r_{n_k}\}$ with $r_{n_k} > k$ for each positive integer k and therefore $\lim_{k\to\infty} r_{n_k} = +\infty$.

11.9. (a) Suppose $\{x_n\}$ is a sequence in $[a, b]$ and converges to a limit L outside of $[a, b]$. Either $L < a$ (in which case, for each n, $|x_n - L| = x_n - L \geq a - L$) or $L > b$ (in which case for each n, $|x_n - L| = L - x_n \geq L - b$); in either case, $|x_n - L|$ is prevented from converging to zero (take $\epsilon = a - L$ or $L - b$ respectively) so any limit of a sequence in $[a, b]$ lies in $[a, b]$ implying that $[a, b]$ is closed.

(b) By Theorem 11.8, to show that no such set (with $(0, 1)$ as set of sequential limits exists), it suffices to show that $(0, 1)$ is not a closed set, i.e. we need to construct a sequence $\{s_n\}$ in $(0, 1)$ which converges to a point outside of $(0, 1)$. There are many ways to do this; one of the easiest is to set $s_n = \frac{1}{n+1}$ for each n (and note $0 < s_n \leq \frac{1}{2} < 1$ for each n), which clearly converges to the point 0, outside of $(0, 1)$. Therefore, the answer to this part must be NO.

12.1. Let S_1 be the limit inferior of the s_n. For each $\epsilon > 0$ there exists $N_1 > 0$ such that $s_n > S_1 - \epsilon$ for $n > N_1$. Letting N_2 be the maximum of N_1 and N_0 we have that for $n \geq N_2$, $t_n \geq s_n > S_1 - \epsilon$ so $\inf_{m>n} t_m \geq S_1 - \epsilon$. This tells us that lim inf $s_n - \epsilon \leq$ lim inf t_n. As ϵ was arbitrary we indeed have lim inf $s_n \leq$ lim inf t_n.

Let T_1 be the limit superior of the t_n. For each $\epsilon > 0$ there exists $M_1 > 0$ such that $t_n < T_1 + \epsilon$ for $n > M_1$. Letting M_2 be the maximum of M_1 and N_0 we have that for $n \geq M_2$, $s_n \leq t_n < T_1 + \epsilon$ so $\sup_{m>n} s_m \leq T_1 + \epsilon$. This tells us that lim sup $s_n \leq$ lim sup $t_n + \epsilon$. As ϵ was arbitrary we indeed have lim sup $s_n \leq$ lim sup t_n.

12.2. "if": If lim $s_n = 0$ then lim $|s_n| = 0$ (by exercise 8.6a from the second homework), which implies that lim sup $|s_n| = 0$ (by Theorem 11.7).

"only if": If lim sup $|s_n| = 0$ then because $|s_n| \geq 0$ for each n, we can use the preceding exercise to conclude that lim inf $|s_n| = 0$. Theorem 11.7 tells us that as the set of sequential limits of $|s_n|$, lim $|s_n| = 0$ so (by Exercise 8.6a), lim $s_n = 0$.

12.3. (a) As $\{s_n\}$ repeats the cycle $\{0, 1, 2, 1\}$ infinitely often, it has infinitely many zeroes, ones, and twos and no other elements so its subsequential limit set is $\{0, 1, 2\}$ meaning that (by Theorem 11.7) lim inf $s_n = 0$.

As $\{t_n\}$ repeats the cycle $\{2, 1, 1, 0\}$ often, it has infinitely many zeroes, ones, and twos and no other elements so its subsequential limit set is $\{0, 1, 2\}$ meaning that (by Theorem 11.7) lim inf $t_n = 0$.

Therefore, lim inf $s_n +$ lim inf $t_n = 0 + 0 = 0$.

(b) As $\{s_n + t_n\}$ repeats the cycle $\{2, 2, 3, 1\}$ infinitely often, it has infinitely many ones, and twos, and threes and no other elements so its subsequential limit set is $\{1, 2, 3\}$ meaning that (by Theorem 11.7) lim inf $(s_n + t_n) = 1$.

(c) In part (a) we had already calculated lim inf s_n to be zero; in this part we had also determined that the subsequential limit set of $\{t_n\}$ is $\{0, 1, 2\}$ so (again by Theorem 11.7) lim sup $t_n = 2$.

Therefore, lim inf $s_n +$ lim sup $t_n = 0 + 2 = 2$.

(d) In part (b) we had already calculated the subsequential limit set of $\{s_n + t_n\}$ to be $\{1, 2, 3\}$ meaning that (by Theorem 11.7) lim sup $(s_n + t_n) = 3$.

(e) In part (a) we had already determined the subsequential limit set of $\{s_n\}$ is $\{0, 1, 2\}$ so (by Theorem 11.7), lim sup $s_n = 2$; in part (c) we had computed that lim sup $t_n = 2$.

Therefore, lim sup $s_n +$ lim sup $t_n = 2 + 2 = 4$.

(f) As $\{s_n t_n\}$ repeats the cycle $\{0, 1, 2, 0\}$ infinitely often, it has infinitely many zeroes, ones, and twos and no other elements so its subsequential limit set is $\{0, 1, 2\}$ meaning that (by Theorem 11.7) lim inf $(s_n t_n) = 0$.

(g) In the preceding part we had already determined the subsequential limit set of $\{s_n t_n\}$ to be $\{0, 1, 2\}$ meaning that (by Theorem 11.7) lim sup $(s_n t_n) = 2$.

12.4. For each positive integer n, N with $n > N$ we note that $s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$; as the RHS is therefore an upper bound for $\{s_n + t_n : n > N\}$ we conclude that

sup $\{s_n + t_n : n > N\} \leq$ sup $\{s_n : n > N\}+$ sup $\{t_n : n > N\}$.

We note that if $\{a_n\}$ and $\{b_n\}$ are sequences converging to a and b respectively such that $a_n \leq b_n$ for each n then $a \leq b$; otherwise there exists $\epsilon > 0$ with $a > b + 2\epsilon$ (and there exists N such that for $n > N$, $a_n > a - \epsilon$ and $b_n < b + \epsilon$ so $a_n > b + \epsilon > b_n$ contradicting $a_n \leq b_n$) so, taking limits of our preceding equation (and using additivity of limits from Section 9),

lim sup $\{s_n + t_n : n > N\} \leq$ lim sup $\{s_n : n > N\}+$ lim sup $\{t_n : n > N\}$. (All limits involved exist and are finite because of the boundedness of $\{s_n\}$ and $\{t_n\}$).

12.8. For each positive integer n, N with $n > N$ we note that $s_n * t_n \leq \sup\{s_n : n > N\} * \sup\{t_n : n > N\}$ (as all numbers involved are nonnegative); as the RHS is therefore an upper bound for $\{s_n * t_n : n > N\}$ we conclude that

sup $\{s_n * t_n : n > N\} \leq$ sup $\{s_n : n > N\}*$ sup $\{t_n : n > N\}$.

We note that if $\{a_n\}$ and $\{b_n\}$ are sequences converging to a and b respectively such that $a_n \leq b_n$ for each n then $a \leq b$ (we proved this as part of problem 12.4) so, taking limits of our preceding equation (and using multiplicativity of limits from Section 9),

lim sup $\{s_n * t_n : n > N\} \leq$ lim sup $\{s_n : n > N\}*$ lim sup $\{t_n : n > N\}$. (All limits involved exist and are finite because of the boundedness of $\{s_n\}$ and $\{t_n\}$).

12.10. For both directions we prove the contrapositive.

"only if": Suppose that lim sup $|s_n|$ is infinite. Then, for each positive integer k there exists M such that the supremum of the $|s_n|$ over $n > M$ is greater than $k + 1$ (In fact this supremum is infinite, but I'm just using the direct definition of lim sup here). This means there exists some $n > M$ with $|s_n| > k$; as we can do this for each k, $|s_n|$ is arbitrarily large so $\{s_n\}$ is unbounded.

"if": Suppose lim sup $|s_n| < \infty$; call this limit S. There exists M such that the supremum of $|s_n|$ over $n > M$ is bounded above by $S + 1$; letting Z be the maximum of $|s_1|, \ldots, |s_n|$, and $S + 1$ (a finite maximum of finite numbers is finite) we have that $|s_k| \leq Z$ for all k so $\{s_n\}$ is bounded.

12.12. (a) We prove the inequalities from right to left.

lim sup $\sigma_n \leq$ lim sup s_n: Suppose $M > N$. If $k > M$ then

$$\sigma_k \leq \frac{s_1 + \ldots + s_N}{k} + \frac{\sup\{s_n : n > N\}(k - N)}{k}$$

(grouping together the first N terms and the last $k - N$)

$$\leq \frac{s_1 + \ldots + s_N}{M} + \sup\{s_n : n > N\}$$

(as the s_N are nonnegative and $n > M$) which implies that

$$\sup\{\sigma_n : n > M\} \leq \frac{s_1 + \ldots + s_N}{M} + \sup\{s_n : n > N\}.$$

Fixing N and letting M go to infinity (we can do this as $N > M$) tells us

$$\text{lim sup}\{\sigma_n\} \leq \sup\{s_n : n > N\}$$

(the first term on the RHS vanished as M went to infinity). Now, letting N go to infinity indeed tells us that

$$\text{lim sup } \sigma_n \leq \text{lim sup } s_n$$

as desired.

lim inf $\sigma_n \leq$ lim sup σ_n: Trivial (for example, by Theorem 11.7).

lim inf $s_n \leq$ lim inf σ_n: Suppose $M > N$. If $k > M$ then

$$\sigma_n \geq \frac{s_1 + \ldots + s_N}{k} + \frac{\inf\{s_n : n > N\}(k - N)}{k}$$

(grouping together the first N terms and the last $k - N$); as the right hand side converges to $\inf s_n : n > N$ as k goes to infinity (the first term becomes arbitrarily small and $\frac{k-N}{k}$ converges to 1) we conclude

$$\text{lim inf } \sigma_n \geq \text{lim inf } s_n$$

as desired (using Exercise 12.1 to compare the lim infs of the LHS and RHS and noting that if the lim inf of $\{a_n\}$ is a and the limit of $\{b_n\}$ is b then the lim inf of $\{a_n + b_n\}$ is $a + b$).

(b) If lim s_n exists then (by Theorem 11.7), the lim inf of s_n is equal to the lim sup of the s_n so all four limits in (a) are equal; in particular, lim inf $\sigma_n =$ lim sup σ_n so (Theorem 11.7 again) we indeed have that lim σ_n exists and is equal to lim s_n (all four limits in (a) are equal).