Problems graded: 1.1.4(a) (10 points), 1.2.5 (20 points; 10 for each part), 1.3.1(c) (10 points), 1.4.9 (10 points); all other problems were worth 5 points each for completion (except that 1.2.3bde and 1.3ef were 10 points each for completion)

Section 1.1

1. (c) As $f = x^2 y$ and $g = y \sin z$, $fg = x^2 y^2 \sin z$ so $\frac{\delta(fg)}{\delta z} = x^2 y^2 \cos z$ and therefore

$$\frac{\delta^2(fg)}{\delta y \delta z} = \frac{\delta(\frac{\delta(fg)}{\delta z})}{\delta y} = 2x^2 y \cos z.$$

4. (a) By the Chain Rule,

$$\frac{\delta f}{\delta x} = \frac{\delta h}{\delta g_1}\frac{\delta g_1}{\delta x} + \frac{\delta h}{\delta g_2}\frac{\delta g_2}{\delta x} + \frac{\delta h}{\delta g_3}\frac{\delta g_3}{\delta x}$$

$$= 2g_1 * 1 - g_3 * 0 - g_2 * 1 = 2(x+y) * 1 - y^2 * 1 = 2x + 2y - y^2$$

(note $h(g_1, g_2, g_3) = g_1^2 - g_2 * g_3$ while $g_1 = x + y$, $g_2 = y^2$, and $g_3 = x + z$.)

Section 1.2

3. (b) $(p_1, p_3 - p_1, 0) = p_1(1,0,0)_p + (p_3 - p_1)(0,1,0)_p + 0(0,0,1)_p = p_1 U_1(p) + (p_3 - p_1)U_2(p) + 0U_3(p)$.

(d) At a given point $p$, $V(p) = (1 + p_1, p_2 p_3, p_2) - (p_1, p_2, p_3) = (1, p_2 p_3 - p_2, p_2 - p_3) = 1U_1(p) + (p_2 p_3 - p_2)(p) + (p_2 - p_3)U_3(p)$.

(e) As the vector from $p$ to the origin is simply $-p$, we have that $V(p) = (-p_1, -p_2, -p_3) = -p_1 U_1(p) + (-p_2)U_2(p) + (-p_3)U_3(p)$.

5. (a) Suppose $V_1(p), V_2(p)V_3(p)$ were linearly dependent at a point $p = (x, y, z)$. Then there would exist scalars $a, b, c$ (not all zero) with

$$0 = aV_1 + bV_2 + cV_3 = a(U_1 - xU_3) + bU_2 + c(xU_1 + U_3)$$

which implies that $a + cx = 0$, $b = 0$, and $-ax + c = 0$. Multiplying the first equation by $a$, multiplying the third equation by $c$, and adding tells us $a^2 + c^2 = 0$ so $a = c = 0$ contradicting the assumption of linear independence; therefore, these vectors are indeed linearly independent.

(b) We note that $U_1 = \frac{1}{1+x^2}V_1 + \frac{x}{1+x^2}V_3$, $U_2 = V_2$, and $U_3 = \frac{-x}{1+x^2}V_1 + \frac{1}{1+x^2}V_3$ so $xU_1 + yU_2 + zU_3 = \frac{x - xz}{1+x^2}V_1 + yV_2 + \frac{x^2+z}{1+x^2}V_3$.

Section 1.3

1. (c) We note that

$$v_p(f) = \frac{d}{dt}(f(v+tp))|_{t=0} = \frac{d}{dt}f(2+2t, 0-1t, -1+3t)|_{t=0}$$

$$= \frac{d}{dt}(e^{2+2t}\cos(-t))|_{t=0} = 2e^2.$$

3. (e) We note that $V[f^2+g^2] = 2fV[f] + 2gV[g]$ (by Corollary 3.4; note that $V[f](p) = \frac{d}{dt}((x+ty^2)y)|_{t=0} = y^3$ while $V[g](p) = \frac{d}{dt}(z-tx)^3|_{t=0} = -3xz^2$) $= 2xy * y^3 + 2z^3 * -3xz^2 = 2xy^4 - 6x^5$.

(f) As $V[f] = y^3$ we have that $V[V[f]](p) = V[y^3](p) = \frac{d}{dt}y^3|_{t=0} = 0$.

4. Suppose $V \neq \Sigma V[x_i]U_i$. By the linear independence of $U_1(p), U_2(p)$, and $U_3(p)$ at each point, there exists a point $p$ such that writing $V(p) = \Sigma a_i U_i(p)$ where $a_i$ is a scalar for $i = 1, 2, 3$ there exists an $i \in \{1, 2, 3\}$ with $a_i \neq V[x_i](p)$. However, by Lemma 3.2, we have that for each such $i$, $V[x_i](p) = V_p[x_i] = a_i$ producing a contradiction so we indeed have $V = \Sigma V[x_i]U_i$.

5. This follows directly from 4: if $V[f] = W[f]$ for every function $f$, $V[x_i] = W[x_i]$ for $i = 1, 2, 3$ so $V = \Sigma V[x_i]U_i = \Sigma W[x_i]U_i = W$.

Section 1.4

2. In order for $\alpha'(t) = (t^2, t, e^t)$ we need $\alpha(t) = (\frac{t^3}{3}, \frac{t^2}{2}, e^t) + \alpha(0)$; as $\alpha(0) = (1, 0, 5)$ we conclude $\alpha(t) = (\frac{t^3}{3} + 1, \frac{t^2}{2}, e^t + 5)$.

3. If $0 < s < 1$, we know from the identity $\sin^2 x + \cos^2 x = 1$ that $\sin(\cos^{-1} s) = \sqrt{1-s^2}$. Also, from the identity $\cos 2x = 1 - 2\sin^2 x$ (which gives us that $\sin^2(x/2) = \frac{1-\cos x}{2}$) we know that $\sin(\frac{\cos^{-1} s}{2}) = \sqrt{\frac{1-s}{2}}$. From this, because $\alpha(t) = (1+\cos t, \sin t, 2\sin\frac{t}{2})$ and $h(s) = \cos^{-1}(s)$ on $(0, 1)$, we conclude

$$\beta(s) = \alpha \circ h(s) = (1+s, \sqrt{1-s^2}, \sqrt{2-2s}).$$

6. Suppose $\alpha$ is a curve with $\alpha(0) = p$ and $\alpha'(0) = v_p$ and $\beta$ is the straight line with $\beta(t) = p + tv_p$. Because $\alpha'(0) = \beta'(0)$ as tangent vectors, Lemma 4.6 tells us that $\frac{d(f(\alpha))}{dt}(0) = \frac{d(f(\beta))}{dt}(0)$. As the RHS is the definition of the directional derivative of $f$ at $p$ in the direction of $v_p$, the LHS (which is the same thing with the straight line $\beta$ replaced by $\alpha$) evaluates to the same value so we can indeed replace the straight line $\beta$ with $\alpha$.

9. Because $\alpha(t) = (2\cos t, 2\sin t, t)$ and $\alpha'(t) = (-2\sin t, 2\cos t, 1)$, the tangent line to the helix $\alpha(t)$ at a fixed $\alpha(t_0)$ is simply $(2\cos t_0 - 2u\sin t_0, 2\sin t_0 + 2u\cos t_0, t_0 + u_0)$. At $t_0 = 0$ this becomes $(2, 2u, u)$; at $t_0 = \frac{\pi}{4}$ this becomes $(\sqrt{2} - u\sqrt{2}, \sqrt{2} + u\sqrt{2}, \frac{\pi}{4} + u)$.