Graded problems: 1.4 (15 points), 2.3 (10 points), 3.6 (10 points), 4.8 (15 points); the other problems were marked for completion only and worth five points each (except that 1.12 and 4.1, which were somewhat longer, were worth ten points).

1.3. To proceed by induction, we prove the result for $n = 1$ and then, assuming the result for n, derive the result for $n + 1$.

$n = 1 : 1^3 = 1 = 1^2$.

n implies $n + 1$: Suppose the result holds for a given natural n,

i.e. $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$. Then

$$1^3 + 2^3 + \ldots + n^3 + (n + 1)^3 = (1 + 2 + \ldots + n)^2 + (n + 1)^3$$

$$= (1 + 2 + \ldots + n)^2 + [(n + 1)^3 - (n + 1)^2] + (n + 1)^2$$

$$= (1 + 2 + \ldots + n)^2 + (n + 1)^2[n + 1 - 1] + (n + 1)^2$$

$$= (1 + 2 + \ldots + n)^2 + 2(n + 1)n(n + 1)/2 + (n + 1)^2$$

$$= (1 + 2 + \ldots + n)^2 + 2(n + 1)(1 + 2 + \ldots + n) + (n + 1)^2$$

(by Example 1)

$$= (1 + 2 + \ldots + (n + 1))^2$$

(by the rule $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 1 + 2 + \ldots + n$ and $b = n + 1$) showing the result for $n + 1$, so induction gives us exactly what we need.

1.4(a): $n = 1$: The expression is simply 1.

$n = 2$: The expression is $1 + 3 = 4$.

$n = 3$: The expression is $1 + 3 + 5 = 9$.

$n = 4$: The expression is $1 + 3 + 5 + 7 = 16$.

For each of these, we note $1 + 3 + \ldots + (2n - 1) = n^2$, so this is our proposed formula.

(b): We already have the $n = 1$ result from part (a); therefore, it suffices to show the formula for $n+1$ given the formula for some n. As $2(n+1)-1 = 2n-1$, we have (assuming the result for n)

$$1 + 3 + \ldots + (2(n + 1) - 1) = 1 + 3 + \ldots + (2n - 1) + (2n + 1)$$

$$= n^2 + 2n + 1 = (n + 1)^2$$

as desired, giving us the formula.

1.10. We could prove this formula by induction; however, there is a quicker way.

We observe that

$$(2n+1)+(2n+3)+\ldots+(4n-1) = (1+3+\ldots+(4n-1))-(1+3+\ldots+(2n-1))$$

$$= (1+3+\ldots+2(2n)-1)-(1+3+\ldots+(2n-1))$$

$$= (2n)^2 - n^2$$

(by Problem 1.4's formula)

$$= 4n^2 - n^2 = 3n^2$$

as desired.

1.12(a). $n = 1$: The left hand side is $(a+b)^1 = a+b$; the right hand side is $a^1 + b^1 = a + b$ so the theorem holds.

$n = 2$: The left hand side is $(a+b)^2 = a^2 + 2ab + b^2$; the right hand side is $a^2 + 2ab + b^2$ so the theorem holds.

$n = 3$: The left hand side is

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b)$$

$$= a^3 + 2ab^2 + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

the right hand side is $a^3 + 3a^2b + 1/2 * 3 * 2 * ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$ so the theorem holds.

(b). NOTE: For the rest of the problem, we use the notation nCk to denote the binomial coefficient $\frac{n!}{k!(n-k)!}$.

We note that

$$nCk + nC(k-1) = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n! * k}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = (n+1)Ck$$

as desired.

(c). We already know the binomial theorem for $n = 1$ by part (a); to complete the induction, we assume the theorem for n and show it for $n + 1$.

Given the result for some n we note that

$$(a + b)^{n+1} = (a + b)^n(a + b) = (nC0a^n + \ldots + nCnb^n)(a + b).$$

We can write this sum as

$$c_0a^{n+1}b^0 + c_1a^nb^1 + \ldots + c_ka^{n+1-k}b^k + \ldots + c_{n+1}a^0b^{n+1}$$

where $c_0, \ldots, c_{n+1}$ are constants (and we want to show that $c_k = (n+1)Ck$ for $k = 0, \ldots, n + 1$).

For c_0 and c_{n+1} we note that both coefficients are equal to 1 (the c_0 term comes from $a^n * a$ and the c_{n+1} term comes from $b^n * b$) which is $(n + 1)C0 = (n + 1)C(n + 1)$ so we look at c_k for $1 \leq k \leq n$.

To find c_k we note that the $a^{n+1-k}b^k$ term comes from

multiplying $nCka^{n-k}b^k$ by a, multiplying $nC(k - 1)a^{n-k+1}b^{k-1}$ by b, and adding these two products.

We get $nCk+nC(k-1) = (n+1)Ck$ by part (b); therefore, as $c_k = (n+1)Ck$ for each $k = 0, \ldots, n + 1$, we therefore have the binomial theorem for $n + 1$ and therefore for all n by induction.

2.3. Letting $x = (2+\sqrt{2})^{1/2}$, we note that $x^2 = 2+\sqrt{2}$ so $x^2 - 2 = \sqrt{2}$ and $(x^2-2)^2 = 2$, i.e. $x^4 - 4x^2 + 2 = 0$.

By the Rational Zeroes Theorem, if x were rational, it would need to be expressible in the form $\frac{p}{q}$ where p divides 2 (i.e. is ± 1 or ± 2) and q divides 1 (i.e. is ± 1), so x would need to be either $1, 2, -1$, or -2. However, plugging 1 into $x^4 - 4x^2 + 2$ yields $1 - 4 + 2 = -1$, plugging in 2 gives $16 - 16 + 2 = 2$, plugging in -1 gives $1 - 4 + 2 = -1$, and plugging in -2 gives $16 - 16 + 2 = 2$. As none of these are zero, none of the four rational candidates can be roots, so x cannot represent a rational number.

3.3 (iv): We note that

$$(-a)(-b) + (-a)(b) = (-a)(-b+b)$$

(by DL)

$$= -a * 0 = 0$$

(by (ii)) so $(-a)(-b) = -[(-a)(b)]$. Because $(-a)(b) = -(ab)$ by (iii) we have $(-a)(-b) = --ab$ which equals ab ($--x = x$ because x is an inverse of $-x$; $-x + x = x + -x = 0$ by commutativity and additive inverses are unique by (i)) as desired.

(v): If $ac = bc$ and $c \neq 0$ then subtractig bc from both sides, we get $ac - bc = 0$, i.e. $ac + (-b)c = 0$ (by (iii)), from which commutativity gives $ca + c(-b) = 0$ so $c(a + -b) = 0$; (vi) gives us that $a + -b = 0$ so $-b$ is an additive inverse for a; as a and b are both additive inverses of $-b$ by commutativity and (i) we have that $a = b$ as desired.

NOTE: It may seem inappropriate to use (vi) in the proof of (v); however, this creates no problem because (v) was not used in the proof of (vi).

3.4. (v) Because $1 = 1^2$, (iv) gives that $0 \leq 1$; as we assumed $0 \neq 1$ we have $0 < 1$.

(vii) If $0 < a < b$ then $0 < b^{-1}$ by (vi); it suffices to show $b^{-1} < a^{-1}$. Assume to the contrary, i.e. that $b^{-1} \geq a^{-1}$. Then, $1 = b * b^{-1} \geq b * a^{-1}$ (by O5 and commutativity) $> a * a^{-1}$ (we have $\geq$ by O5; inequality is strict by Theorem 3.1 because $b \neq a$ and $a^{-1} > 0$ by (vi)) $= 1$ producing a contradiction (as $1 = 1$) so $0 < b^{-1} < a^{-1}$ as desired.

3.6(a). We note that

$$|a+b+c| = |(a+b)+c| \leq |a+b| + |c|$$

(by the triangle inequality) $\leq |a| + |b| + |c|$ (by O4; $|a+b| \leq |a+b|$ by the triangle inequality) as desired.

(b). As always, we start with $n = 1$, which gives us $|a_1|$ on both sides of the inequality (so the inequality reduces to an equality, which clearly holds).

Assuming the result for a given n, we show the result for $n+1$ as follows:

$$|a_1 + \ldots + a_{n+1}| \leq |a_1 + \ldots + a_n| + |a_{n+1}|$$

(by the triangle inequality)

$$\leq |a_1| + \ldots + |a_n| + |a_{n+1}|$$

(by the induction hypothesis and O4) as desired, so our inequality holds for all n as desired.

4.1. NOTE: For this part, a 'correct' answer does not require you to find the supremum; you merely need to show that the three numbers you choose are indeed upper bounds. However, for this part, my solutions will involve finding the supremum; any solution with three numbers at least this big is correct.

(b) $(0,1)$ has supremum 1; 1 is greater than any element in the set, but if $0 < x < 1$ then $(x+1)/2$ is in the set but greater than x.

(d) As $\pi > e$, π is the supremum of $\{\pi, e\}$.

(f) 0 is clearly the supremum of $\{0\}$.

(h) $\bigcup_{n=1}^{\infty}[2n, 2n+1]$ contains all positive integers greater than 1 so it is not bounded above.

(j) For each $n \in \mathbf{N}$, $1 - 1/3^n < 1$ so 1 is an upper bound of $\{1 - 1/3^n\}$. Because $3^n > n$ for each $n \in \mathbf{N}$ (use induction) we have $1 - 1/3^n > 1 - 1/n$; if $x < 1$ then (by the Archimedean property) there exists $m \in \mathbf{N}$ with $(1-x)m > 1$; this yields $x < 1 - 1/n < 1 - 1/3^n$ and therefore x is not an upper bound so 1 is the supremum.

(l) By denseness of $\mathbf{Q}$, 2 is the supremum of $\{r \in \mathbf{Q} : r < 2\}$.

(n) By denseness of $\mathbf{Q}$ (and the fact that $x^2 > y^2$ whenever $x > y > 0$, which can be shown as $x^2 - y^2 = (x+y)(x-y)$) $\sqrt{2}$ is the supremum of $\{r \in \mathbf{Q} : r^2 < 2\}$.

(p) As $3 < \pi < 4$ and $9 < \pi^2 < 10$, the supremum of $\{1, \pi/3, \pi^2, 10\}$ is 10.

(r) By the density of the rationals, $\bigcap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n) = \{1\}$, whose supremum is 1.

(t) By denseness of $\mathbf{Q}$ (and the fact that $x^3 > y^3$ whenever $x > y > 0$, which can be shown as $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$) 2 is the supremum of $\{x \in \mathbf{Q} : x^3 < 8\}$.

(v) Because the cosine function is bounded above by 1, $\{\cos(n\pi/3) : n \in \mathbf{N}\}$ is bounded above by 1. As the set contains 1 (plug in $n = 6$), 1 is indeed the supremum.

4.6. (a) Let $s \in S$ (as S is nonempty). If a is the inf of S and b is its sup then $a \leq s$ (as a is a lower bound of S) and $b \geq s$ (as b is an upper bound of S) so $a \leq b$ by transitivity.

(b) If S has two distinct elements x, y we can suppose by relabeling that $x < y$. As in (a), $a \leq x$ and $y \leq b$ so transitivity now gives $a < b$. Therefore, if $\inf S = \sup S$ then S can only have one element (and single-element sets $\{t\}$ have t as both inf and sup).

4.7. (a) If $S \subset T$ then any lower bound for T is a lower bound for S; in particular, the greatest lower bound for T is a lower bound for S (so it is smaller than or equal to the greatest lower bound for S, i.e. $\inf T \leq \inf S$). We have $\inf S \leq \sup S$ by 4.6.(a); as any upper bound for T is an upper bound for S, the least upper bound for T is an upper bound for S (so it is greater than or equal to the smallest upper bound for S, i.e. $\sup S \leq \sup T$).

(b) Because $S \cup T$ contains both S and T, $\sup(S \cup T)$ is greater than or equal to both $\sup S$ and $\sup T$ (and therefore their maximum) by the preceding part. Now, if X is the maximum of $\sup S$ and $\sup T$ then X, being at least $\sup S$, is an upper bound for S. Being at least $\sup T$, it is an upper bound for T. However, this means X is greater than or equal to every element in S or T (and therefore in $S \cup T$), i.e. X is an upper bound for $S \cup T$ and therefore greater than or equal to its supremum. As we have shown that X is both less than or equal to and greater than or equal to $\sup(S \cup T)$ we have equality as desired.

4.8.(a) Choose an element $s' \in S$ and an element $t' \in T$. By assumption, $s \leq t'$ for $s \in S$ so t' is an upper bound of S; also, $s' \leq t$ for $t \in S$ so s' is a lower bound of T.

(b) Assume to the contrary, i.e. $\inf T < \sup S$. Then by density of the rationals, there exists some rational x with $\inf T < x < \sup S$; x is neither an upper bound for S nor a lower bound for T. Therefore, there exist $s^* \in S, t^* \in T$ such that $t^* < x$ and $x < s^*$; by transitivity, $t^* < s^*$ contradicting our initial assumption so $\sup S \leq \inf T$ as desired.

(c) Answers will vary; for example, if $S = T = \{0\}$ then as $0 \leq 0$ the hypothesis holds and $S \cap T = 0$.

(d) Answers will vary; for example, if $S = \{x : x < 0\}$ and $T = \{x : x > 0\}$, $\sup S = \inf T = 0$ but $S \cap T$ is empty.