Problems graded: 3.3.2 (10 points), 3.4.4 (10 points), 3.6.1 (10 points); all other problems were graded for completeness to be worth five points each (except that each part of 3.3.6 and 3.4.1 was considered as a problem) for a grand total of 75 points for this assignment.

Section 3.3

2. Suppose G is another orientation-reversing isometry; so is H_0^{-1} (as $(H_0^{-1})_* = ((H_0)_*)^{-1}$, sgn $H_0^{-1} = (\text{sgn} H_0)^{-1} = (-1)^{-1} = -1$) so writing $F = (H_0^{-1}G)$ gives that sgn F = sgn H_0^{-1} sgn $G = (-1) * (-1) = 1$ so F is an orientation-preserving matrix; therefore, $G = H_0(H_0^{-1}G) = H_0F$ where F is orientation preserving. However, if $G = H_0K$ for some isometry K, then left-multiplying by H_0 gives $K = (H_0)^{-1}G = F$ so this expansion is unique.

4. If C is a rotation then its characteristic polynomial is of the form $p(\lambda) = \lambda^3 + b_0\lambda^2 + c_0\lambda + d_0 = 0$ where d_0 is the additive inverse of the determinant of C (by linear algebra). However, since $d_0 = -1$, $p(0) = -1 < 0$ and $p(\lambda)$ approaches $+\infty$ with λ so p must have a positive real root corresponding to an eigenvalue of C, i.e. C has a nonzero eigenvector v with $Cv = \lambda v$ for some $\lambda > 0$. However, as C is orthogonal, $< v, v >=< Cv, Cv >=< \lambda v, \lambda v >= \lambda^2 < v, v >$ so $\lambda^2 = 1$ and therefore $\lambda = 1$ so v is a fixed point of C. We now let $e_3 = \frac{v}{||v||}$ and $\{e_1, e_2\}$ be an orthonormal basis of its complement (chosen so that $\{e_1, e_2, e_3\}$ is positively oriented).

As C preserves dot products, $< Ce_1, e_3 >=< Ce_1, Ce_3 >=< e_1, e_3 >= 0$ so $Ce_1 = \alpha e_1 + \beta e_2$ for some α, β with $\alpha^2 + \beta^2 = 1$.

Similarly, $< Ce_2, e_3 >= 0$ and $< Ce_2, Ce_1 >= 0$ so Ce_2 (being a unit vector) is one of the two unit vectors in the plane spanned by e_1 and e_2 orthogonal to Ce_2, i.e. either $-\beta e_1 + \alpha e_2$ or $\beta e_1 - \alpha e_2$. However, in the second case the determinant of the matrix representing C would be $-\alpha^2 - \beta^2 = -1$ so the first case must hold. Therefore, we let θ be the angle with $\cos\theta = \alpha$, $\sin\theta = \beta$.

Consequently, $C(e_1) = \cos\theta e_1 + \sin\theta e_2$, $C(e_2) = -\sin\theta e_1 + \cos\theta e_2$, $C(e_3) = e_3$, and $e_i \bullet e_j = \delta_{ij}$ as desired.

6. (a) If G, H are rotations then GH is orientation-preserving (the sign of GH is equal to $det((GH)_*) = det(G_*H_*) = det(G_*)det(H_*)$, the product of the signs of G and H) and orthogonal (by Exercise 3.1.8 from the last homework) so GH is a rotation.

Clearly I is a rotation (as it is orthogonal and orientation-preserving); if G is a rotation then G^{-1} is orientation-preserving (the sign is equal to $1^{-1} = 1$) and orthogonal (by Exercise 3.1.8 again) so $O^+(3)$ satisfies all of the properties required to be a subgroup of the orthogonal group and therefore is indeed such a subgroup.

(b) If G and H are orientation-preserving then so is GH (copy the argument above for the sign of GH), as are G^{-1} and the identity map so $E^+(3)$ indeed satisfies the requirements for being a subgroup of $E(3)$.

Section 3.4

1. (a) If β is a cylindrical helix (with curvature κ, torsion τ, and $\frac{\kappa}{\tau}$ constant), then $F(\beta)$ has curvature κ and torsion $\epsilon\tau$ (where ϵ is the sign of F) by Theorem 4.2 of this section; as $\frac{\kappa}{\epsilon\tau}$ is itself constant, $F(\beta)$ is a cylindrical helix.

(b) At any point t, $\sigma(t) = \beta'(t)$ so the spherical image of $F \circ \beta$ at the point is $F_*(\beta'(t)) = F_*(\sigma(t)) = C(t)$ by Theorem 2.1; therefore, $F(\beta)$ has spherical image $C(\sigma)$.

4. Note that if γ is a curve, then its arc length is $\int |\gamma'(t)|dt$ so the arc length of $F(\gamma)$ is

$$\int |(F \circ \gamma)'(t)|dt = \int |F_*(\gamma'(t))|dt = \int |\gamma'(t)|dt$$

(because F_* preserves dot products, it preserves absolute value as $|v| = \sqrt{v \bullet v}$), which is the same as the arc lenght of γ.

As the distance between two points p and q is equal to the length of the shortest curve between p and q (by Exercise 2.2.11 from the third homework, which also identifies this shortest curve as a straight line), F preserves distance and therefore is an isometry.

5. Writing $F = TC$ with T a translation and C orthogonal, one notes that

$$\overline{\nabla_V(W)}F(p) = F_*(\nabla_V(W)(p))$$

(by definition of $\bar{V}$)

$$= F_*(\frac{d}{dt}(W(p + tV))|_{t=0})$$

(by definition of covariant derivative)

$$= \frac{d}{dt}(F_*(W(p + tV)))|_{t=0})$$

(as linear maps commute with differential operators)

$$= \frac{d}{dt}(\bar{W} \circ F(p + tV)))|_{t=0})$$

(by definition of $\bar{W}$)

$$= \frac{d}{dt}(\bar{W} \circ (F(p) + tC(V))|_{t=0})$$

(as $F(x + y) = F(x) + C(y)$)

$$= \nabla_{C \circ V(p)}(\bar{W})(F(p))$$

(by definition of covariant derivative)

$$= \nabla_{F_*(V)(p)}(\bar{W})(F(p))$$

(as $F_* = C$)

$$= \nabla_{\bar{V}}(\bar{W})(F(p))$$

(by definition of $\bar{V}$; note that $\bar{V}$ is evaluated at $F(p)$ here) as desired.

Section 3.5

1. "only if": Suppose α is congruent to β; there exists an isometry F such that $F \circ \alpha = \beta$. Writing $F = TC$ where T is a translation and C is orthogonal, one uses p to denote $T(0)$ (T is translation by p), $\{v_i\}$ to denote the standard orthonormal basis of $\mathbf{R}^3$, and sets $e_i = C(v_i)$ (an orthonormal basis as C is orthogonal, i.e. $e_i \bullet e_j = \delta_{ij}$); then

$$\beta(t) = TC(\alpha_1(t)v_1 + \alpha_2(t)v_2 + \alpha_3(t)v_3) = T(\alpha_1(t)e_1 + \alpha_2(t)v_2 + \alpha_3(t)e_3)$$

$$= p + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \alpha_3(t)e_3$$

as desired.

"if": Suppose $\beta(t) = p + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \alpha_3(t)e_3$ where $e_i \bullet e_j = \delta_{ij}$; as $\{e_i\}$ is an orthonormal basis of $\mathbf{R}^3$ there exists some orthonormal C sending the standard orthonormal basis of $\mathbf{R}^3$ to this new basis; letting T denote translation by p, clearly $\beta = TC(\alpha)$ showing congruence as TC is an isometry.

2. Letting $\{E_i\}$ denote this frame field, we let $\{G_i\}$ denote the induced frame field on α and $\{F_i\}$ denote the induced frame field on F.

We verify the conditions of Theorem 5.7 (all functions are evaluated at a fixed time t) of this section.

Then, for each i, $\alpha' \bullet G_i = \theta_i(\alpha') = \theta_i(\beta') = \beta' \bullet G_i$ (by the definition of θ_i using our assumption for the middle equality), which is the first condition of 5.7.

Also, with i and j fixed,

$$G_i' \bullet G_j = \nabla_{\alpha'} G_i \bullet G_j$$

(by Exercise 2.5.5 from the fourth homework)

$$= \omega_{ij}(\alpha') = \omega_{ij}(\beta')$$

(by definition of ω_{ij} and our assumption)

$$= \nabla_{\beta'} F_i \bullet F_j = F_i \bullet F_j$$

verifying the second condition; by Theorem 5.7, α and β are indeed congruent.

3. Because $\beta(t) = (t + \sqrt{3}\sin t, 2\cos t, \sqrt{3}t - \sin t)$,

$\beta'(t) = (1+\sqrt{3}\cos t, -2\sin t, \sqrt{3}-\cos t)$ (with norm equal to the square root of

$$(1 + 3\cos^2 t + 2\sqrt{3}\cos t) + 4\sin^2 t + (3 + \cos^2 t - 2\sqrt{3}\cos t)$$

$= 4 + 4\cos^2 t + 4\sin^2 t = 8$, i.e. the curve has speed $2\sqrt{2}$),

$\beta''(t) = (-\sqrt{3}\sin t, -2\cos t, \sin t)$,

$\beta'''(t) = (-\sqrt{3}\cos t, 2\sin t, \cos t)$,

$\beta'(t) \times \beta''(t)$

$$= (-2\sin^2 t - 2\cos^2 t + 2\cos t\sqrt{3}, -3\sin t + \sqrt{3}\sin t\cos t - \sin t - \sqrt{3}\sin t\cos t,$$

$$-2\cos t - 2\sqrt{3}\cos^2 t - 2\sqrt{3}\sin^2 t)$$

$$= (-2 + 2\sqrt{3}\cos t, -4\sin t, -2\sqrt{3} - 2\cos t)$$

(with norm equal to the square root of

$$(4 - 8\sqrt{3}\cos t + 12\cos^2 t) + 16\sin^2 t + (12 + 8\sqrt{3}\cos t + 4\cos^2 t)$$

$= 16 + 16\sin^2 t + 16\cos^2 t = 32$), and

$$(\beta'(t) \times \beta''(t)) \bullet \beta'''(t) = (2\sqrt{3}\cos t - 6\cos^2 t) - 8\sin^2 t + (-2\sqrt{3}\cos t - 2\cos^2 t)$$

$$= -6\cos^2 t - 8\sin^2 t - 2\cos^2 t = -8.$$

This gives curvature of $\frac{\sqrt{32}}{(\sqrt{8})^3} = \frac{1}{4}$ and torsion of $\frac{-8}{(\sqrt{32})^2} = -\frac{1}{4}$; as κ, τ are constant with $\kappa > 0$ the curve is indeed a helix.

Now, by Example 2.3.3, the helix $\alpha(t) = (2\cos t, 2\sin t, -2t)$ (which has the same speed, $2\sqrt{2}$, as our new helix) has curvature equal to $\frac{1}{4}$ and torsion equal to $-\frac{1}{4}$ so the isometry must send the Frenet apparatus of α at $t = 0$ to the Frenet apparatus of β at $t = 0$.

Although one could find an isometry F with $F(\alpha) = \beta$ by computing the Frenet apparatuses of the two curves at $t = 0$, it is computationally easier to play around with the components of α and β (denote $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and similarly for β). One notices that $\beta_1 = \frac{\alpha_2\sqrt{3}-\alpha_3}{2}$, $\beta_2 = \alpha_1$, and $\beta_3 = \frac{-\alpha_2-\alpha_3\sqrt{3}}{2}$, so letting F be the linear transformation sending (x, y, z) to $(\frac{y\sqrt{3}-z}{2}, x, \frac{-y-z\sqrt{3}}{2})$, clearly $F(\alpha) = \beta$; further, F sends the standard orthornormal basis of $\mathbf{R}^3$ to the set whose first element is $(0, 1, 0)$, whose second element is $(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2})$, and whose third element is $(-\frac{1}{2}, 0, \frac{-\sqrt{3}}{2})$ which is indeed an orthonormal basis so F is orthogonal and therefore an isometry.

6. (a) WLOG suppose α, β are parametrized by arc length and therefore are unit speed.

Consequently, letting T_α be the unit tangent vector of α, N_α be its unit normal vector, T_β be the unit tangent vector of β and N_β be its unit normal vector (all normals are taken in the sense of plane curvature), we let F be the isometry of the plane that sends the frame $\{T_\alpha(0), N_\alpha(0)\}$ at $\alpha(0)$ to the frame $\{T_\beta(0), N_\beta(0)\}$ at $\beta(0)$ (this can be done by Theorem 3.2.3; the proof works just as well for two dimensions).

Now let the function f be equal to $\bar{T_\alpha} \bullet T_\beta + \bar{N_\alpha} \bullet N_\beta$ (where $\bar{T_\alpha} \circ F = F_* \circ T_\alpha$, for example) and note that $f(0) = 2$ (as $\bar{T_\alpha} = T_\beta$ and similarly for N by initial construction).

However, $f'(t)$ is equal to

$$\bar{T_\alpha}' \bullet T_\beta + \bar{T_\alpha} \bullet T_\beta' + \bar{N_\alpha}' \bullet N_\beta + \bar{N_\alpha} \bullet N_\beta'$$

(evaluated at t)

$$\tilde{\kappa}_\alpha \bar{N_\alpha} \bullet T_\beta + \bar{T_\alpha} \bullet \tilde{\kappa}_\beta N_\beta - \tilde{\kappa}_\alpha \bar{T_\alpha} \bullet N_\beta - \bar{N_\alpha} \bullet \tilde{\kappa}_\beta T_\beta$$

(by Exercise 2.3.8, using $\tilde{\kappa}_\alpha$ to denote the plane curvature of α and similarly for β; using Theorem 3.4.2 to note that the portions of the Frenet apparatus used here are invariant under isometry))

$$\tilde{\kappa}(\bar{N_\alpha} \bullet T_\beta + \bar{T_\alpha} \bullet N_\alpha - \bar{T_\alpha} \bullet N_\beta - \bar{N_\alpha} \bullet T_\beta)$$

(using $\tilde{\kappa}$ to denote $\tilde{\kappa}_\alpha = \tilde{\kappa}_\beta$; these two are equal by assumption)

$= 0$ so f is constant at two, which means that everywhere (by the strict form of Cauchy-Schwarz) $\bar{T_\alpha} = T_\beta$, i.e. $(F \circ \alpha)' = \beta'$ so $F \circ \alpha$ and β, which have the same first derivative and agree at one point, are the same curve, giving the desired congruence.

(b) The easiest way to show $\alpha(t) = (\sqrt{2}t, t^2, 0)$ and $\beta(t) = (-t, t, t^2)$ are congruent is to use the α_i and β_i notation as before, note that $\beta_1 = -\frac{\alpha_1}{\sqrt{2}}$ while $\beta_2 = \frac{\alpha_1}{\sqrt{2}}$ and $\beta_3 = \alpha_2$ so if F is the map sending (x, y, z) to $(\frac{-x-z}{\sqrt{2}}, \frac{x-z}{\sqrt{2}}, y)$ then the standard orthonormal basis is sent to the basis whose first element is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, whose second element is $(0, 0, 1)$, and whose third element is $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, which is an orthonormal basis so F is an isometry with $F \circ \alpha = \beta$ giving the desired congruences. (One could also do this by noting that α is a plane curve and β lies in the plane $x + y = 0$; noting that both curves pass through the origin, one sends plane tangents to plane tangents, plane normals to plane normals, and completes the respective bases with vectors orthogonal to the appropriate planes).