

Problems graded: 2.4.2 (15 points), 2.5.3 (10 points), 2.6.1 (10 points), 2.7.2 (15 points). The other problems were graded for completeness to be worth three points each (except 2.6.2 counted as two problems); five points were added to everyone's score to make the total score 100.

Section 2.4

2. Because $\alpha(t) = (\cosh t, \sinh t, t)$, we compute $\alpha'(t) = (\sinh t, \cosh t, 1)$, $\alpha''(t) = (\cosh t, \sinh t, 0)$, and $\alpha'''(t) = (\sinh t, \cosh t, 0)$.

Then, the Frenet formula for curvature tells us that the curvature at t is $\frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$;

as the cross product of $\alpha'(t)$ and $\alpha''(t)$ is equal to $(-\sinh t, \cosh t, -1)$ (note $\cosh^2 t = \sinh^2 t + 1$) with norm $\sqrt{(-\sinh t)^2 + (\cosh t)^2 + 1}$

$$= \sqrt{(\sinh t)^2 + (\cosh t)^2 + 1} = \sqrt{2(\cosh t)^2} = \sqrt{2} \cosh t$$

and the absolute value of $|\alpha'(t)|$ is

$$\sqrt{(\sinh t)^2 + (\cosh t)^2 + 1} = \sqrt{2(\cosh t)^2} = \sqrt{2} \cosh t$$

, the curvature at t is $\frac{1}{2 \cosh^2 t}$.

To express this in terms of arc length, note that the arc length of α from 0 to t is $\int_0^u \sqrt{2} \cosh t \, dt = \sqrt{2} \sinh t$ so $2 \cosh^2 t = 2 + s(t)^2$ where s denotes arc length, giving a curvature of $\frac{1}{2+s^2}$.

Now, the formula for torsion at t is $\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'''(t)|^2}$, of which the numerator is $(-\sinh t, \cosh t, -1) \cdot (\sinh t, \cosh t, 0) = \cosh^2 t - \sinh^2 t = 1$ and the denominator is $(\sqrt{2} \cosh t)^2$, yielding torsion equal to $\frac{1}{2 \cosh^2 t} = \frac{1}{2+s^2}$.

3. (a) As in the preceding problem, because $\alpha(t) = (t \cos t, t \sin t, t)$; we compute $\alpha'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$, $\alpha''(t) = (-2 \sin t - t \cos t, 2 \cos t - t \sin t, 0)$, and $\alpha'''(t) = (-3 \cos t + t \sin t, -3 \sin t - t \cos t, 0)$; at $t = 0$ these become $(1, 0, 1)$, $(0, 2, 0)$, and $(-3, 0, 0)$ respectively.

Also, $\alpha'(0) \times \alpha''(0) = (-2, 0, 2)$.

Consequently, at $t = 0$, $T = \frac{\alpha'(0)}{|\alpha'(0)|} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $B = \frac{\alpha'(0) \times \alpha''(0)}{|\alpha'(0) \times \alpha''(0)|} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $N = B \times T = (0, 1, 0)$,

$$\kappa = \frac{|\alpha'(0) \times \alpha''(0)|}{|\alpha'(0)|^3} = \frac{2\sqrt{2}}{(\sqrt{2})^3} = 1, \text{ and}$$

(because $(\alpha'(0) \times \alpha''(0)) \cdot \alpha'''(0) = 6$),

$$\tau = \frac{6}{|\alpha'(0) \times \alpha''(0)|^2} = \frac{6}{8} = \frac{3}{4}.$$

(b) (The graph should have vertical coordinate exactly equal to the time t ; the horizontal component should spiral outward from the origin as t increases from 0 to 2π and again as t decreases from 0 to -2π).

At $t = 0$, T should point along the line $x = z; y = 0$ [in the direction of increasing z], N should point along the y axis, and B should point along the line $-x = z; y = 0$ [in the direction of increasing z].

9. Because $\kappa > 0$ and τ are constant, so is $\frac{\tau}{\kappa}$ so, by Theorem 4.6 of this chapter, α is a cylindrical helix, i.e. there exists a unit vector v such that $\alpha \bullet v$ is equal to some constant c which can be expressed as $\cos \theta$ for some $\theta \in [0, \pi]$.

To show that α is a circular helix it suffices to show that the projection of α onto the orthogonal complement of v is contained in a circle. To this end, we (assuming WLOG that α is parametrized with respect to arclength) write $\alpha(t) = \alpha(0) + t \cos \theta v + M(t)$ where $M(t)$ is orthogonal to v (and $|M'| = \sin \theta$).

As $\alpha'(t) = \cos \theta v + M'(t)$ is the unit tangent, we note that the unit normal to α is $\frac{M''(t)}{|M''(t)|}$; further, M'' has norm κ .

Letting w_t be the projection of $\alpha(0)$ onto the complement of v , we conclude $w_t = w_0 + M(t)$ where M is a plane curve such that $|M'| = \sin \theta$ and $|M''| = \kappa$; therefore, as M has constant curvature (equal to $\frac{\kappa}{\sin^2 \theta}$), Lemma 3.6 of this chapter tells us that M is a circular curve; therefore, so is w_t (so we can conclude α is indeed a circular helix).

12. NOTE: For this problem we use N to denote the unit normal in the usual sense and \tilde{N} to denote the unit normal in the sense of plane curvature.

If α is regular, using v to denote its speed, $\alpha' = vT$ where T is the unit tangent so $T = v^{-1}\alpha'$ and $T' = -v^{-2}v'\alpha' + v^{-1}\alpha''$ (note that the first term is orthogonal to $J(\alpha')$).

From Problem 8a of the preceding section (on the last homework), we know that

$$\begin{aligned} \tilde{\kappa} &= (v^{-1}T') \bullet \tilde{N} = (v^{-1}T') \bullet J(T) \\ &= (v^{-1}T') \bullet J(\alpha'v^{-1}) = (v^{-2}\alpha'') \bullet (v^{-1}J(\alpha')) = \frac{\alpha'' \bullet J(\alpha')}{v^3}; \end{aligned}$$

the numerator is $(x'', y'') \bullet (-y', x') = x'y'' - x''y'$ while the denominator is $(x'^2 + y'^2)^{3/2}$ giving the second expression:

$$\frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

17. (b) The helix in example 3.3, which is unit-speed, was computed to have constant curvature $\frac{a}{a^2+b^2} > 0$; therefore, its total curvature is the integral of a positive constant over an infinite interval which is clearly ∞ .

(c) The curve from Exercise 2 was computed to have curvature $\frac{1}{s^2+2}$ when parametrized by arclength; therefore, its total curvature is $\int_{-\infty}^{\infty} \frac{1}{s^2+2}$; as the integrand has antiderivative $\tan^{-1}(\frac{s}{\sqrt{2}}) * \frac{1}{\sqrt{2}}$ (which approaches $\pm \frac{\pi}{2\sqrt{2}}$ as s approaches $\pm\infty$) the integral defining total curvature evaluates to be equal to $\frac{\pi}{\sqrt{2}}$.

18. We suppose that the convex closed plane curve α is parametrized by arc length of spherical image.

Lemma: The total curvature of α is a **positive integral multiple** of 2π .

Proof: Let β denote the spherical image of α (which is a curve on the unit circle of the plane, parametrized by arc length). By Exercise 2.1.12 (note: this exercise was not assigned) one can create an angle function $\theta(t)$ with $\beta(t) = (\cos \theta(t), \sin \theta(t))$. Note that

$$\kappa(t) = |\beta'(t)| = |\theta'(t)(-\sin \theta(t), \cos \theta(t))| = |\theta'(t)| > 0$$

so θ' (being continuous) is either always positive or always negative (WLOG we assume the former; otherwise we replace $\alpha(t)$ by $\alpha(-t)$ in our arguments).

Letting t_0 be the starting point of the curve and t_1 be the ending point, we note that as $t_1 = t_0$, $\beta(t_0) = \beta(t_1)$ so $\theta(t_1)$ is congruent to $\theta(t_0)$ modulo 2π . However, as $\theta(t_1) - \theta(t_0) = \int_{t_0}^{t_1} \kappa(t) dt$ we conclude that the total curvature is itself a multiple of 2π (which is clearly positive as curvature is greater than zero), proving the lemma.

Armed with the lemma, we seek to show that the total curvature is actually equal to 2π .

We now fix a point z_1 on the curve where the y-component of α attains its minimum value and note that α' points either in the positive x -direction or in the negative x -direction; by passing to $\alpha(-t)$ if necessary we assume the former is the case. Letting β be the spherical image of α (and, by reparametrizing if necessary) supposing both curves are parametrized by arc length of the spherical image β we note that $\beta(s) = (\cos s, \sin s)$ (because α begins by pointing in the positive horizontal direction, we can assume $s = 0$ at this point; as the vertical component is positive shortly after that point, $s > 0$ shortly after that point, so reparametrization of β by arc length would indeed be by s). The total curvature is equal to the first point $s > 0$ where $\alpha(s) = \alpha(0)$; suppose $s > 2\pi$. For each positive integer n we define α_n to be the image of $[(n-1)\pi, n\pi]$ under α (counting only the points for which this image is defined). We note that the y-coordinate is strictly decreasing along α_n for n odd and strictly increasing for n even.

One notes that α_2 (except for endpoints) is strictly to the left of α_1 in the sense that for each point in α_2 there exists a point in α_1 with the same vertical coordinate and greater horizontal coordinate (as the x component has negative derivative near $s = \pi$, this happens near $s = \pi$; as $\alpha(0)$ provides the minimum vertical coordinate for α_1 and $\alpha(\pi)$ provides the maximal vertical coordinate for α_1 , the infimum of the set of s where the above statement fails would be an intersection of α_1 and α_2 , which can only happen at $\alpha(0)$.)

Further, if $s > 2\pi$ then $\alpha(2\pi)$ is either strictly above $\alpha(0)$ or strictly to the left of $\alpha(\pi)$.

In either case, if $k > 2$, then α_k (except for the initial endpoint) is strictly between α_{k-1} and α_{k-2} . For k odd, this means that in the sense that for each point in α_k there exists a point in α_{k-1} with the same vertical coordinate and greater horizontal coordinate and a point in α_{k-2} with the same vertical coordinate and smaller horizontal coordinate; for k even, this means that for

each there exists a point in α_{k-1} with the same vertical coordinate and smaller horizontal coordinate and a point in α_{k-2} with the same vertical coordinate and greater horizontal coordinate; this is proved by induction on k (taking derivatives of the x component near $s = (k-1)\pi$, the α_{k-1} statement follows near $s = (k-1)\pi$ directly and the α_{k-2} statement follows by induction; if either statement fails along any point of α_k , the infimum of the set of s where the above statement fails, provided that the vertical coordinate of a point on α_k is not beyond the range of either α_{k-1} or α_{k-2} , would be an intersection of α_k and α_{k-1} or α_{k-2} , which can only happen at $\alpha(0)$.) One notes that because the vertical coordinate is increasing in the case of k odd or decreasing in the case of k even, if the vertical coordinate of a point on α_k were to be beyond the range of either α_{k-1} or α_{k-2} , at some point α_k would therefore had to cross the image point where $s = \pi(k-2)$, forcing α to self-intersect at a point other than $\alpha = 0$ and creating a contradiction.)

In the first case ($\alpha(2\pi)$ strictly above $\alpha(0)$), the y-coordinate of $\alpha(0)$ will never be attained for $s > 0$ because, by induction on k , the minimal α_k coordinate is strictly above $\alpha(0)$ for $k > 1$.

In the second case, the y-coordinate of $\alpha(0)$ will never be attained for $s > 2\pi$ because the y-coordinate is strictly increasing on α_3 (ruling out $(2\pi, 3\pi]$) and the betweenness result therefore states that the only way α_4 can reach this coordinate is by self-intersection with α_2 at $s = 2\pi$ (producing a contradiction); as this is impossible, betweenness shows (by induction on k) that α_k does not hit this minimum y-coordinate beyond $s = 2\pi$.

In any event, $\alpha(0)$ is never attained again (its height only hits the minimum for $t = 0$ can only be hit at $t = 2\pi$, where $\alpha(0) \neq \alpha(2\pi)$), producing a contradiction.

Therefore, by the lemma, the total curvature is indeed equal to 2π .

Section 2.5

2. (a) As $V = -yU_1 + xU_3$ and $W = \cos xU_1 + \sin xU_2$,

$$\nabla_V W = \frac{d}{dt}(\cos(x - ty)U_1 + \sin(x - ty)U_2)|_{t=0}$$

$$= y \sin xU_1 - y \cos xU_2.$$

(b)

$$\nabla_V V = \frac{d}{dt}(-yU_1 + (x - ty)U_3)|_{t=0} = -yU_3.$$

(c)

$$\nabla_V(z^2W) = \nabla_V(-z^2yU_1 + z^2xU_3) = \frac{d}{dt}(-(z+tx)^2yU_1 + (z+tx)^2(x-ty)U_2)|_{t=0}$$

$$= -2xyzU_1 + (2zx^2 - z^2y)U_2.$$

(d)

$$\nabla_W(V) = \frac{d}{dt}(-(y + t \sin x)U_1 + (x + t \cos x)U_2)|_{t=0}$$

$$= -\sin xU_1 + \cos xU_2.$$

(e) As $\nabla_V W$ was already computed in part (a) to be

$= y \sin xU_1 - y \cos xU_2$, we have that

$$\nabla_V(\nabla_V W) = \frac{d}{dt}(y \sin(x - ty)U_1 - y \cos(x - ty)U_2)|_{t=0}$$

$$= -y^2 \cos xU_1 - y^2 \sin xU_2.$$

(f) We note that $V[x] = \frac{d}{dt}(x - ty)|_{t=0} = -y$ while $V[z] = \frac{d}{dt}(z + tx)|_{t=0} = x$.
Consequently,

$$\nabla_V(xV) = V[x]V + x\nabla_V(V) = -y(-yU_1 + xU_3) + x(-yU_3) = y^2U_1 - xyU_3$$

(using the value of $\nabla_V V$, calculated in part (b), and Corollary 5.4(3).)

Similarly,

$$\nabla_V(zW) = V[z]W + z\nabla_V(W) = x(\cos xU_1 + \sin xU_2) + z(y \sin xU_1 - y \cos xU_2)$$

$$= (x \cos x + yz \sin x)U_1 + (x \sin x - yz \cos x)U_2$$

(using the value of $\nabla_V W$, calculated in part (a), and Corollary 5.4(3).).

We conclude by linearity that

$$\nabla_V(xV - zW) = \nabla_V(xV) - \nabla_V(zW)$$

$$= (y^2 - x \cos x - yz \sin x)U_1 + (yz \cos x - x \sin x)U_2 + (-xy)U_3.$$

3. Because $\|W\|$ is constant, so is its square $W \bullet W$. Therefore, using the product rule of Corollary 5.4(4),

$$0 = V[W \bullet W] = \nabla_V W \bullet W + W \bullet \nabla_V W = 2\nabla_V W \bullet W$$

so as $\nabla_V W \bullet W = 0$, the covariant derivative $\nabla_V W$ is everywhere orthogonal to W .

4. By Lemma 5.2,

$$\delta_V X = \Sigma_i V[x_i]U_i = \Sigma_i dx_i[V]U_i = \Sigma_i v_i U_i = V$$

(where we write $V = \Sigma_i v_i U_i$ in terms of its components.)

5. (a) By definition,

$$\nabla_{\alpha'(t)} W = \frac{d}{ds}(W(\alpha(t) + s\alpha'(t)))|_{s=0} = W'(\alpha(t)) * \alpha'(t)$$

(by the chain rule)

$$= (W \circ \alpha)'(t)$$

(by the chain rule again)

$$= (W_\alpha)'(t).$$

(b) If α is a curve with initial velocity v (based at the point $\alpha(t)$) and β is the straight line sending s to $\alpha(t) + vs$, then we note that (using the preceding part and the fact that β has initial velocity $v = \alpha'$)

$$(W_\beta)'(t) = \nabla_{\beta'(t)} W = \nabla_{\alpha'(t)} W = (W_\alpha)'(t);$$

as the LHS is the definition of covariant derivative in the text and the RHS is the same definition replacing β by α , we may indeed replace the straight line β with any α with initial velocity v . Therefore, from the preceding part, one could claim that $\nabla_{\alpha'} Y$ serves as a 'derivative' of Y along α .

Section 2.6

1. We note that $\|E_1\| = \frac{\|V\|}{\|V\|} = 1$ (by linear independence V is nonzero at each point); as \tilde{W} is nonzero at each point (otherwise W would be a scalar multiple of E_1 and therefore of V at that point) and $\|E_2\| = \frac{\|\tilde{W}\|}{\|\tilde{W}\|}$. Also, \tilde{W} is orthogonal to E_1 because $\tilde{W} \bullet E_1 = W \bullet E_1 - (W \bullet E_1)(E_1 \bullet E_1) = 0$ as $E_1 \bullet E_1 = \|E_1\|^2 = 1$; by linearity of dot products, $E_1 \bullet E_2$ is a scalar multiple of $E_1 \bullet \tilde{W} = 0$ so E_1 and E_2 is orthogonal.

Therefore, as E_3 is the cross product of the two orthogonal unit vectors E_1 and E_2 , $\|E_3\| = \|E_1\|\|E_2\| = 1$ and E_3 is orthogonal to E_1 and E_2 . Therefore, E_1, E_2, E_3 are mutually orthogonal unit vectors at each point and therefore form a frame at each point (and a frame field in general).

2. (a) (i) In cylindrical coordinates, $E_1 = \cos \theta U_1 + \sin \theta U_2$, $E_2 = -\sin \theta U_1 + \cos \theta U_2$, and $E_3 = U_3$; therefore, one notes that $\cos \theta E_1 = \cos^2 \theta U_1 + \sin \theta \cos \theta U_2$ and $-\sin \theta E_2 = \sin^2 \theta U_1 - \sin \theta \cos \theta U_2$; adding yields $\cos \theta E_1 - \sin \theta E_2 = U_1$.

(ii) In spherical coordinates, $F_1 = \cos \phi E_1 + \sin \phi E_3$, $F_2 = E_2$, and $F_3 = -\sin \phi E_1 + \cos \phi E_3$ so, by an identical computation to part (i), $\cos \phi F_1 - \sin \phi F_3 = E_1$ which yields

$$\begin{aligned} U_1 &= \cos \theta E_1 - \sin \theta E_2 = \cos \theta (\cos \phi F_1 - \sin \phi F_3) - \sin \theta F_2 \\ &= \cos \theta \cos \phi F_1 - \sin \theta F_2 - \cos \theta \sin \phi F_3 \end{aligned}$$

(note of course that $F_2 = E_2$).

(b) (i) Because $\cos \theta U_1 + \sin \theta U_2 = E_1$ and $U_3 = E_3$, $\cos \theta U_1 + \sin \theta U_2 + U_3 = E_1 + E_3$.

(ii) In part (a) we already computed $E_1 = \cos \phi F_1 - \sin \phi F_3$; however, note that $\sin \phi F_1 = \sin \phi \cos \phi E_1 + \sin^2 \phi E_3$ while $\cos \phi F_3 = -\sin \phi \cos \phi E_1 + \cos^2 \phi E_3$; adding yields $\sin \phi F_1 + \cos \phi F_3 = E_3$ so

$$\cos \theta U_1 + \sin \theta U_2 + U_3 = E_1 + E_3 = (\cos \phi + \sin \phi) F_1 + (\cos \phi - \sin \phi) U_3.$$

(c) (i) Because $x = r \cos \theta$, $y = r \sin \theta$, we have

$$\begin{aligned} xU_1 + yU_2 + zU_3 &= r \cos \theta U_1 + r \sin \theta U_2 + zU_3 \\ &= r(\cos \theta U_1 + \sin \theta U_2) + zE_3 = rE_1 + zE_3. \end{aligned}$$

(ii) Because $r = \rho \cos \phi$, $z = \rho \sin \phi$, we have $xU_1 + yU_2 + zU_3 = rE_1 + zE_3$ (from part (i))

$$= \rho \cos \phi E_1 + \rho \sin \phi E_3 = \rho(\cos \phi E_1 + \sin \phi E_3) = \rho F_1.$$

3. We know $E_1 = \cos x U_1 + \sin x \cos z U_2 + \sin x \sin z U_3$ (which has norm 1 by direct computation) take $E_2 = -\sin x U_1 + \cos x \cos z U_2 + \cos x \sin z U_3$ and note that $E_1 \bullet E_2 = 0$ by direct computation and $\|E_2\| = 1$ by direct computation.

If one finishes with $E_3 = -\sin z U_2 + \cos z U_3$, one can easily see (as the projection of E_1 or E_2 onto the U_2 - U_3 plane is orthogonal to E_3) that E_3 has norm 1 and is orthogonal to the other two vectors at each point, finishing the frame field.

Section 2.7

1. As $E_1 = (\sin f U_1 + U_2 - \cos f U_3)/\sqrt{2}$, $\|E_1\|^2 = \frac{\sin^2 f + 1 + \cos^2 f}{2} = \frac{2}{2} = 1$ so $\|E_1\| = 1$.

As $E_2 = (\sin f U_1 - U_2 - \cos f U_3)/\sqrt{2}$, $\|E_2\|^2 = \frac{\sin^2 f + 1 + \cos^2 f}{2} = \frac{2}{2} = 1$ so $\|E_2\| = 1$.

As $E_3 = \cos f U_1 + \sin f U_3$, $\|E_3\|^2 = \cos^2 f + \sin^2 f = 1$ so $\|E_3\| = 1$.

Now, $E_1 \bullet E_2 = \frac{1}{2}(\sin^2 f - 1 + \cos^2 f) = 0$, $E_1 \bullet E_3 = \frac{\sin f \cos f - \cos f \sin f}{\sqrt{2}} = 0$, and $E_2 \bullet E_3 = \frac{\sin f \cos f - \cos f \sin f}{\sqrt{2}} = 0$ so E_1, E_2, E_3 form an orthonormal basis at each point and therefore a frame field.

Now, Theorem 7.3 of this section tells us (writing $E_i = \sum_j a_{ij}$ that $\omega_{ij} = \sum_k a_{jk} da_{ik}$; this tells us that

$$\begin{aligned} \omega_{12} &= a_{21} da_{11} + a_{22} da_{12} + a_{23} da_{13} = \frac{\sin f d(\sin f) + -1 * d(1) + -\cos f d(\cos f)}{2} \\ &= \frac{\sin f \cos f df + -\cos f \sin f df}{2} = 0. \end{aligned}$$

Also,

$$\omega_{13} = a_{31} da_{11} + a_{32} da_{12} + a_{33} da_{13} = \frac{\cos f * (\cos f df) + \sin f * (\sin f df)}{\sqrt{2}} = \frac{df}{\sqrt{2}}.$$

Further,

$$\omega_{23} = a_{32} da_{21} + a_{32} da_{22} + a_{33} da_{23} = \frac{\cos f * (\cos f df) + \sin f * (\sin f) df}{\sqrt{2}} = \frac{df}{\sqrt{2}}.$$

By Lemma 7.1 this is all we need; $\omega_{ii} = 0$ for all i and $\omega_{ij} = -\omega_{ji}$ so $\omega_{21} = 0$, $\omega_{31} = -\frac{df}{\sqrt{2}}$, and $\omega_{32} = -\frac{df}{\sqrt{2}}$.

4. To begin, it is necessary to recall

$$F_1 = \cos \phi \cos \theta U_1 + \cos \phi \sin \theta U_2 + \sin \phi U_3,$$

$$F_2 = -\sin \theta U_1 + \cos \theta U_2,$$

$$F_3 = -\sin \phi \cos \theta U_1 - \sin \phi \sin \theta U_2 + \cos \phi U_3.$$

Therefore,

$$\omega_{12} = d(\cos \phi \cos \theta)(-\sin \theta) + d(\cos \phi \sin \theta)(\cos \theta)$$

$$= (-\sin \phi \cos \theta d\phi - \cos \phi \sin \theta d\theta)(-\sin \theta) + (-\sin \phi \sin \theta d\phi + \cos \phi \cos \theta d\theta)(\cos \theta)$$

$$= (\sin \phi \sin \theta \cos \theta - \sin \phi \sin \theta \cos \theta) d\phi + (\cos \phi \sin^2 \theta + \cos \phi \cos^2 \theta) d\theta = \cos \phi d\theta,$$

$$\begin{aligned}
\omega_{13} &= d(\cos \phi \cos \theta)(-\sin \phi \cos \theta) + d(\cos \phi \sin \theta)(-\sin \phi \sin \theta) + d(\sin \phi)(\cos \phi) \\
&= (-\sin \phi \cos \theta d\phi - \cos \phi \sin \theta d\theta)(-\sin \phi \cos \theta) \\
&\quad + (-\sin \phi \sin \theta d\phi + \cos \phi \cos \theta d\theta)(-\sin \phi \sin \theta) + (\cos \phi d\phi)(\cos \phi) \\
&= (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi)d\phi \\
&\quad + (\sin \phi \cos \phi \sin \theta \cos \theta - \sin \phi \cos \phi \sin \theta \cos \theta)d\theta \\
&= (\sin^2 \phi + \cos^2 \phi)d\phi = d\phi, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\omega_{23} &= d(-\sin \theta)(-\sin \phi \cos \theta) + d(\cos \theta)(-\sin \phi \sin \theta) \\
&= (-\cos \theta d\theta)(\sin \phi \cos \theta) + (\sin \theta d\theta)(\sin \phi \sin \theta) \\
&= (\cos^2 \theta \sin \phi + \sin^2 \theta \sin \phi)d\theta = \sin \phi d\theta.
\end{aligned}$$

5. One notes that

$$\nabla_V W = \Sigma_i (\nabla_V f_i E_i) = \Sigma_i (V[f_i] E_i + f_i (\nabla_V E_i))$$

(by standard properties of covariant derivatives)

$$= (\Sigma_i V[f_i] E_i) + \Sigma_i (f_i \Sigma_j \omega_{ij} E_j)$$

(by definition of ω_{ij})

$$= (\Sigma_j V[f_j] E_j) + \Sigma_{i,j} (f_i \omega_{ij} E_j)$$

$$= \Sigma_j \{V[f_j] + \Sigma_i f_i \omega_{ij}(V)\} E_j$$

(by regrouping) as desired (note that O'Neill's formula states $V[f_i]$ instead of $V[f_j]$, but the only thing left to sum over at this point is a j).

6. At the end of this section, it was computed that $\nabla_V E_1 = V[\theta]E_2 = E_2$ (by assumption), $\nabla_V E_2 = -V[\theta]E_1 = -E_1$, and $\nabla_V E_3 = 0$.

Also, $V(r \cos \theta) = V(r) \cos \theta + rV(\cos \theta) = r \cos \theta - r \sin \theta$ while $V(r \sin \theta) = V(r) \sin \theta + rV(\sin \theta) = r \sin \theta + r \cos \theta$ (note $V(r) = 1$ and $V(\theta) = 0$).

Therefore, by the covariant derivative formula from the previous problem, $\nabla_V(r \cos \theta E_1 + r \sin \theta E_3)$ has E_1 component $V[r \cos \theta] + (r \cos \theta \omega_{11}(V) + 0 * \omega_{21}(V) + r \sin \theta \omega_{31}(V)) = -r \sin \theta + r \cos \theta$ (note the ω terms vanish as the only nonzero one is multiplied by zero),

E_2 component

$$V(0) + (r \cos \theta \omega_{12}(V) + 0 * \omega_{22}(V) + r \sin \theta \omega_{32}(V)) = r \cos \theta d\theta(V)$$

(as the only nonzero ω , ω_{12} , equals $d\theta$) $= r \cos \theta V(\theta) = r \cos \theta$,
and E_3 component

$$V(r \sin \theta) + (r \cos \theta \omega_{13}(V) + 0 * \omega_{23}(V) + r \sin \theta \omega_{33}(V)) = r \sin \theta + r \cos \theta$$

as the ω terms all vanish. This gives that $\nabla_V(r \cos \theta E_1 + r \sin \theta E_3)$ is equal to

$$(-r \sin \theta + r \cos \theta)E_1 + r \cos \theta E_2 + (r \sin \theta + r \cos \theta)E_3.$$

8. In this case, Exercise 5.5(a) tells us that for any vector field W , $\nabla_T(W) = W'$ along our curve. To find ω_{1i} for $i = 1, 3$ we want to find $\nabla_T(E_1) = \nabla_T(T) = T' = \kappa N = \kappa E_2$ which tells us that $\omega_{12}(T) = \kappa$, $\omega_{13}(T) = 0$. For ω_{23} note $\nabla_T(E_2) = \nabla_T(N) = N' = -\kappa T + \tau B = -\kappa E_1 + \tau E_3$ so $\omega_{23} = \tau$.

Now, assuming these connection equations, we note that

$$T' = \nabla_T(T) = \nabla_T(E_1) = \sum_i \omega_{1i}(T)E_i = \kappa N$$

(the first Frenet equation; note that $\omega_{12}(T) = \kappa$ and $\omega_{13}(T) = 0$ by assumption whereas $\omega_{11}(T) = 0$ by Lemma 7.1),

$$N' = \nabla_T(N) = \nabla_T(E_2) = \sum_i \omega_{2i}(T)E_i = -\kappa T + \tau B$$

(the second Frenet equation; note that $\omega_{21}(T) = -\omega_{12}(T) = -\kappa$ by Lemma 7.1, $\omega_{22}(T) = 0$ by Lemma 7.1, and $\omega_{23}(T) = \tau$ by assumption), and

$$B' = \nabla_T(B) = \nabla_T(E_3) = \sum_i \omega_{3i}(T)E_i = -\tau N$$

(the third Frenet equation; note that $\omega_{31}(T) = -\omega_{13}(T) = 0$ by Lemma 7.1, $\omega_{32}(T) = -\omega_{23}(T) = -\tau$ by Lemma 7.1, and $\omega_{33}(T) = 0$ by Lemma 7.1).