UNRAMIFIED ELEMENTS IN CYCLE MODULES

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ABSTRACT. Let X be an algebraic variety over a field F. We study the functor taking a cycle module M over F to the group of unramified elements $M\big(F(X)\big)_{nr}$ of $M\big(F(X)\big)$. We prove that this functor is represented by a cycle module. The existence of pull-back maps on $M\big(F(X)\big)_{nr}$ for rational maps (under a mild condition) is established. An application to the R-equivalence on classifying varieties of algebraic groups is given.

The unramified Galois cohomology of the function field of a smooth proper variety X over a field is a birational invariant of X, so it can be used to detect non-rationality of algebraic varieties (cf. [2]). Galois cohomology is a special case of a cycle module that M. Rost developed in [7] more generally. The group of unramified elements can be defined in the context of cycle modules, and it still represents a birational invariant of a smooth proper variety (cf. [7, Cor. 12.10]). We prove that for any smooth proper variety X over a field F, there is a universal cycle module K^X representing the functor that takes a cycle module M over F to the subgroup of unramified elements $M\big(F(X)\big)_{nr}$ of $M\big(F(X)\big)$. The cycle module K^X is defined by means of algebraic cycles of dimension zero on X. As a corollary, we show (cf. Theorem 2.11) that $M\big(F(X)\big)$ has only trivial unramified elements for all cycle modules M over F, i.e., the natural homomorphism $M(F) \to M\big(F(X)\big)_{nr}$ is an isomorphism if and only if the degree map $\mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism for every field extension L/F, where $\mathrm{CH}_0(X_L)$ is the Chow group of zero-dimensional cycles on X_L modulo rational equivalence.

For a fixed cycle module M, we study functorial properties of the assignment $X \mapsto M\big(F(X)\big)_{nr}$ for all varieties X, not necessarily smooth or proper. We show that under mild restrictions on a rational morphism $f : Y \dashrightarrow X$, there exists a pull-back homomorphism $f^* : M\big(F(X)\big)_{nr} \to M\big(F(Y)\big)_{nr}$. We use this construction to show that the R-equivalence on the set of isomorphism classes of torsors of an algebraic group G dominates unramified elements of the function field of a classifying variety of G (cf. Proposition 4.1). As an example, we show that the classifying variety of the group $\mathbf{SL}_8/\boldsymbol{\mu}_2$ has only trivial unramified elements in any cycle module.

In this paper the word "scheme" means a localization of a separated scheme of finite type over a field and a "variety" an integral scheme. For a scheme

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X and an integer $i \geq 0$, we let $X^{(i)}$ and $X_{(i)}$ denote the set of points of X of codimension and dimension i respectively. An algebraic group over a field F is a smooth group scheme of finite type over F.

1. Models and valuations

Let L/F be a finitely generated field extension. A *model of* L/F is a proper variety X over F together with an isomorphism $\varphi : F(X) \xrightarrow{\sim} L$ of fields over F. A *morphism* of models (X, φ) and (X', φ') is a morphism $f : X \to X'$ over F such that the composition

$$F(X') \xrightarrow{f^*} F(X) \xrightarrow{\varphi} L$$

coincides with φ'. There exists at most one morphism between two models of L/F. We write $X \succ X'$ if there is a morphism from (X, φ) to (X', φ') and say that X *dominates* X'. The relation $\succ$ yields an ordering on the set of isomorphism classes of all models of L/F. This is a directed ordered set. Indeed let X and X' be two models of L/F. Choose a nonempty scheme Y over F isomorphic to open subschemes of X and X' and let X'' be the closure of the image of the diagonal embedding of Y into $X \times X'$. The projections of X'' onto X and X' yield $X \prec X'' \succ X'$.

Let v be a valuation on L over F and X a model of L/F. The valuation v dominates a unique point $x \in X$, i.e., $O_{X,x} \subset O_v$ and $\mathfrak{m}_{X,x} = O_{X,x} \cap \mathfrak{m}_v$, where $\mathfrak{m}_v$ is the maximal ideal of the valuation ring O_v of v. In particular, we have an extension of residue fields $F(x) \subset F(v)$. If X' is another model dominating X and $x' \in X'$ is the point dominated by v then x' is *above* x, i.e., x' maps to x under the morphism $X' \to X$. Moreover, $O_{X,x} \subset O_{X',x'} \subset O_v$ and $F(x) \subset F(x') \subset F(v)$.

Lemma 1.1. *Let v be a valuation on L/F, X is a model of L/F and $f \in O_v$. Then there is a model $X' \succ X$ and a point $x' \in X'$ above x satisfying $f \in O_{X',x'}$.*

Proof. We can view f as a rational morphism from X to the projective line $\mathbb{P}^1$ over F. Let X' be closure of the graph of f in $X \times \mathbb{P}^1$. We have projections $X' \to X$ and $X' \to \mathbb{P}^1$, so $X' \succ X$ and we can view f (and also $1/f$) as a morphism $X' \to \mathbb{P}^1$. Let v dominate $x' \in X'$. As $f \in O_v$, we have $1/f \notin \mathfrak{m}_v$. Hence $1/f \notin \mathfrak{m}_{X',x'}$, i.e., $(1/f)(x') \neq 0$. Therefore, $f(x') \neq \infty$ and $f \in O_{X',x'}$. □

It follows from Lemma 1.1 that O_v is the union of the rings $O_{X,x}$ over all models X of L/F and $x \in X$ dominated by v. As the set of all models of L is directed, we have the following:

Corollary 1.2. *Let v be a valuation on L over F with residue field finitely generated over F. Then there is model X of L/F such that $F(x) = F(v)$ for the point $x \in X$ dominated by v.*

Corollary 1.3. *Let v and v' be distinct valuations of L/F. Then there is a model X of L/F such that v and v' dominate distinct points in X.*

Proof. We may assume that $O_v \not\subseteq O_{v'}$. Choose $f \in O_v$ such that $f \notin O_{v'}$. By Lemma 1.1, there is a model X of L/F with $f \in O_{X,x}$, where $x \in X$ is dominated by v. If v' dominates $x' \in X$ then $f \notin O_{X,x'}$, therefore, $x \neq x'$. □

Let v be a valuation on L over F with residue field E and let u be a valuation on E over F. The pre-image of the valuation ring O_u under the natural homomorphism $O_v \to E$ is a valuation ring of L. We write $u \circ v$ for the corresponding valuation on L over F and call it the *composition of v and u*.

Let v and v' be two valuations on L/F. We say that v *divides* v' and write $v|v'$ if $O_{v'} \subset O_v$. We have $v|v'$ if and only if $v' = u \circ v$ for a (unique) valuation u on $F(v)$.

Let L/F be a finitely generated field extension and v a valuation on L over F with residue field E. The transcendence degree $\operatorname{tr.deg}_F(E)$ is called the *dimension* $\dim(v)$ *of* v. The *rank* $\operatorname{rank}(v)$ *of* v is the largest integer r such that there is a sequence of distinct valuations $v_1|v_2|\dots|v_r = v$ on L over F. By [9, Ch. VI, Th. 3, Cor. 1], we have

$$\operatorname{rank}(v) + \dim(v) \leq \operatorname{tr.deg}_F(L).$$

A valuation v is called *geometric* if $\operatorname{rank}(v) + \dim(v) = \operatorname{tr.deg}_F(L)$.

Proposition 1.4. *Let L/F be a finitely generated field extension and E a field with $F \subset E \subset L$. A geometric discrete valuation v on L of rank 1 is either trivial on E or restricts to a geometric discrete valuation on E of rank 1.*

Proof. Let w be the restriction of v on E. Clearly, $\operatorname{rank}(w) \leq 1$. By [9, Ch. VI, Lemma 2, Cor. 1], we have

$$\dim(v) - \dim(w) \leq \operatorname{tr.deg}_E(L) = \operatorname{tr.deg}_F(L) - \operatorname{tr.deg}_F(E).$$

It follows that $\dim(w) \geq \operatorname{tr.deg}_F(E) - 1$ and hence w is either trivial or a geometric discrete valuation of rank 1. □

Proposition 1.5. [9, Ch. VI, Th. 31, Cor.] *The residue field E of a geometric valuation v on L over F is a finitely generated field extension of F with* $\operatorname{tr.deg}_F(E) = \operatorname{tr.deg}_F(L) - \operatorname{rank}(v)$. *Every geometric valuation of rank r is a unique composition of r geometric discrete valuations of rank 1.*

Example 1.6. Let x be a regular point of codimension r of a variety X. Let $a_1, a_2, \dots, a_r$ be regular parameters of the regular local ring $O_{X,x}$. For every $i = 0, 1, \dots, r$, let R_i be the factor ring of R by the ideal generated by $a_1, \dots, a_i$ and let L_i be the quotient field of R_i. We have $L_0 = F(X)$ and $L_r = F(x)$. Denote by v_i the discrete valuation on L_i with residue field L_{i+1} and set $v = v_r \circ \dots \circ v_2 \circ v_1$. Then v is a geometric valuation of rank r on $F(X)$ with residue field $F(x)$.

Proposition 1.7. *Let v be a discrete valuation of rank 1 on a finitely generated field extension L over F. Then the following conditions are equivalent:*

(1) *The valuation v is geometric.*
(2) *There is a normal model X of L/F such that the point x dominated by v is (regular) of codimension 1 and $O_v = O_{X,x}$.*

Proof. (1) $\Rightarrow$ (2): By Proposition 1.5, $F(v)/F$ is a finitely generated field extension. It follows from Corollary 1.2, that there is a model X such that $F(v) = F(x)$ for the point x dominated by v. By assumption, $\operatorname{tr.deg}_F\big(F(v)\big) = \operatorname{tr.deg}_F(L) - 1$, i.e., x is of codimension 1 in X. Replacing X by its normalization in L, we may assume that X is normal and hence x is regular. Therefore, $O_{X,x}$ is a DVR that is contained in the DVR O_v. Hence $O_v = O_{X,x}$.

(2) $\Rightarrow$ (1): It follows from the equality $\operatorname{tr.deg}_F\big(F(v)\big) = \operatorname{tr.deg}_F\big(F(x)\big) = \dim x = \dim X - 1 = \operatorname{tr.deg}_F(L) - 1$ that v is geometric. $\square$

Lemma 1.8. *Let $v|v'$ be two geometric valuations on L/F of rank 1 and 2 respectively. Write $v' = u \circ v$ for a geometric discrete valuation u on $F(v)$ of rank 1. Then there is a model X of L/F such that v and v' dominate points x and x' of codimension 1 and 2 respectively with:*

(1) *$F(x) = F(v)$ and $F(x') = F(v')$,*
(2) *v is the only valuation on L/F dominating x,*
(3) *u is the only valuation on $F(v)/F$ dominating x'.*

Proof. By Proposition 1.7 and Corollary 1.2, there is a model X of L/F such that (1) holds with x a regular point of X of codimension 1 and $O_{X,x} = O_v$. Hence (2) also holds. There are finitely many geometric discrete valuations $u = u_1, u_2, \dots, u_n$ on $F(x)$ over F dominating x' in the closure of $\{x\}$ in X. By Corollary 1.3, there is model $X' \succ X$ such that the valuations $u_1 \circ v, u_2 \circ v, \dots, u_n \circ v$ dominate distinct points in X'. Clearly, X' satisfies all the conditions. $\square$

2. Cycle modules

Cycle modules were introduced by M. Rost in [7]. By definition, a ($\mathbb{Z}$-graded) *cycle module over F* is a functor $M = M_*$ from the category of finitely generated field extensions over F to the category of graded abelian groups equipped with the following datum:

(1) A *norm homomorphism* $N_{L/E} : M_*(L) \to M_*(E)$ for any finite field extension L/E of finitely generated field extensions of F,
(2) A structure of a left graded module on $M_*(L)$ over the Milnor ring $K_*(L)$ for any finitely generated field extension L/F,
(3) A *residue homomorphism* $\partial_v : M_*(L) \to M_{*-1}(E)$ for any finitely generated field extension L/F and a geometric discrete valuation v on L over F of rank 1 with residue field E.

All the structures should satisfy various compatibility conditions (cf. [7, §2]). We write K_* for the cycle module taking a field L to the Milnor K-group $K_*(L)$.

Let M be a cycle module over F. For an element $\rho \in M(E)$ and a field extension L/E over F we write ρ_L for the image of ρ in $M(L)$.

For a cycle module M over F and an integer d let $M[d]$ denote the *shifted cycle module* defined by $M[d]_n(L) = M_{n+d}(L)$.

A *morphism of cycle modules* M *and* N *of degree* d is a morphism of functors $M \to N[d]$ commuting with the structures (1), (2) and (3). All cycle modules over F and morphisms of cycle modules of arbitrary degree form the *category of cycle modules* $\mathrm{CM}(F)$. We write $\mathrm{Hom}^d_{\mathrm{CM}(F)}(M,N)$ for the group of all degree d morphisms of cycle modules M and N.

Let L/F be a finitely generated field extension, v a geometric discrete valuation on L over F of rank 1 with residue field E and $\pi \in L$ a uniformizer, i.e., $v(\pi) = 1$. The map

$$s_v^\pi : M_*(L) \to M_*(E), \qquad s_v^\pi(\alpha) = \partial_v\big(\{-\pi\} \cdot \alpha\big)$$

is called a *specialization homomorphism.*

Let X be a scheme and M a cycle module over F. In [7, §2], using the norm and residue maps, Rost constructed a complex

$$\ldots \to \coprod_{x \in X_{(i+1)}} M_{d-i+1}\big(F(x)\big) \to \coprod_{x \in X_{(i)}} M_{d-i}\big(F(x)\big) \to \coprod_{x \in X_{(i-1)}} M_{d-i-1}\big(F(x)\big) \to \ldots.$$

If $x \in X_{(i)}$, the x-component ∂_x of the differential of the complex is defined as follows. If $x' \in X_{(i-1)}$ the x'-component $\partial_x^{x'}$ of ∂_x is trivial if x' does not belong to the closure of $\{x\}$ and is equal to $\sum N_{F(v)/F(x')} \circ \partial_v$ otherwise, where the sum is taken over all geometric discrete valuations v of $F(x)$ of rank 1 dominating x'.

We shall consider the following homology group of the complex:

$$A_0(X, M_d) := \mathrm{Coker}\Big(\coprod_{x \in X_{(1)}} M_{d+1}\big(F(x)\big) \to \coprod_{x \in X_{(0)}} M_d\big(F(x)\big)\Big).$$

In particular, $A_0(X, K_d) = \mathrm{CH}_0(X)$.

If X is equidimensional, we set

$$A^0(X, M_d) := \mathrm{Ker}\Big(\coprod_{x \in X^{(0)}} M_d\big(F(x)\big) \to \coprod_{x \in X^{(1)}} M_{d-1}\big(F(x)\big)\Big).$$

If X is a variety, we have $A^0(X, M_d) \subset M_d\big(F(X)\big)$.

2.1. **Gysin homomorphism.** Let M be a cycle module over a field F, $f : Y \to X$ a closed regular embedding of equidimensional schemes and $p : N \to Y$ the normal bundle of f. The *Gysin homomorphism* $f^\star$ is defined as the composition

$$A^0(X, M) \xrightarrow{\sigma} A^0(N, M) \xrightarrow{(p^*)^{-1}} A^0(Y, M),$$

where σ is the deformation homomorphism (denoted by J in [7, §11]) and p^* is the pull-back isomorphism with respect to the flat morphism p (cf. [7, §4]).

Let x be a point of a variety X. We have

$$A^0(X,M)\subset A^0(\operatorname{Spec}O_{X,x},M)\subset M\big(F(X)\big).$$

If x is a regular (non-singular) point of X then the closed embedding $f:\operatorname{Spec}F(x)\to\operatorname{Spec}O_{X,x}$ is regular. For any $\rho\in A^0(X,M)$, we write $\rho(x)$ for the element $f^\bigstar(\rho)$ in $A^0(F(x),M)=M\big(F(x)\big)$ and call it the *value of* ρ *at* x.

Example 2.1. Let x be the generic point of a variety X. As $O_{X,x}=F(x)=F(X)$, we have $\rho(x)=\rho$ for any $\rho\in A^0(X,M)$.

Let $f:Y\to X$ be a morphism of equidimensional schemes over F with X smooth. The morphism f factors into the composition of the regular embedding $g=(\mathrm{id},f):Y\to X\times Y$ and the flat projection $h:X\times Y\to X$ of constant relative dimension. The pull-back homomorphism

$$f^*:A^0(X,M)\to A^0(Y,M)$$

is defined as the composition $f^*=g^\bigstar\circ h^*$.

Lemma 2.2. *Let $i:Y'\to Y$ be a regular closed embedding of equidimensional schemes and $f:Y\to X$ a morphism with X equidimensional and smooth. Then $(f\circ i)^*=i^\bigstar\circ f^*$.*

Proof. Consider the factorizations $f=h\circ g$ and $f\circ i=h'\circ g'$ as above. In the diagram

$$\begin{array}{ccccc}
A^0(X,M) & \xrightarrow{h^*} & A^0(Y\times X,M) & \xrightarrow{g^\bigstar} & A^0(Y,M)\\
\Big\| & & \Big\downarrow{\scriptstyle (i\times 1_X)^\bigstar} & & \Big\downarrow{\scriptstyle i^\bigstar}\\
A^0(X,M) & \xrightarrow{h'^*} & A^0(Y'\times X,M) & \xrightarrow{g'^\bigstar} & A^0(Y',M)
\end{array}$$

the left square is commutative by [7, Lemma 11.4] and the right square is commutative by the functoriality of Gysin homomorphisms (cf. [3, Lemme 3.10]). □

2.2. **Unramified elements.** Let M be a cycle module over F and let L/F be a finitely generated field extension. An element $a\in M(L)$ is called *unramified* if $\partial_v(a)=0$ for all geometric discrete valuations v on L of rank 1 over F. We write $M(L)_{nr}$ for the subgroup of all unramified elements. For any $b\in M(F)$, the element b_L in $M(L)$ is unramified by [7, R3c], hence we have a canonical homomorphism $M(F)\to M(L)_{nr}$. If this map is as an isomorphism, we simply write $M(L)_{nr}=M(F)$ and say that $M(L)$ *has only trivial unramified elements.* We also say that L *has trivial unramified cohomology* if $M(L)$ has only trivial unramified elements for every cycle module M over F.

Example 2.3. A purely transcendental field extension L/F has trivial unramified cohomology. Indeed, $L\simeq F(\mathbb{P}^n)$ for some n and $M(L)_{nr}=A^0(\mathbb{P}^n,M)=M(F)$ by [7, Prop. 8.6] for any cycle module M over F.

Let X be a variety over F. We have

$$M\big(F(X)\big)_{nr} \subset A^0(X, M),$$

and equality holds if X is smooth and proper (cf. [7, §12]).

Let L'/L be an extension of finitely generated fields over F. It follows from Proposition 1.4 and [7, R3a, R3c] that the restriction homomorphism $M(L) \to M(L')$ takes $M(L)_{nr}$ to $M(L')_{nr}$.

Lemma 2.4. *Let L/F be a finitely generated field extension, v a geometric discrete valuation on L of rank 1 over F with residue field E. Let M be a cycle module over F. Then the restriction of the specialization homomorphism s_v^π on $M(L)_{nr}$ does not depend on the choice of the prime element π.*

Proof. Let $\pi' = u\pi$ for some $u \in L^\times$ with $v(u) = 0$. By [7, R3e], we have

$$s_v^{\pi'}(\rho) - s_v^\pi(\rho) = \partial_v\big(\{-\pi'\}\rho\big) - \partial_v\big(\{-\pi\}\rho\big) = \partial_v\big(\{u\}\rho\big) = -\{\bar{u}\}\partial_v\big(\rho\big) = 0$$

for any $\rho \in M(L)_{nr}$. □

We write s_v for the restriction of the specialization homomorphism s_v^π on $M(L)_{nr}$.

2.3. **The cycle module M^X.** Let M be a cycle module and X a scheme over F. By [7, Th. 7.1], the assignment

$$M^X(L) := A_0(X_L, M)$$

defines the cycle module M^X over F. In particular, $K_0^X(L) = \mathrm{CH}_0(X_L)$ and $M^{\operatorname{Spec} F} = M$.

If X is proper, the norm morphisms in Milnor K-theory induce a natural morphism

$$\mathrm{Norm}_X : M^X \to M.$$

If a smooth proper X is stably rational then Norm_X is an isomorphism.

We now turn to the case $M = K$. For every point $x \in X$, the group $K_0^X\big(F(x)\big)$ has a distinguished element ξ_x – the class of the image of the diagonal embedding $\operatorname{Spec} F(x) \to X_{F(x)}$.

Suppose that X is a variety. If x is the generic point, we simply write ξ for $\xi_x \in K_0^X\big(F(X)\big) = \mathrm{CH}_0(X_{F(X)})$. Trivially, ξ is an unramified element in $K_0^X\big(F(X)\big)$.

The next lemma follows from [7, Cor. 12.4].

Lemma 2.5. *For any $x \in X$, we have $\xi(x) = \xi_x$.*

Lemma 2.6. *The group $K^X(F) = A_0(X, K)$ is generated by the elements of the form $N_{F(x)/F}(a \cdot \xi_x)$ over all $x \in X_{(0)}$ and $a \in K\big(F(x)\big)$.*

Proof. This follows from the fact that the norm $N_{F(x)/F}(a \cdot \xi_x)$ in $A_0(X, K)$ is represented by $a \in K\big(F(x)\big) \subset \coprod_{x \in X_{(0)}} K\big(F(x)\big)$. □

The following statement shows that a morphism $K^X \to M$ of cycle modules is determined by the value at the generic point of X.

Proposition 2.7. *Let $\alpha : K^X \to M$ be a morphism of cycle modules over F (of arbitrary degree). If $\alpha(\xi) = 0$ then $\alpha = 0$.*

Proof. By Lemma 2.6, it is sufficient to show that $\alpha\big(N_{F(x)/F}(a \cdot \xi_x)\big) = 0$ for all $x \in X_{(0)}$ and $a \in K\big(F(x)\big)$. As α commutes with products, norms and evaluation homomorphisms, by Lemma 2.5,

$$\alpha\big(N_{F(x)/F}(a \cdot \xi_x)\big) = N_{F(x)/F}\big(a \cdot \alpha(\xi)(x)\big) = 0. \qquad \square$$

2.4. **A pairing.** Let X be a smooth proper variety over F, M a cycle module over F and $\rho \in M\big(F(X)\big)_{nr}$. Let $i : C \to X$ be the closed embedding of an integral curve and $f : C' \to C$ the normalization morphism. Let $\rho' \in A^0(C', M) = M\big(F(C)\big)_{nr}$ be the pull-back of ρ with respect to the composition $i \circ f$.

Lemma 2.8. *Let $\rho \in M\big(F(X)\big)_{nr}$, $x \in C_{(0)}$ and $a \in K\big(F(C)\big)$. Then*

$$\partial^C_x(a \cdot \rho') = \partial^C_x(a) \cdot \rho(x)$$

in $M\big(F(x)\big)$.

Proof. Let $y \in C'$ be a point over x. As $\partial^{C'}_y(\rho') = 0$, the Rule [7, R3f] yields:

$$\partial^{C'}_y(a \cdot \rho') = \partial^{C'}_y(a) \cdot s^\pi_y(\rho'), \tag{1}$$

where s^π_y is the specialization homomorphism for an uniformizer π of y on C'.

It follows from [7, Lemma 11.2] that $s^\pi_y(\rho') = \rho'(y)$. By Lemma 2.2, $\rho'(y)$ is the pull-back of ρ with respect to the composition $\operatorname{Spec} F(y) \to C' \to X$. Since this composition coincides with the composition $\operatorname{Spec} F(y) \to \operatorname{Spec} F(x) \to X$, by transitivity of the pull-back homomorphisms and [7, Prop. 12.3], we have $\rho'(y) = \rho(x)_{F(y)}$. Therefore, it follows from (1) that

$$\partial^{C'}_y(a \cdot \rho') = \partial^{C'}_y(a) \cdot \rho(x)_{F(y)}. \tag{2}$$

We have:

$$\begin{aligned}
\partial^C_x(a \cdot \rho') &= \sum_{y|x} N_{F(y)/F(x)}\big(\partial^{C'}_y(a \cdot \rho')\big) && \text{(definition of } \partial^C_x) \\
&= \sum_{y|x} N_{F(y)/F(x)}\big(\partial^{C'}_y(a) \cdot \rho(x)_{F(y)}\big) && \text{(by (2))} \\
&= \sum_{y|x} N_{F(y)/F(x)}\big(\partial^{C'}_y(a)\big) \cdot \rho(x) && \text{(by [7, R2c])} \\
&= \partial^C_x(a) \cdot \rho(x). && \text{(definition of } \partial^C_x) \qquad \square
\end{aligned}$$

The lemma together with the Reciprocity Law [7, Prop. 2.2] yield:

Corollary 2.9. *Let $\rho \in M(F(X))_{nr}$ and $C \subset X$ a closed curve. Then for any $a \in K\big(F(C)\big)$, we have*

$$\sum_{x \in C_{(0)}} N_{F(x)/F}\big(\partial^C_x(a) \cdot \rho(x)\big) = 0.$$

We have a pairing

$$\Big(\coprod_{x\in X_{(0)}} K_*F(x)\Big)\otimes M_d(F(X))_{nr}\to M_{*+d}(F)$$

defined by $\sum a_x\otimes\rho_x\mapsto\sum N_{F(x)/F}\big(a_x\cdot\rho_x(x)\big)$. By Corollary 2.9, this pairing factors through a pairing

$$A_0(X,K_*)\otimes M_d(F(X))_{nr}\to M_{*+d}(F). \tag{3}$$

The pairing (3) over field extensions of F yield a homomorphism

$$\varphi_{X,M}:M_d(F(X))_{nr}\to \operatorname{Hom}^d_{\mathrm{CM}(F)}(K^X,M).$$

A morphism $\alpha:K^X\to M$ of cycle modules of degree d induces a homomorphism

$$A^0(X,K_0^X)\to A^0(X,M_d)=M_d\big(F(X)\big)_{nr}.$$

We denote the image of ξ in $M_d\big(F(X)\big)_{nr}$ by $\psi(\alpha)$. Thus we get a homomorphism

$$\psi_{X,M}:\operatorname{Hom}^d_{\mathrm{CM}(F)}(K^X,M)\to M_d\big(F(X)\big)_{nr}.$$

Theorem 2.10. *For every smooth proper variety X and cycle module M over F, the maps $\varphi_{X,M}$ and $\psi_{X,M}$ are isomorphisms inverse to each other. In other words, for a fixed X, the functor $M\mapsto M\big(F(X)\big)_{nr}$ from the category* CM(F) *of cycle modules to the category of abelian groups is represented by the cycle module K^X.*

Proof. The statement is obvious in the case $X=\operatorname{Spec}F$, i.e., the top map in the commutative diagram

$$\begin{array}{ccc} M_d(F) & \xrightarrow{\varphi_{SpecF,M}} & \operatorname{Hom}^d_{\mathrm{CM}(F)}(K,M)\\ \downarrow & & \downarrow\\ M_d(F(X))_{nr} & \xrightarrow{\varphi_{X,M}} & \operatorname{Hom}^d_{\mathrm{CM}(F)}(K^X,M)\end{array} \tag{4}$$

is an isomorphism. The composition $\psi_{X,M}\circ\varphi_{X,M}$ takes a $\rho\in M_d\big(F(X)\big)_{nr}$ to $\rho(x)$, where x is the generic point of X. By Example 2.1, the latter element is equal to ρ, hence the composition $\psi_{X,M}\circ\varphi_{X,M}$ is the identity. By Proposition 2.7, $\psi_{X,M}$ is injective. Hence $\varphi_{X,M}$ and $\psi_{X,M}$ are isomorphisms inverse to each other. □

Theorem 2.11. *Let X be a smooth proper variety over a field F. Then the following conditions are equivalent:*

1. *For every cycle module M over F, we have $M\big(F(X)\big)_{nr}=M(F)$, i.e., $M\big(F(X)\big)$ has only trivial unramified elements.*
2. *The map* $\operatorname{Norm}_X:K^X\to K$ *is an isomorphism of cycle modules.*
3. *The degree map* $\mathrm{CH}_0(X_L)\to\mathbb{Z}$ *is an isomorphism for every field extension L/F.*

(4) *The class* ξ *in* $\mathrm{CH}_0(X_{F(X)})$ *is defined over* F*, i.e.,* ξ *belongs to the image of the natural map* $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_{F(X)})$*.*

Proof. (1) ⇒ (4): Apply (1) to $M = K^X$.

(4) ⇒ (3): We may assume that $L = F$. In the case $M = K^X$ and $d = 0$ in diagram (4), we have $\varphi_{X,M}(\xi) = \mathrm{id}_M$. By assumption and Theorem 2.10, there is a morphism $\alpha : K \to K^X$ such that $\alpha \circ \mathrm{Norm}_X$ is the identity of K^X. Let $a \in \mathrm{CH}_0(X)$ be the image of 1 under α. For every $b \in \mathrm{CH}_0(X)$, we then have

$$b = \alpha\big(\mathrm{Norm}_X(b)\big) = \deg(b) \cdot \alpha(1) = \deg(b) \cdot a,$$

hence $\mathrm{CH}_0(X)$ is generated by a. Taking the degree, we have $\deg(a) = 1$. Therefore, the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$ is an isomorphism.

(3) ⇒ (2): Let $a \in \mathrm{CH}_0(X)$ be a cycle of degree 1. Then for every $u \in K(F)$, we have $\mathrm{Norm}_X(u \cdot a) = u$, i.e., the norm map $K^X \to K$ is surjective. To prove injectivity, it is sufficient to show that for every $x \in X_{(0)}$ and $u \in K\big(F(x)\big)$, we have $u \cdot [x] = N_{F(x)/F}(u) \cdot a$ in $A_0(X, K)$. As the degree of the cycle $\xi_x \in \mathrm{CH}_0(X_{F(x)})$ is 1 and the degree map $\mathrm{CH}_0(X_{F(x)}) \to \mathbb{Z}$ is injective, we have $\xi_x = a_{F(x)}$, hence $u \cdot \xi_x = u \cdot a_{F(x)}$. Therefore,

$$u \cdot [x] = N_{F(x)/F}(u \cdot \xi_x) = N_{F(x)/F}(u \cdot a_{F(x)}) = N_{F(x)/F}(u) \cdot a.$$

(2) ⇒ (1): In the commutative diagram (4), the vertical maps are isomorphisms. □

2.5. **Specialization of unramified elements.** Let L/F be a finitely generated field extension with $\mathrm{tr.deg}_F(L) = d$. Let M be a cycle module over F and v a geometric discrete valuation of L over F of rank 1 with residue field E.

Proposition 2.12. *For every* $\alpha \in M(L)_{nr}$*, we have* $s_v(\alpha) \in M(E)_{nr}$*.*

Proof. Let u be a geometric discrete valuation of E/F of rank 1. Set $v' = u \circ v$. Choose a model X and two points x and x' of L/F satisfying the conditions of Lemma 1.8. The local 2-dimensional ring $O_{X,x'}$ may not be regular. By Lipman's resolution of singularities in dimension 2 (cf. [1]), there is a regular connected 2-dimensional scheme Y together with a birational isomorphism $f : Y \to \operatorname{Spec} O_{X,x'}$. The valuations v and v' dominate points y and y' of Y over x and x' respectively with $\kappa(y) = \kappa(x) = E$ and $\kappa(y') = \kappa(x') = F(v')$ (we write κ to denote the residue field).

Let Z be the closure of y in Y. As the local ring $O_{Y,y'}$ is regular, it is a UFD. Therefore, the closed embedding $f : \operatorname{Spec} O_{Z,y'} \to \operatorname{Spec} O_{Y,y'}$ is regular. Consider the Gysin homomorphism

$$f^\star : A^0(\operatorname{Spec} O_{Y,y'}, M) \to A^0(\operatorname{Spec} O_{Z,y'}, M).$$

By [7, Cor. 12.4], $f^\star = s_v$ since $O_{Y,y} = O_v$. On the other hand, u is the only valuation on $\kappa(y) = F(v)$ dominating y'. Hence the y'-component $\partial_y^{y'}$ of the differential ∂_y coincides with ∂_u. Finally,

$$\partial_u\big(s_v(\alpha)\big) = \partial_y^{y'}\big(f^{\bigstar}(\alpha)\big) = 0. \qquad \square$$

Let v be a rank r geometric valuation on a field L over F. Write $v = v_1 \circ v_2 \circ \cdots \circ v_r$ where v_i are (unique) geometric discrete valuations of rank 1. We define the *specialization homomorphism*

$$s_v := s_{v_1} \circ \cdots \circ s_{v_r} : M_*(L)_{nr} \to M_*\big(F(v)\big)_{nr}.$$

Let $Z \subset X$ be a closed subvariety of a variety X over F such that the generic point of Z is regular in X. Then the closed embedding $f : \operatorname{Spec} F(Z) \to \operatorname{Spec} O_{X,Z}$ is regular, so we have the Gysin homomorphism

$$f^{\bigstar} : A^0(\operatorname{Spec} O_{X,Z}, M) \to A^0(\operatorname{Spec} F(Z), M).$$

Let v be the geometric valuation on $F(X)$ with residue field $F(Z)$ given by a regular sequence of the regular local ring $\operatorname{Spec} O_{X,Z}$ (cf. Example 1.6). It follows from [7, Cor. 12.4] that $f^{\bigstar} = s_v$. Hence

$$f^{\bigstar}\big(M(F(X))\big)_{nr} \subset M\big(F(Z)\big)_{nr}.$$

Let $g : Y \dashrightarrow Z$ be a dominant rational morphism of varieties over F. The induced embedding of fields $F(Z) \hookrightarrow F(Y)$ induces a homomorphism

$$g^* : M\big(F(Z)\big)_{nr} \to M\big(F(Y)\big)_{nr}.$$

By [7, R1a], the operation $g \mapsto g^*$ is compatible with compositions.

We now consider a more general situation. Let $f : Y \dashrightarrow X$ be a rational morphism of varieties over F satisfying the following condition:

($\diamond$) *The image of the generic point of Y is a regular point of X.*

For example, a dominant morphism satisfies ($\diamond$). The morphism f factors into the composition of the rational dominant morphism $g : Y \dashrightarrow Z$, where Z is the closure of the image of f, and the closed embedding $h : Z \to X$. We define the *pull-back homomorphism* as the composition

$$f^* : M\big(F(X)\big)_{nr} \xrightarrow{h^{\bigstar}} M\big(F(Z)\big)_{nr} \xrightarrow{g^*} M\big(F(Y)\big)_{nr}.$$

The pull-back homomorphisms satisfy the following functorial properties.

Proposition 2.13. *Let $f : Y \dashrightarrow X$ be a rational morphism of varieties over F satisfying the condition ($\diamond$). Let $d : Y' \dashrightarrow Y$ be a dominant rational morphism of varieties over F. Then $f \circ d$ satisfies ($\diamond$) and $(f \circ d)^* = d^* \circ f^*$.*

Proof. Let $f = h \circ g$ be the factorization as above. Then $f \circ d = h \circ (g \circ d)$ and

$$(f \circ d)^* = (g \circ d)^* \circ h^{\bigstar} = d^* \circ (g^* \circ h^{\bigstar}) = d^* \circ f^*. \qquad \square$$

Proposition 2.14. *Let $f : Y \dashrightarrow X$ be a rational morphism of varieties over F satisfying the condition ($\diamond$). Let $e : X \dashrightarrow X'$ be a dominant rational morphism of varieties over F such that the composition $e \circ f$ is well defined and dominant. Then $(e \circ f)^* = f^* \circ e^*$.*

Proof. Let $f = h \circ g$ be the factorization as above. By assumption, e yields an embedding of $F(X')$ into the local ring $O_{X,Z}$. The surjection $O_{X,Z} \to F(Z)$ makes $F(Z)$ a field extension of $F(X')$. We have $h^\star = s_v$ for some geometric valuation v on $F(X)$ dominating Z. The restriction of v on $F(X')$ is trivial. It follows from [7, R3d] that $h^\star(e^*(\rho)) = s_v(\rho_{F(X)}) = \rho_{F(Z)}$ for any $\rho \in M\big(F(X')\big)_{nr}$. Hence

$$(e \circ f)^*(\rho) = \rho_{F(Y)} = g^*(\rho_{F(Z)}) = g^*\big(h^\star(e^*(\rho))\big) = (f^* \circ e^*)(\rho). \qquad \square$$

2.6. **Retract rational.** A variety X over F is called *retract rational* if there are rational morphisms $f : X \dashrightarrow \mathbb{P}^n$ and $g : \mathbb{P}^n \dashrightarrow X$ for some n such that the composition $g \circ f$ is defined and is equal to the identity of X.

The following proposition generalizes Example 2.3.

Proposition 2.15. *Let X be a retract rational over F. Then $F(X)$ has trivial unramified cohomology.*

Proof. Let M be a cycle module over F and f and g the morphisms in the definition of the retract rational X. As $g \circ f$ is the identity, g is a dominant morphism. By Proposition 2.14, $f^* \circ g^*$ is the identity on $M\big(F(X)\big)_{nr}$. Let $p : X \to \operatorname{Spec} F$ be the structure morphism. By Example 2.3, the composition $g^* \circ p^*$ is an isomorphism. Hence g^*, and therefore, p^* are isomorphisms, i.e., $M\big(F(X)\big)_{nr} = M(F)$. $\square$

3. R-equivalence

Let $\mathcal{C}(F)$ be the category of semilocal commutative F-algebras. We consider functors

$$P : \mathcal{C}(F) \to Sets.$$

Example 3.1. Let X be a scheme over F. We view X as a functor via $X(A) = \operatorname{Mor}_F(\operatorname{Spec} A, X)$.

Example 3.2. Let G be an algebraic group over F. We write G-*Torsors* for the functor taking a commutative F-algebra A to the set of isomorphism classes $H^1_{et}(A, G)$ of G-torsors over $\operatorname{Spec} A$.

We write H_F for the semilocal ring of all rational functions $f(t)/g(t) \in F(t)$ such that $g(0)$ and $g(1)$ are nonzero. In other words, H_F is the localization of $F[t]$ at 0 and 1. Let $x \in P(H_F)$ and $i = 0$ or 1. We write $x(i)$ for the image of x under the map $P(H_F) \to P(F)$ induced by the ring homomorphism $H_F \to F$ taking a function h to $h(i)$.

Two elements x_0 and x_1 in $P(F)$ are *strictly R-equivalent* if there is an $x \in P(H_F)$ such that $x(0) = x_0$ and $x(1) = x_1$. The equivalence relation $\sim$ on $P(F)$ generated by the relation of strict R-equivalence is called *R-equivalence.* We write $P(F)/R$ for the set of equivalence classes. If L/F is a field extension, let $P(L)/R$ denote the set of equivalence classes of the restriction of the functor P on $\mathcal{C}(L)$.

Example 3.3. Let X be a scheme over F. We can view X as the functor in Example 3.1. Then the notion of R-equivalence in $X(F)$ coincides with the classical one defined in [5].

We say that a functor $P : \mathcal{C}(F) \to Sets$ is *R-trivial* if for any field extension L/F, the set $P(L)/R$ has only one element. In particular note that $P(L)$ is not empty.

A morphism of functors $P \to Q$ induces a morphism of sets $P(L)/R \to Q(L)/R$ for any L. Call a morphism of functors $P \to Q$ *surjective* if the map $P(A) \to Q(A)$ is surjective for all A in $\mathcal{C}(F)$. If $f : P \to Q$ is a surjective morphism and P is R-trivial then so is Q.

For an element $q \in Q(F)$, we write P_q for the subfunctor of P defined by

$$P_q(A) := \{p \in P(A) \quad \text{such that} \quad f(p) = q_A\}.$$

We call P_q the *fiber of f over q*.

Proposition 3.4. *Let $f : P \to Q$ be a surjective morphism of functors from $\mathcal{C}(F)$ to Sets and L/F a field extension. If the fibers P_q are R-trivial for all $q \in Q(L)$ then the canonical map $P(L)/R \to Q(L)/R$ is a bijection.*

Proof. Since f is surjective, the map $P(L)/R \to Q(L)/R$ is also surjective. Let $p_0, p_1 \in P(L)$ and set $q_i = f(p_i)$. Suppose that q_0 and q_1 are R-equivalent. We may assume that they are simply R-equivalent. There exists $q \in Q(H_L)$ such that $q(0) = q_0$ and $q(1) = q_1$. As f is surjective, there is $p \in P(H_L)$ with $f(p) = q$. Set $p'_0 = p(0)$ and $p'_1 = p(1)$. We have $p'_0 \sim p'_1$. The elements p_0 and p'_0 belong to the same fiber. Since the fibers are R-trivial, we have $p_0 \sim p'_0$ and similarly, $p_1 \sim p'_1$. Finally, $p_0 \sim p'_0 \sim p'_1 \sim p_1$. □

Let G be an algebraic group over F, $C \subset G$ a central subgroup and $G' = G/C$. For every $A \in \mathcal{C}(F)$, we have an exact sequence

$$G\text{-}\mathsf{Torsors}(A) \to G'\text{-}\mathsf{Torsors}(A) \xrightarrow{\partial} H^2_{et}(A, C)$$

with $\partial : H^1_{et}(A, G') \to H^2_{et}(A, C)$ the connecting map. Let P be the subfunctor of G'-$\mathsf{Torsors}$ consisting of all elements p such that $\partial(p) = 0$. The group C-$\mathsf{Torsors}(A)$ acts transitively on the fibers of the natural surjection G-$\mathsf{Torsors}(A) \to P(A)$ (cf. [4, Cor. 28.6]). Therefore, Proposition 3.4 yields:

Corollary 3.5. *Suppose that the functors C-$\mathsf{Torsors}$ and P are R-trivial. Then G-$\mathsf{Torsors}$ is also R-trivial.*

4. Classifying variety

Let G be an algebraic group over F. We view G as a closed subgroup of a group $S = \mathbf{GL}_{n_1} \times \mathbf{GL}_{n_2} \times \cdots \times \mathbf{GL}_{n_k}$ over F. The factor variety S/G is denoted by BG and called a *classifying variety for G*. The variety BG has the distinguished rational point G/G. The stable birational equivalence class of BG is independent of the choice of S and the embedding of G into S (cf. [6, 2.1]).

Let L be a field extension of F. By Hilbert Theorem 90, the map

$$f_L : BG(L) \to G\text{-}\mathsf{Torsors}(L),$$

taking a point $\operatorname{Spec} L \to BG$ in $BG(L)$ to the class of the G-torsor $S \times_{BG} \operatorname{Spec} L$ over L, is surjective. The group $S(L)$ acts on $BG(L)$, and this action is transitive on the fibers of f_L by [4, Cor. 28.2]. As the variety of the group S is R-trivial, the fibers of f are also R-trivial. Hence by Proposition 3.4, the map f induces a bijection

$$BG(L)/R \simeq G\text{-}\mathsf{Torsors}(L)/R.$$

Proposition 4.1. *Consider the following properties:*

(1) *The classifying variety BG is stably rational.*
(2) *The functor $G\text{-}\mathsf{Torsors}$ is R-trivial.*
(3) *For any cycle module M over F, we have $M\big(F(BG)\big)_{nr} = M(F)$, i.e., the function field $F(BG)$ has trivial unramified cohomology.*

Then (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2): Replacing S by $S \times \mathbf{GL}_n$ for sufficiently large n we may assume that BG is rational. Let $U \subset BG$ be a non-empty open subvariety isomorphic to an open subvariety of an affine space. Since every two points in $U(F)$ are R-equivalent, it is sufficient to prove that every point x in $BG(F)$ is R-equivalent to a point in $U(F)$. Consider the surjective morphism $S \to BG$, $g \mapsto gx$. As rational points are dense in S, the set $S(F)x$ is dense in BG and hence intersects U, i.e., there is a $g \in S(F)$ such that $gx \in U(F)$. As the points 1 and g are R-equivalent in $S(F)$, the points x and gx are R-equivalent in $BG(F)$.

(2) ⇒ (3): The homomorphism $M(F) \to M\big(F(BG)\big)$ is injective as it is split by the evaluation at any rational point. Let L/F be a field extension and $\rho \in M\big(L(BG)\big)_{nr}$. We claim that for every two points $x_0, x_1 \in BG(L)$, we have $\rho(x_0) = \rho(x_1)$. We may assume that x_0 and x_1 are strictly R-equivalent, i.e., there is a morphism $f : W \to BG$ for an open nonempty subscheme $W \subset \mathbb{P}^1$ containing points 0 and 1 and satisfying $f(0) = x_0$ and $f(1) = x_1$. Let $\mu = f^*(\rho)$. It follows from Example 2.3 that $\mu \in M\big(F(W)\big)_{nr} = M(F)$. Hence

$$\rho(x_0) = \mu(0) = \mu(1) = \rho(x_1).$$

We apply the claim to the distinguished point $x_0 \in BG(L)$ and the generic point x_1 of BG over the function field $L = F(BG)$. We have $\rho(x_0) \in \operatorname{Im}\big(M(F) \to M(L)\big)$ and $\rho(x_1) = \rho$. It follows from the claim that $\rho \in \operatorname{Im}\big(M(F) \to M(L)\big)$. □

5. An example

Let F be a field of characteristic not two, $G = \mathbf{SL}_8/\boldsymbol{\mu}_2$, $C = \boldsymbol{\mu}_8/\boldsymbol{\mu}_2 \simeq \boldsymbol{\mu}_4$ the center of G and A a semilocal commutative F-algebra. We have $G/C \simeq \mathbf{PGL}_8$

and $\mathbf{PGL}_8$ - *Torsors*(A) is the set of isomorphism classes of Azumaya A-algebras of rank 8. The connecting map

$$\partial : G/C\text{-}\mathit{Torsors}(A) \to H^2_{et}(A, C) = \mathrm{Br}_4(A)$$

takes the class $[\Lambda]$ of an Azumaya A-algebra Λ to $2[\Lambda]$ in the Brauer group of A. Let P be the functor in Corollary 3.5, i.e., $P(A)$ is the set of isomorphism classes of Azumaya A-algebras of rank 8 and exponent 2.

The functor C - *Torsors*$(A) = A^\times / A^{\times 4}$ is R-trivial. We shall show that P is also R-trivial.

Let P' be the subfunctor of P comprised of algebras that can be factored into a tensor product of three quaternion algebras. We show that P' is R-trivial. Write (a, b), where $a, b \in F^\times$, for the quaternion algebra over F generated by elements i and j subject to the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

Let Q and Q' be two algebras in $P'(F)$, i.e., $Q = (a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3)$ and $Q' = (a'_1, b'_1) \otimes (a'_2, b'_2) \otimes (a'_3, b'_3)$ for some $a_i, a'_i, b_i, b'_i \in F^\times$. Consider the Azumaya algebra

$$B = \otimes_{i=1}^3 \big(ta_i + (1-t)a'_i, tb_i + (1-t)b'_i\big)$$

over the ring H_F. We have $B(0) \simeq Q'$ and $B(1) \simeq Q$, so $Q \sim Q'$.

In general, by [8], every algebra Q' in $P(F)$ is similar to an algebra

$$Q = (a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3) \otimes (c, d),$$

where (c, d) (and hence Q) is split over $L = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$. Consider the Azumaya algebra

$$B = \big(a_1, tb_1 + (1-t)\big) \otimes (a_2, b_2) \otimes (a_3, b_3) \otimes (c, d)$$

over H_F. Since (c, d) is split over L, the algebra B is split over $R \otimes_F L$. Hence B is equivalent to an algebra B' in $P(H_F)$. We have $B(1) = Q$, hence $B'(1) = Q'$ and

$$B'(0) = (a_2, b_2) \otimes (a_3, b_3) \otimes (c, d) \in P'(F).$$

Thus every point of $P(F)$ is R-equivalent to a point in $P'(F)$. Hence P is R-trivial. It follows from Corollary 3.5 that the functor G - *Torsors* is R-trivial and therefore, by Proposition 4.1, the function field $F(BG)$ has trivial unramified cohomology.

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