Homework 3

NOTE: This was a half-length assignment, so only two problems were graded: 8.8 (worth 10 points) and 9.6 (worth 15 points). The other problems were graded for completion (worth 5 points each excpt for 9.1, which was worth 10) to bring the total score to 50 points. Scores were than doubled to make this assignment worth 100 points like the others.

8.8. Before beginning, we prove the following lemma.

Lemma: Suppose A and B are strictly positive numbers and $\{c_n\}$ is a sequence of strictly positive numbers which converges to C. Then $\frac{A}{B+c_n}$ converges to $\frac{A}{B+C}$.

Proof: We note that

$$|\frac{A}{B+c_n} - \frac{A}{B+C}| = |\frac{A(B+C) - A(B+c_n)}{(B+c_n)(B+C)}| \leq A\frac{|C-c_n|}{B(B+C)};$$

it suffices to show that this last expression converges to zero. Fixing $\epsilon > 0$ we can pick N such that for $n > N$, $|C - c_n| < \epsilon\frac{B(B+C)}{A}$ (as $c_n \to C$); this tells us that $A\frac{|C-c_n|}{B(B+C)} < \epsilon\frac{A}{B(B+C)}\frac{B(B+C)}{A} = \epsilon$ so our sequence is within ϵ of its desired limit; letting ϵ go to zero proves the lemma.

Armed with this lemma, we now establish our limits.

(a) $|\sqrt{n^2+1} - n| = \frac{1}{|\sqrt{n^2+1}+n|}$ (multiplying numerator and denominator by $\sqrt{n^2+1}+n$) $\leq \frac{1}{n+n}$ (as $\sqrt{}$ is an increasing function and $1/x < 1/y$ if $x > y > 0$) $= \frac{1}{2n}$; picking $\epsilon > 0$ we therefore have that if $n > \frac{1}{2\epsilon}$ then $|(\sqrt{n^2+1}-n)-0| < \epsilon$ showing that the limit of $\sqrt{n^2+1} - n$ is indeed 0.

(b) $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}$ (multiplying numerator and denominator by $\sqrt{n^2+n}+n$)
$= \frac{1}{1+\sqrt{1+n^{-1}}}$ (dividing numerator and denominator by n). Because $1+n^{-1}$ clearly converges to 1 (if $\epsilon > 0$ then for $N > \epsilon^{-1}$ the distance between $1+n^{-1}$ and 1 is less than ϵ) we have that $\sqrt{1+n^{-1}}$ converges to 1 by Example 5 of this section. Therefore, the lemma shows that $\sqrt{n^2+n} - n$ converges to $\frac{1}{1+1} = \frac{1}{2}$.

(c) $\sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n}+2n}$ (multiplying numerator and denominator by $\sqrt{4n^2+n}+2n$)
$= \frac{1}{2+\sqrt{4+n^{-1}}}$ (dividing numerator and denominator by n). Because $4+n^{-1}$ clearly converges to 4 (if $\epsilon > 0$ then for $N > \epsilon^{-1}$ the distance between $4+n^{-1}$ and 4 is less than ϵ) we have that $\sqrt{4+n^{-1}}$ converges to $\sqrt{4} = 2$ by Example 5 of this section. Therefore, the lemma shows that $\sqrt{4n^2+n} - 2n$ converges to $\frac{1}{2+2} = \frac{1}{4}$.

8.10. Let S refer to the limit of s_n. Because $S > a$ we have that $\frac{S-a}{2} > 0$ so there exists N such that for $n > N$, $|s_n - S| < \frac{S-a}{2}$. For such n,

$$s_n > S - \frac{S-a}{2} = a + (S-a) - \frac{S-a}{2} = a + \frac{S-a}{2} > a$$

as desired.

NOTE: Results will often be quoted just by number; "by 9.7a", or "by 9.3" shall refer to the corresponding theorem, lemma, or part of Basic Example 9.7.

9.1. (a) Because the limit of $1/n$ is zero (by 9.7a) and $(n+1)/n = 1 + \frac{1}{n}$, $\lim[(n+1)/n] = \lim[1] + \lim[1/n]$ (by 9.3; as both those limits exist) $= 1 + 0 = 1$.

(b) $\lim[(3n+7)/(6n+5)] = \lim[(3+7/n)/(6+5/n)]$ (dividing numerator and denominator by n) $= \lim[3+7/n]/\lim[6+5/n]$ (by 9.6 provided both these limits exist).

We quickly note that for k constant, $\lim[k/n] = k\lim[1/n]$ (by 9.2) $= k * 0$ (by 9.7a) $= 0$ so

$\lim[3+7/n] = \lim[3] + \lim[7/n]$ (by 9.3 as both these limits exist) $= 3+0 = 3$

and $\lim[6+5/n] = \lim[6] + \lim[5/n]$ (by 9.3 as both these limits exist) $= 6+0 = 6$ so we indeed have $\lim[(3n+7)/(6n+5)] = \frac{3}{6} = \frac{1}{2}$.

(c) $\lim[(17n^5+73n^4-18n^2+3)/(23n^5+13n^3)] = \lim[(17+73n^{-1}-18n^{-3}+3n^{-5})/(23+13n^{-2})]$ (dividing both numerator and denominator by n^5) $= \lim[(17+73n^{-1}-18n^{-3}+3n^{-5})]/\lim[(23+13n^{-2})]$ (by 9.6, providing both these limits exist).

We quickly note that for k constant and $p > 0$, $\lim[k/n^p] = k\lim[1/n^p]$ (by 9.2) $= k * 0$ (by 9.7a) $= 0$ so $\lim[17+73n^{-1}-18n^{-3}+3n^{-5}] = \lim[17] + \lim[73n^{-1}] + \lim[-18n^{-3}] + \lim[3n^{-5}]$ (by 9.3) $= 17+0+0+0 = 17$ and $\lim[(23+13n^{-2})] = \lim[23] + \lim[13n^{-2}] = 23+0 = 23$ and therefore we can conclude

$\lim[(17n^5+73n^4-18n^2+3)/(23n^5+13n^3)] = 17/23$.

9.3. We note that $\lim[a_n^2] = \lim[a_n]\lim[a_n]$ (by 9.4) $= a * a = a^2$ so

$\lim[a_n^3] = \lim[a_n^2]\lim[a_n]$ (by 9.4) $= a^2 * a = a^3$.

Also, $\lim[4a_n] = 4\lim[a_n]$ (by 9.2) $= 4a$ so

$$\lim[a_n^3 + 4a_n] = \lim[a_n^3] + \lim[4a_n]$$

(by 9.3) $= a^3 + 4a$.

Similarly, we note that $\lim[b_n^2] = \lim[b_n]\lim[b_n]$ (by 9.4) $= b * b = b^2$ so

$$\lim[b_n^2 + 1] = \lim[b_n^2] + \lim[1]$$

(by 9.3) $= b^2 + 1$.

We finally can conclude that $\lim[s_n] = \lim[a_n^3+4a_n]/\lim[b_n^2+1]$ (by 9.6; note $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$) $= (a^3+4a)/(b^2+1)$ as desired.

9.6.(a) Suppose $a = \lim[x_n]$. Then, $a^2 = \lim[x_n^2]$ (a limit I have already proved in Problem 3 of this section). Therefore, (using 9.2), $3a^2 = \lim[3x_n^2] = \lim[x_{n+1}]$. However, $\{x_{n+1}\}$ converges to a as well (if $\epsilon > 0$ there exists N such that $|x_n - a| < \epsilon$ for $n > N$; this works just as well for x_{n+1}) so $a = 3a^2$, i.e. $a(3a-1) = 0$. This means that either $3a - 1 = 0$ ($a = 1/3$) or $a = 0$ by the field properties of the real numers.

(b) $\lim x_n = \infty$ because $x_n \geq n$ for each positive integer n (this can be proven by induction; if $x_n \geq n$ then $x_{n+1} \geq 3n^2 \geq 3n \geq n+1$) so whenever M is a number we can pick a positive integer $N > M$; for $n > N$, $x_n > N > M$.

(c) Part b does not contradict part a because part a said that if the sequence had a FINITE limit it must equal 1/3 or 0 and b says that it has an INFINITE limit.

9.15. Letting N be a positive integer greater than $|a|$ we let $C = \frac{a^N}{N!}$.

Then we have that if $n = N + r$,

$$|a^n/n!| \leq |a^N/N!|(\frac{|a|}{N})^r.$$

By 9.7b we note that $(\frac{a}{N})^r$ converges to 0 as r goes to infinity. Therefore, we let $\epsilon > 0$. There exists M such that for $m > M$, $(\frac{|a|}{N})^m < \epsilon N!/|a|^N$ so, for $n > N + M$, we have

$$|a^n/n!| < \epsilon|a^N/N!|N!/|a|^N = \epsilon;$$

as we can do this for each $\epsilon > 0$, we conclude that $|a^n/n!|$ approaches 0 as n approaches ∞ and therefore so does $a^n/n!$.