

Problems graded: 2.8.3 (15 points), 3.1.7 (10 points), 3.1.4a (10 points); the other problems were graded based on completeness to be worth five points each, for a grand total of 60 points (this is fewer than the earlier assignments because there were only eight problems.)

Section 2.8

1. Because

$$\phi = \sum f_i \theta_i, d\phi = (\sum_i df_i \wedge \theta_i) + \sum_i (f_i d\theta_i)$$

(by Theorem 1.6.4)

$$= (\sum_j df_j \wedge \theta_j) + \sum_i (f_i * \sum_j \omega_{ij} \wedge \theta_j)$$

(by the first structural equations)

$$= \sum_j (df_j + \sum_i f_i * \omega_{ij}) \wedge \theta_j$$

as desired by the associativity of wedge products.

One notes (by comparing coefficients) that this is the dual formulation of equation 2.7.5: recall that this problem involved $W = \sum f_i E_i$, whose dual element in the space of 1-forms is indeed ϕ .

2. In the spherical frame field we first note that because $x_1 = \rho \cos \phi \cos \theta$, $x_2 = \rho \cos \phi \sin \theta$, $x_3 = \rho \sin \theta$,

$$dx_1 = \cos \phi \cos \theta d\rho - \rho \sin \phi \cos \theta d\phi - \rho \cos \phi \sin \theta d\theta,$$

$$dx_2 = \cos \phi \sin \theta d\rho - \rho \sin \phi \sin \theta d\phi + \rho \cos \phi \cos \theta d\theta, \text{ and}$$

$$dx_3 = \sin \phi d\rho + \rho \cos \phi d\phi.$$

Also, the attitude matrix has $(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)$, $(-\sin \theta, 0, \cos \theta)$, and $(-\sin \phi \cos \theta, -\sin \phi \sin \theta, \cos \phi)$ as its rows F_1, F_2, F_3 in order.

This tells us that, letting $d\xi$ be as in the text (i.e. with the i th row of dx_i) and $d\psi$ be the analogue of $d\xi$ with polar coordinates (first row $d\rho$, second row $d\phi$, third row $d\theta$), $d\xi = Bd\psi$ where the first column of B is F_1 , the second column of B is ρF_3 , and the third column of B is $\rho \cos \phi F_2$.

As $\theta = Ad\xi = ABd\psi$, where (by orthogonality) AB has first column $(1, 0, 0)$, second column $(0, 0, \rho)$, and third column $(0, \rho \cos \phi, 0)$, one reads off $\theta_1 = d\rho$, $\theta_2 = \rho \cos \phi d\theta$, $\theta_3 = \rho d\phi$.

One recalls that in Exercise 2.7.4 (from the previous homework), it was shown $\omega_{12} = \cos \phi d\theta$, $\omega_{13} = d\phi$, $\omega_{23} = \sin \phi d\theta$.

For the first structural equation, one checks
 $d\theta_1 = d(d\rho) = 0$ while

$$\omega_{11} \wedge \theta_1 + \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \theta_3 = 0 + \cos \phi d\theta \wedge \rho \cos \phi d\theta + d\phi \wedge \rho d\phi = 0.$$

Also, one checks
 $d\theta_2 = d(\rho \cos \phi d\theta) = \cos \phi d\rho d\theta - \rho \sin \phi d\phi d\theta$ while

$$\begin{aligned} \omega_{21} \wedge \theta_1 + \omega_{22} \wedge \theta_2 + \omega_{23} \wedge \theta_3 &= -\cos \phi d\theta \wedge d\rho + \sin \phi d\theta \wedge \rho d\phi \\ &= \cos \phi d\rho d\theta - \rho \sin \phi d\phi d\theta \end{aligned}$$

(by the alternating nature of wedge products); $d\theta_3$ was developed in the text.

For the second structural equations, as these equations clearly obey the desired bilinearity, it suffices to consider $d\omega_{12}$ (done in the text), $d\omega_{13}$, and $d\omega_{23}$.

For $d\omega_{13}$, one gets $dd\phi = 0$ while

$$\omega_{11}\omega_{13} + \omega_{12}\omega_{23} + \omega_{13}\omega_{33} = \cos \phi d\theta \wedge \sin \phi d\theta = 0.$$

For $d\omega_{23}$, one gets $d(\sin \phi d\theta) = \cos \phi d\phi d\theta$ while

$$\omega_{21}\omega_{13} + \omega_{22}\omega_{23} + \omega_{23}\omega_{33} = -\cos \phi d\theta \wedge d\phi = \cos \phi d\phi d\theta$$

as desired.

3. (a) Because $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, one notes that $dx = \cos \theta dr - r \sin \theta d\theta$, $dy = \sin \theta dr + r \cos \theta d\theta$, $dz = dz$; left-multiplying by the attitude matrix (whose rows are $(\cos \theta, \sin \theta, 0)$, $(-\sin \theta, \cos \theta, 0)$, and $(0, 0, 1)$ respectively) yields a matrix whose i th row is θ_i ; we conclude

$$\theta_1 = \cos^2 \theta dr - r \sin \theta \cos \theta d\theta + \sin^2 \theta dr + r \sin \theta \cos \theta d\theta = dr,$$

$$\theta_2 = -\sin \theta \cos \theta dr + r \sin^2 \theta d\theta + \sin \theta \cos \theta dr + r \cos^2 \theta d\theta = r d\theta,$$

and $\theta_3 = dz$.

Therefore, $E_1[r] = dr(E_1) = \theta_1(E_1) = 1$, $E_2[\theta] = d\theta(E_2) = r^{-1}\theta_2(E_2) = r^{-1}$, $E_3[z] = dz(E_3) = \theta_3(E_3) = 0$, and as for the other possibilities:

$$E_1[\theta] = d\theta(E_1) = r^{-1}\theta_2(E_1) = 0,$$

$$E_1[z] = dz(E_1) = \theta_3(E_1) = 0,$$

$$E_2[r] = dr(E_2) = \theta_1(E_2) = 0,$$

$$E_2[z] = dz(E_2) = \theta_3(E_2) = 0,$$

$$E_3[r] = dr(E_3) = \theta_1(E_3) = 0, \text{ and}$$

$$E_3[\theta] = d\theta(E_3) = r^{-1}\theta_2(E_3) = 0.$$

(c) We note that for any vector field V ,

$$V[f] = df(V) = (f_r dr + f_\theta d\theta + f_z dz)V = f_r V[r] + f_\theta V[\theta] + f_z V[z],$$

from which, by (b), we conclude

$$E_1[f] = f_r = \frac{\delta f}{\delta r}, \quad E_2[f] = r^{-1} f_\theta = \frac{1}{r} \frac{\delta f}{\delta \theta}, \quad \text{and} \quad E_3[f] = f_z = \frac{\delta f}{\delta z}.$$

4.(a) One observes that as $E_1 = \cos \psi U_1 + \sin \psi U_2$, $E_2 = -\sin \psi U_1 + \cos \psi U_2$, the attitude matrix A has rows equal to $(\cos \psi, \sin \psi)$ and $(-\sin \psi, \cos \psi)$, so dA has rows equal to $(-d\psi \sin \psi, d\psi \cos \psi)$ and $(-d\psi \cos \psi, -d\psi \sin \psi)$ which gives that multiplying out dA by A^t yields a first row of

$$(-d\psi \sin \psi \cos \psi + d\psi \cos \psi \sin \psi, d\psi \sin^2 \psi + d\psi \cos^2 \psi) = (0, d\psi)$$

and a second row of

$$(-d\psi \cos^2 \psi - d\psi \sin^2 \psi, d\psi \cos \psi \sin \psi - d\psi \sin \psi \cos \psi) = (-d\psi, 0)$$

from which we can read off (although the results in the past three sections were written in terms of \mathbf{R}^3 , the proofs go over just as well in \mathbf{R}^2) that $\omega_{12} = d\psi = \psi_x dx + \psi_y dy$ (and $\omega_{21} = -d\psi = -\psi_x dx - \psi_y dy$, with the others being trivial).

Also, one has $\theta_1 = \cos \psi dx + \sin \psi dy$, $\theta_2 = -\sin \psi dx + \cos \psi dy$. (Extending to a frame field in \mathbf{R}^3 would add a bottom row and bottom column, both $(0, 0, 1)$, to the attitude matrix, and a third equation $\theta_3 = dz$).

(b) The first structural equations are

$$d\theta_1 = \omega_{11}\theta_1 + \omega_{12}\theta_2,$$

which holds as both sides equal $(\psi_y \sin \psi + \psi_x \cos \psi) dx dy$, and

$$d\theta_2 = \omega_{21}\theta_1 + \omega_{22}\theta_2,$$

which holds as both sides equal $(\psi_y \cos \psi - \psi_x \sin \psi) dx dy$.

(The structural equation for θ_3 , if considered, would be completely trivial).

For the second structural equation one only needs to check

$d\omega_{12} = \omega_{11}\omega_{12} + \omega_{12}\omega_{22}$; however, the RHS is trivially zero and the LHS is zero as $\omega_{12} = d\psi$. (Once again, the structural equations involving an index of 3 would be trivial as well.)

Section 3.1

7. Composition of functions sends isometries to isometries by Lemma 1.3. The associative law holds because composition of functions is associative. For the identity, let e denote the identity map on \mathbf{R}^3 sending every point to itself; clearly $e \circ g(p) = g \circ e(p) = g(p)$ for each $g \in E(3), p \in \mathbf{R}^3$ so $e \circ g = g \circ e = g$ and e is unique because if $g \in E(3)$ is another identity element then $e = e \circ g = g$. For inverses, if g is an isometry let g^{-1} be the inverse map sending $p \in \mathbf{R}^3$ to the point $z \in \mathbf{R}^3$ with $g(z) = p$ (this is well-defined because G is one-to-one and onto by Theorem 1.7); note that g^{-1} is an isometry because for $p, q \in \mathbf{R}^3$, $d(p, q) = d(g \circ g^{-1}(p), g \circ g^{-1}(q)) = d(g^{-1}(p), g^{-1}(q))$.

8. Translations forms a subgroup because if g, h are translations (say, by v and w respectively), $g \circ h(p) = g(p + w) = p + v + w$ so $g \circ h$ is translation by $v + w$ and also, g^{-1} is translation by $-v$.

Orthogonal transformations form a subgroup by their matrix representations: if g, h are orthogonal transformations represented by matrices A, B respectively, $g \circ h$ is multiplication by the matrix AB (which is orthogonal because $(ABv \bullet ABw) = Bv \bullet Bw = v \bullet w$ for $v, w \in \mathbf{R}^3$) and g^{-1} is multiplication by A^{-1} (which is orthogonal because for $p, q \in \mathbf{R}^3$, $p \bullet q = (AA^{-1}p) \bullet (AA^{-1}q) = A^{-1}p \bullet A^{-1}q$).

If g is both a translation and an orthogonal transformation, it fixes the origin (being orthogonal) and therefore translates by $0 - 0 = 0$ (the zeroes refer to vectors in \mathbf{R}^3); this means that it must be the identity transformation which is clearly in both subgroups.

Section 3.2

1. If T is a translation, its decomposition (as in Theorem 3.1.7) is $T = TI$ (where I is the identity mapping), i.e. its 'orthogonal part' is I , so Theorem 3.2.1 says that for a given tangent vector v_p , $T_*(v_p) = I(v)_{F_p} = v_{F_p}$ by definition of I . In other words, $T_*(v_p)$ and v_p both have the Euclidean coordinates of v .

4. (a) An arbitrary point in \mathbf{R}^3 can be written in the form $p + w$; it is in the plane through p orthogonal to q iff $w \bullet q = 0$. Now,

$$F(p + w) = TC(p + w) = T(Cp + Cw) = Cp + Cw + z$$

(where z is the vector such that $T(r) = r + z$ for each vector r ; note that $C(p + w) = Cp + Cw$ by linearity)

$$= (Cp + z) + Cw = T(Cp) + Cw = F(p) + C(w);$$

as $C(w) \bullet C(q) = w \bullet q$ (by orthogonality of C), $p + w$ is in the plane through p orthogonal to q if and only if its image under F is in the plane orthogonal to $C(q)$, i.e. (as F is surjective), F indeed carries the plane through p orthogonal to q to the plane through $F(p)$ orthogonal to $C(q)$.