Homework 2

Problems graded: 4.12 (10 points), 5.2 (10 points), 7.3 (15 points), 8.6 (15 points). The other problems were graded for completion to be worth five points each (and another five points were given to all students to make the total point value 100).

4.10. Suppose $a > 0$. By the Archimedean property there exists a natural m_1 with $m_1 a > 1$; dividing by m_1, $a > 1/m_1$. However (by 3.2.vi), $a^{-1} > 0$; by the Archimedean property there exists a natural m_2 with $m_2 a^{-1} > 1$, i.e. $m_2 > a$. Letting n be the maximum of m_1 and m_2, $n \in \mathbf{N}$ with $a > 1/m_1 \geq 1/n$ (by 3.2.vii as $n \geq m_1$) and $n \geq m_2 > a$, i.e. $1/n < a < n$.

4.11. Suppose $a < b$ are real. We construct an infinite increasing sequence of rationals $\{q_i\}$ in (a, b) by the density of $\mathbf{Q}$ as follows:

Step 1: Let q_1 be a rational in (a, b).

Step k (assuming step $1, \ldots, k-1$ are done): Let q_k be a rational in (q_{k-1}, b).

After completing this procedure, we have that the q_i are distinct rationals with $a < q_1 < \ldots < q_i < \ldots < b$.

4.12. We begin by noting that if r is rational then $r + \sqrt{2}$ is irrational (as otherwise $(r + \sqrt{2}) - r = \sqrt{2}$, being the difference of two rationals, is rational; note $\mathbf{Q}$ forms a field).

Now, if $a < b$, $a - \sqrt{2} < b - \sqrt{2}$ so there exists a rational y with $a - \sqrt{2} < y < b - \sqrt{2}$. Adding $\sqrt{2}$, we have that $y + \sqrt{2}$ is irrational with $a < y + \sqrt{2} < b$.

4.14. (a) If α is the supremum (which is an upper bound) of A and β is the supremum (which is an upper bound) of B then for each $s \in S$ there exist $a \in A, b \in B$ with $a + b = s$ so $s = a + b \leq \alpha + \beta$. ($\alpha$ and β are finite as A, B are nonempty bounded). This tells us that sup $S \leq$ sup $A+$ sup B.

For equality we let $\epsilon > 0$; there exist $a', b' \in A, B$ respectively with $\alpha - a'$, $\beta - b'$ both less than $\epsilon/2$. Therefore, $a' + b'$ is an element of S with $a' + b' > \alpha + \beta - \epsilon$. Letting ϵ go to zero we have sup $S \geq$ sup $A+$ sup B; this indeed gives us sup $S =$ sup $A+$ sup B.

(b) If α is the infimum (which is a lower bound) of A and β is the infimum (which is a lower bound) of B then for each $s \in S$ there exist $a \in A, b \in B$ with $a + b = s$ so $s = a + b \geq \alpha + \beta$. ($\alpha$ and β are finite as A, B are nonempty bounded). This tells us that inf $S \geq$ inf $A+$ inf B.

For equality we let $\epsilon > 0$; there exist $a', b' \in A, B$ respectively with $a' - \alpha$, $b' - \beta$ both less than $\epsilon/2$. Therefore, $a' + b'$ is an element of S with $a' + b' < \alpha + \beta + \epsilon$. Letting ϵ go to zero we have inf $S \leq$ inf $A+$ inf B; this indeed gives us inf $S =$ inf $A+$ inf B.

5.2 (a) As $\{x \in \mathbf{R} : x < 0\}$ contains arbitrarily small numbers (for each real r, the minimum of r and -1 is in the set), its infimum is $-\infty$. Now, 0 is clearly an upper bound for this set; to show that it is the least upper bound, note that if $x < 0$ then $x/2 > x$ is in the set so 0 is the supremum.

(b) To find the infinmum and supremum of $\{x \in \mathbf{R} : x^3 \leq 8\}$ we note that if $x \leq 0$ then $x^2 \geq 0$ so $x^3 = x^2 * x \leq 0 < 8$. If $x \in (0, 2)$ then $x^3 = x * x * x < 2 * x * x < 2 * 2 * x < 2 * 2 * 2 = 8$ so x lies in our set. Further, if $x = 2$, $x^3 = 8$ so x is still in the set. Finally, if $x > 2$, $x^3 = x * x * x > 2 * x * x > 2 * 2 * x > 2 * 2 * 2 = 8$ so x is NOT in our set. Therefore, our set can be written as $(-\infty, 2]$. Clearly the set is unbounded from below (if r is real, the minimum of r and 0 lies in our set) so its infimum is negative infinity. For its supremum note that 2 is clearly an upper bound; as 2 lies in the set (as the maximum element) it is the supremum as well.

5.6. As $S \subset T$, any lower bound of T (such as the infimum of T) is also a lower bound of S (and therefore bounded above by its greatest lower bound) so $\inf T \leq \inf S$. Letting $s \in S$, clearly $\inf S \leq s \leq \sup S$ so $\inf S \leq \sup S$. Finally, any upper bound of T (such as the supremum of T) is also an upper bound of S (and therefore bounded below by its least upper bound) so $\sup S \leq \sup T$. Putting it all together, we have $\inf T \leq \inf S \leq \sup S \leq \sup T$. 4.7 (which is 5.6 when S, T are bounded) is a special case which (as we recall from HW 1) has the same proof.

7.2. (a) $1/(3n+1)$ clearly converges to 0 as n goes to infinity (note that the denominator gets arbitrarily large).

(b) $(3n+1)/(4n-1) = (3+1/n)/(4-1/n)$ which converges to $3/4$ as n goes to infinity (note the numerator approaches 3 and the denominator approaches 4).

(c) $n/3^n$ converges to 0 as n goes to infinity (the denominator gets multiplied by 3 at each stage; the numerator gets multiplied by 2 or less.)

(d) $\sin(n\pi/4)$ does not converge (the sequence is

$$\{\sqrt{2}/2, 1, \sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0\}$$

repeated infinitely many times).

7.3. (e) $73 + (-1)^n$ alternates between 72 and 74 and therefore does not converge.

(j) $(7n^3 + 8n)/(2n^3 - 31) = (7 + 8n^{-2})/(2 - 31n^{-3})$ which converges to $7/2$ as the numerator converges to 7 and the denominator converges to 2.

(k) $(9n^2 - 18)/(6n + 18) = n(9 - 18n^{-2})/(6 + 18n^{-1})$ which diverges to ∞ (note that the numerator of this new fraction converges to 9 and the denominator converges to 6).

(o) $(1/n)\sin n$ converges to 0 as $|\sin n| \leq 1$.

(p) $(2^{n+1} + 5)/(2^n - 7) = (2 + 5 * 2^{-n})/(1 - 7 * 2^{-n})$ converges to 2(note that the numerator converges to 2 and the denominator converges to 1).

7.5. (a) We note that

$$s_n = \sqrt{n^2+1} - n = (\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)/(\sqrt{n^2+1} + n) = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

(as $(a+b)(a-b) = a^2 - b^2$) $= \frac{1}{\sqrt{n^2+1}+n}$ which approaches 0 as n approaches ∞ because the denominator gets arbitrarily large.

(b) We note that

$$\sqrt{n^2+n} - n = (\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)/(\sqrt{n^2+n} + n) = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$$

$= \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{1+\sqrt{1+n^{-1}}}$ which approaches $1/2$ as n approaches ∞ because the denominator approaches 2 as n approaches ∞.

(c) We note that

$$\sqrt{4n^2+n} - 2n = (\sqrt{4n^2+n} - 2n)(\sqrt{4n^2+n} + 2n)/(\sqrt{4n^2+n} + 2n)$$

$$= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n}$$

$= \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{2+\sqrt{4+n^{-1}}}$ which approaches $1/4$ as n approaches ∞ because the denominator approaches 4 as n approaches ∞.

8.2. (a) We note that $n/(n^2+1) = 1/(n+1/n)$ so (as the denominator gets arbitrarily large) we claim the limit is zero. Formally, let $\epsilon > 0$; if $N > 1/\epsilon$ then for each $n > N$, $a_n = 1/(n+1/n) < 1/n < 1/N < \epsilon$. As a_n is clearly positive for each n we have that for $n > N$, $|a_n - 0| < \epsilon$; as we can do this for all $\epsilon > 0$ the limit is indeed 0.

(b) We note that $(7n-19)/(3n+7) = (7-19/n)/(3+7/n)$; as the numerator approaches 7 and the denominator approaches 3 we claim the limit is 7/3. Formally, we let $\epsilon > 0$ and note that

$$|(7n-19)/(3n+7) - 7/3| = |(21n-57)/(9n+21) - (21n+49)/(9n+21)|$$

$$= |-106/(9n+21)| = 106/(9n+21) < 20/n.$$

Letting $N = 20/\epsilon$, we note that for $n > N, 20/n < \epsilon$ so $|b_n - 7/3| < \epsilon$; as we can do this for all $\epsilon > 0$ the limit is indeed 7/3.

(c) We note that $(4n+3)/(7n-5) = (4+3/n)/(7-5/n)$; as the numerator approaches 4 and the denominator approaches 7 we claim the limit is 4/7. Formally, we let $\epsilon > 0$ and note that

$$|(4n+3)/(7n-5) - 4/7| = |(28n+21)/(49n-35) - (28n-20)/(49n-35)|$$

$$= |41/(49n-35)| = 41/(49n+35) < 1/n.$$

Letting $N = 1/\epsilon$, we note that for $n > N, 1/n < \epsilon$ so $|c_n - 4/7| < \epsilon$; as we can do this for all $\epsilon > 0$ the limit is indeed 4/7.

8.3. Let $\epsilon > 0$. As $\epsilon^2 > 0$, we note that (as s_n is a sequence of nonnegative reals with limit 0) there exists N such that for $n > N$, $s_n < \epsilon^2$. However, this means that for such n, $\sqrt{s_n} < \epsilon$ (if $\sqrt{y} \geq \epsilon$ then $y = \sqrt{y} * \sqrt{y} \geq \sqrt{y} * \epsilon \geq \epsilon * \epsilon = \epsilon^2$) so for each $\epsilon > 0$ our N satisfies $\sqrt{s_n} < \epsilon$ whenever $n > N$; this is the definition of $\lim \sqrt{s_n} = 0$.

8.6. (a) If $\lim s_n = 0$ then for each $\epsilon > 0$ there exists N such that $|s_n - 0| < \epsilon$, i.e. $|s_n| < \epsilon$. However, as $|s_n| \geq 0$ this tells us that $||s_n| - 0| < \epsilon$ as well for such n so $\lim |s_n| = 0$.

Similarly, if $\lim |s_n| = 0$ then for each $\epsilon > 0$ there exists N such that $||s_n| - 0| < \epsilon$, i.e. $|s_n| < \epsilon$ (the absolute value of the nonnegative number $|s_n|$ is itself). However, this says $|s_n - 0| < \epsilon$ for such n so $\lim s_n = 0$.

(b) If $s_n = (-1)^n$ then $|s_n| = 1$ for each n so $\lim s_n = 1$ (for each $\epsilon > 0$, $|s_n - 1| = 0 < \epsilon$ for all n) but $\lim s_n$ does not exist (this is Example 4).