UNRAMIFIED DEGREE THREE INVARIANTS OF REDUCTIVE GROUPS

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Abstract. We prove that if G is a reductive group over an algebraically closed field F, then for a prime integer $p \neq \operatorname{char}(F)$, the group of unramified Galois cohomology $H^3_{\mathrm{nr}}(F(BG), \mathbb{Q}_p/\mathbb{Z}_p(2))$ is trivial for the classifying space BG of G if p is odd or the commutator subgroup of G is simple.

1. Introduction

The notion of a cohomological invariant of an algebraic group was introduced by J-P. Serre in [18]. Let G be an algebraic group over a field F and M a Galois module over F. A *degree d invariant* of G assigns to every G-torsor over a field extension K over F an element in the Galois cohomology group $H^d(K, M)$, functorially in K. In this paper we consider the cohomology groups $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, where $\mathbb{Q}/\mathbb{Z}(d-1)$ is defined as the Galois module of $(d-1)$-twisted roots of unity. The p-part of this module requires special care if $p = \operatorname{char}(F) > 0$. All degree d invariants of G form an abelian group $\operatorname{Inv}^d(G)$. An invariant is *normalized* if it takes a trivial torsor to the trivial cohomology class. The group $\operatorname{Inv}^d(G)$ is the direct sum of the subgroup $\operatorname{Inv}^d(G)_{\mathrm{norm}}$ of normalized invariants and the subgroup of *constant* invariants isomorphic to $H^d(F)$.

The group $\operatorname{Inv}^d(G)_{\mathrm{norm}}$ for small values of d is well understood. The group $\operatorname{Inv}^1(G)_{\mathrm{norm}}$ is trivial if G is connected. There is a canonical isomorphism $\operatorname{Inv}^2(G)_{\mathrm{norm}} \simeq \operatorname{Pic}(G)$ for every reductive group G (see [2, Theorem 2.4]). M. Rost proved (see [6, Part 2]) that if G is simple simply connected then the group $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ is cyclic of finite order with a canonical generator called the *Rost invariant*. The group $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ for an arbitrary semisimple group G was studied in [10].

For a prime integer p, write $H^d(K, p)$ and $\operatorname{Inv}^d(G, p)$ for the p-primary components of $H^d(K)$ and $\operatorname{Inv}^d(G)$ respectively. If v is a discrete valuation of a field extension K/F trivial on F with residue field $F(v)$, then there is defined the *residue* homomorphism $\partial_v : H^d(K, p) \longrightarrow H^{d-1}(F(v), p)$ for every $p \neq \operatorname{char}(F)$. An element $a \in H^d(K, p)$ is *unramified* with respect to v if $\partial_v(a) = 0$. We write $H^d_{\mathrm{nr}}(K, p)$ for the subgroup of all elements unramified

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with respect to every discrete valuation of K over F. An invariant in $\mathrm{Inv}^d(G,p)$ is called *unramified* if all values of the invariant over every K/F belongs to $H^d_{\mathrm{nr}}(K,p)$. We write $\mathrm{Inv}^d_{\mathrm{nr}}(G,p)$ for the group of all unramified invariants.

Let V be a generically free representation of G. There is a nonempty G-invariant open subscheme $U \subset V$ and a *versal* G-torsor $U \longrightarrow X$ for a variety X over F. We think of X as an approximation of the *classifying space* BG of G. The larger the codimension of $V \setminus U$ in V the better X approximates BG. Abusing notation, we will write BG for X. Note that the stable birational type of BG is well defined.

The generic fiber of the versal G-torsor is the *generic* G-torsor over the function field $F(BG)$ of the classifying space. A theorem of Rost and Totaro asserts that the evaluation at the generic G-torsor yields an isomorphism between $\mathrm{Inv}^d(G,p)$ and the subgroup of $H^d(F(BG),p)$ of all elements unramified with respect to the discrete valuations associated with all irreducible divisors of BG. This isomorphism restricts to an isomorphism

$$\mathrm{Inv}^d_{\mathrm{nr}}(G,p) \xrightarrow{\sim} H^d_{\mathrm{nr}}(F(BG),p).$$

A classical question is whether the classifying space BG of an algebraic group G is stably rational. To disprove stable rationality of BG it suffices to show that the map $H^d(F,p) \longrightarrow H^d_{\mathrm{nr}}(F(BG),p)$ is not surjective for some d and p or, equivalently, to find a non-constant unramified invariant of G. For example, D. Saltman disproved in [14] the Noether Conjecture (that V/G is stably rational for a faithful representation V of a finite group G over an algebraically closed field) by proving that $H^2_{\mathrm{nr}}(F(BG),p) \neq H^2(F,p)$ for some G and p, i.e., by establishing a non-constant degree 2 invariant of G. E. Peyre found new examples of finite groups with non-constant unramified degree 3 invariants in [12] . Degree 3 unramified invariants of simply connected groups (over arbitrary fields) were studied in [11] (classical groups) and [7] (exceptional groups).

It is still a wide open problem whether there exists a connected algebraic group G over an algebraically closed field F with the classifying space BG that is not stably rational. Connected groups have no non-trivial degree 1 invariants. F. Bogomolov proved in [3, Lemma 5.7] (see also [2, Theorem 5.10]) that connected groups have no non-trivial degree 2 unramified invariants. In [15] and [16], D. Saltman proved that the projective linear group PGL_n has no non-trivial degree 3 unramified invariants.

In the present paper, we study unramified degree 3 invariants of an arbitrary (connected) reductive group G over an algebraically closed field, or equivalently, the unramified elements in $H^3(F(BG))$. The language of invariants seems easier to work with. The main result is the following theorem (see Theorems 8.4 and 11.3):

Theorem. Let G be a split reductive group over an algebraically closed field F and p a prime integer different from $\mathrm{char}(F)$. Then

$$\mathrm{Inv}^3_{\mathrm{nr}}(G,p) = H^3_{\mathrm{nr}}(F(BG),p) = 0$$

if p is odd or the commutator subgroup of G is (almost) simple.

Let H be the commutator subgroup of a split reductive group G. We have $\operatorname{Inv}^3_{\mathrm{nr}}(G,p) = \operatorname{Inv}^3_{\mathrm{nr}}(H,p)$ (see Proposition 6.1). If H is a simple group, we compare the group $\operatorname{Inv}^3(H)$ with the group $\operatorname{Inv}^3(\widetilde{H}^{\mathrm{gen}})$, where $\widetilde{H}^{\mathrm{gen}}$ is the simply connected cover of H twisted by a generic H-torsor, and use our knowledge of the unramified degree 3 invariants in the simply connected case. The key statement is the injectivity of the homomorphism $\operatorname{Inv}^3(H) \longrightarrow \operatorname{Inv}^3(\widetilde{H}^{\mathrm{gen}})$ (see Section 8).

In general, when H is semisimple but not necessarily simple, we consider an embedding of H into a reductive group G' as the commutator subgroup. Then $\operatorname{Inv}^3(G')$ is identified with a subgroup of $\operatorname{Inv}^3(H)$. If G' is *strict*, i.e., the center of G' is a torus, this subgroup is the smallest possible and is independent of the choice of G'. We write $\operatorname{Inv}^3_{\mathrm{red}}(H)$ for this subgroup. It satisfies

$$\operatorname{Inv}^3_{\mathrm{nr}}(H,p) \subset \operatorname{Inv}^3_{\mathrm{red}}(H,p) \subset \operatorname{Inv}^3(H,p)$$

for every prime $p \neq \operatorname{char}(F)$. The group $\operatorname{Inv}^3_{\mathrm{red}}(H,p)$ is easier to control than $\operatorname{Inv}^3_{\mathrm{nr}}(H,p)$. We show that $\operatorname{Inv}^3_{\mathrm{red}}(H,p) = 0$ which implies that $\operatorname{Inv}^3_{\mathrm{nr}}(H,p)$ is also trivial.

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2. Basic definitions and facts

Let F be a field. If $d \geq 1$, we write $H^d(F)$ for the Galois cohomology group $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$, with $\mathbb{Q}/\mathbb{Z}(d-1)$ the direct sum of $\operatorname{colim}_n \ \mu_n^{\otimes(d-1)}$, where μ_n is the group of roots of unity of degree n, and the p-component if $p = \operatorname{char}(F) > 0$ (see [6, Part 2, Appendix A]). In particular, $H^1(F)$ is the group of (continuous) characters of the absolute Galois group $\operatorname{Gal}(F_{\mathrm{sep}}/F)$ of F and $H^2(F)$ is the Brauer group $\operatorname{Br}(F)$. We view H^d as a functor from the category Fields_F of field extensions of F to the category of abelian groups (or the category Sets of sets).

Let G be a (linear) algebraic group over a field F. The notion of an *invariant* of G was defined in [6] as follows. Consider the functor

$$\operatorname{Tors}_G : \mathit{Fields}_F \longrightarrow \mathit{Sets}$$

taking a field K to the set $\operatorname{Tors}_G(K) := H^1(K,G)$ of isomorphism classes of (right) G-torsors over $\operatorname{Spec} K$. A *degree d cohomological invariant* of G is then a morphism of functors

$$\operatorname{Tors}_G \longrightarrow H^d,$$

i.e., a functorial in K collection of maps of sets $\operatorname{Tors}_G(K) \longrightarrow H^d(K)$ for all field extensions K/F. We denote the group of such invariants by $\operatorname{Inv}^d(G)$.

An invariant $I \in \operatorname{Inv}^d(G)$ is called *normalized* if $I(E) = 0$ for a trivial G-torsor E. The normalized invariants form a subgroup $\operatorname{Inv}^d(G)_{\mathrm{norm}}$ of $\operatorname{Inv}^d(G)$

and there is a natural isomorphism

$$\mathrm{Inv}^d(G) \simeq H^d(F) \oplus \mathrm{Inv}^d(G)_{\mathrm{norm}}.$$

Example 2.1. Let G be a (connected) reductive group over F. It is shown in [2, Theorem 2.4] that there is an isomorphism

$$\beta_G : \mathrm{Pic}(G) \xrightarrow{\sim} \mathrm{Inv}^2(G)_{\mathrm{norm}}.$$

Let G be a split reductive group and H the commutator subgroup of G. Let $\pi : \widetilde{H} \longrightarrow H$ be a simply connected cover with kernel $\widetilde{C}$. There are canonical isomorphisms (see [17, §6])

$$\mathrm{Pic}(G) \xrightarrow{\sim} \mathrm{Pic}(H) \simeq \widetilde{C}^* := \mathrm{Hom}(\widetilde{C}, \mathbb{G}_m). \tag{2.2}$$

Take any character $\chi \in \widetilde{C}^*$ and consider the push-out diagram

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{C} \arrow[r] \arrow[d, "\chi"'] & \widetilde{H} \arrow[r, "\pi"] \arrow[d] & H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & H' \arrow[r] & H \arrow[r] & 1. \end{tikzcd}$$

The isomorphism $\widetilde{C}^* \simeq \mathrm{Pic}(H)$ takes a character χ to the class the line bundle L_χ on H given by the $\mathbb{G}_m$-torsor $H' \longrightarrow H$ in the bottom row of the diagram. For a field extension K/F and an H-torsor E over K, the value of the invariant $\beta_H(L_\chi)$ is equal to $\delta([E]) \in H^2(K, \mathbb{G}_m) = \mathrm{Br}(K)$, where $[E]$ is the class of E in $H^1(K, H)$ and $\delta : H^1(K, H) \longrightarrow H^2(K, \mathbb{G}_m)$ is the connecting map for the bottom exact sequence in the diagram.

If $f : G_1 \longrightarrow G_2$ is a homomorphism of algebraic groups over F and E_1 is a G_1-torsor over a field extension K/F, then $E_2 := (E_1 \times G_2)/G_1$ is the G_2-torsor over K which we denote by $f_*(E_1)$. If I is a degree d invariant of G_2, we define an invariant $f^*(I)$ of G_1 by $f^*(I)(E_1) := I(f_*(E_1))$. Thus, we have a group homomorphism

$$f^* : \mathrm{Inv}^d(G_2) \longrightarrow \mathrm{Inv}^d(G_1) \tag{2.3}$$

taking normalized invariants to the normalized ones.

Let G be an algebraic group over a field F and let V be a generically free representation of G. There is a nonempty G-invariant open subscheme $U \subset V$ such that U is a G-torsor over a variety which we denote by U/G (see [19, Remark 1.4]). We think of U/G as an approximation of the "classifying space" BG of G and abusing notation write $U/G = BG$. The space BG is better approximated by U/G if the codimension of $V \setminus U$ in V is large. For our purposes it suffices to assume that this codimension is at least 3 (see [2]).

Note that by the No-name Lemma, the stable rationality type of BG is uniquely determined by G.

The generic fiber $E^{\mathrm{gen}} \longrightarrow \mathrm{Spec}(F(BG))$ of the projection $U \longrightarrow U/G$ is called the *generic* G-torsor. The value of an invariant of G at the generic torsor E^{gen} yields a homomorphism

$$\operatorname{Inv}^d(G) \longrightarrow H^d(F(BG)).$$

Rost proved (see [6, Part 2, Th. 3.3] or [2, Theorem 2.2]) that this map is injective, i.e., every invariant is determined by its value at the generic torsor.

We decompose the group of invariants into a direct sum of primary components:

$$\operatorname{Inv}^d(G) = \coprod_{p \text{ prime}} \operatorname{Inv}^d(G, p).$$

Let K be a field extension of F. For a prime integer p, write $H^d(K, p)$ for the p-primary component of $H^d(K)$. Let v be a discrete valuation of K over F with residue field $F(v)$. If $\operatorname{char}(F) \neq p$, there is the *residue map* (see [6, Chapter 2])

$$\partial_v : H^d(K, p) \longrightarrow H^{d-1}(F(v), p).$$

An element $a \in H^d(K, p)$ is *unramified with respect to* v if $\partial_v(a) = 0$.

A point x of codimension 1 in BG for an algebraic group G yields a discrete valuation v_x on the function field $F(BG)$ over F. Write $A^0(BG, H^d, p)$ for the group of all elements in $H^d(F(BG), p)$ that are unramified with respect to v_x for all points x of codimension 1 in BG. It is proved in [6, Part 1, Theorem 11.7] that the value of every invariant from $\operatorname{Inv}^d(G, p)$ at the generic G-torsor E^{gen} belongs to $A^0(BG, H^d, p)$. Moreover, we have the following theorem (see [6, Part 1, Appendix C]):

Theorem 2.4. *Let G be an algebraic group over F and p a prime different from* $\operatorname{char}(F)$*. Then the evaluation of an invariant at the generic G-torsor yields an isomorphism*

$$\operatorname{Inv}^d(G, p) \xrightarrow{\sim} A^0(BG, H^d, p).$$

The inverse isomorphism is defined as follows. Let E be a G-torsor over a field extension K/F and $BG = U/G$. We have the following canonical morphisms:

$$\operatorname{Spec} K = E/G \xleftarrow{f} (E \times U)/G \xrightarrow{h} U/G = BG.$$

Note that the groups $H^d(K, p)$ for all d and all field extensions K/F form a *cycle module* in the sense of Rost (see [13]). In particular, we have flat pull-back homomorphisms

$$H^d(K, p) = A^0(\operatorname{Spec} K, H^d, p) \xrightarrow{f^*} A^0((E \times U)/G, H^d, p) \xleftarrow{h^*} A^0(BG, H^d, p).$$

The variety $(E \times U)/G$ is an open subscheme of the vector bundle $(E \times V)/G$ over $\operatorname{Spec} K$. By the homotopy invariance property, the pull-back homomorphism

$$H^d(K, p) = A^0(\operatorname{Spec} K, H^d, p) \longrightarrow A^0((E \times V)/G, H^d, p)$$

is an isomorphism. Since the inclusion of $(E \times U)/G$ into $(E \times V)/G$ is a bijection on points of codimension 1 (by our assumption on the codimension of $V \setminus U$ in V), the restriction homomorphism

$$A^0((E \times V)/G, H^d, p) \longrightarrow A^0((E \times U)/G, H^d, p)$$

is an isomorphism. It follows that f^* is an isomorphism.

Let $a \in A^0(BG, H^d, p)$. The invariant $I \in \operatorname{Inv}^d(G, p)$ defined by $I(E) = (f^*)^{-1} h^*(a)$ is the inverse image of a under the isomorphism in Theorem 2.4.

3. Decomposable invariants

The group of decomposable degree 3 invariants of a semisimple group was defined in [10, §1]. We extend this definition to the class of split reductive groups.

Let G be a split reductive group over F. The $\cup$-product $H^2(K) \otimes K^\times \longrightarrow H^3(K)$ for any field extension K/F yields a pairing

$$\operatorname{Inv}^2(G)_{\mathrm{norm}} \otimes F^\times \longrightarrow \operatorname{Inv}^3(G)_{\mathrm{norm}}.$$

The subgroup of *decomposable invariants* $\operatorname{Inv}^3(G)_{\mathrm{dec}}$ is the image of the pairing.

Proposition 3.1. *Let G be a split reductive group over F. Then the composition*

$$\operatorname{Pic}(G) \otimes F^\times \xrightarrow{\sim} \operatorname{Inv}^2(G)_{\mathrm{norm}} \otimes F^\times \longrightarrow \operatorname{Inv}^3(G)_{\mathrm{dec}}$$

is an isomorphism.

Proof. The surjectivity of the composition follows from the definition. Let H be the commutator subgroup of G. By [10, Theorem 4.2]), the composition is an isomorphism when G is replaced by H. The injectivity of the composition for G follows then from the fact that the map $\operatorname{Pic}(G) \longrightarrow \operatorname{Pic}(H)$ in (2.2) is an isomorphism. $\square$

It follows from the proposition that $\operatorname{Inv}^3(G)_{\mathrm{dec}} = 0$ if $\operatorname{Pic}(G) = 0$ (for example, G is semisimple simply connected) or F is algebraically closed.

We write

$$\operatorname{Inv}^3(G)_{\mathrm{ind}} := \operatorname{Inv}^3(G)_{\mathrm{norm}} / \operatorname{Inv}^3(G)_{\mathrm{dec}}.$$

4. Unramified invariants

Let K/F be a field extension and p a prime integer different from $\operatorname{char}(F)$. We write $H^d_{\mathrm{nr}}(K/F, p)$ for the subgroup of all elements in $H^d(K, p)$ that are unramified with respect to all discrete valuations of K over F. A field extension L/K yields a natural homomorphism $H^d(K) \longrightarrow H^d(L)$ that takes $H^d_{\mathrm{nr}}(K/F, p)$ into $H^d_{\mathrm{nr}}(L/F, p)$ by [6, Part 1, Proposition 8.2].

Let G be an algebraic group over F. An invariant $I \in \operatorname{Inv}^d(G, p)$ is called *unramified* if for every field extension K/F and every $E \in \operatorname{Tors}_G(K)$, we have $I(E) \in H^d_{\mathrm{nr}}(K/F, p)$. Note that the constant invariants are always unramified. We will write $\operatorname{Inv}^d_{\mathrm{nr}}(G, p)$ for the subgroup of all unramified invariants in $\operatorname{Inv}^d(G, p)$.

If $f : G_1 \longrightarrow G_2$ is a group homomorphism, then the map f^* in (2.3) takes $\mathrm{Inv}^d_{\mathrm{nr}}(G_2, p)$ into $\mathrm{Inv}^d_{\mathrm{nr}}(G_1, p)$.

Proposition 4.1. *Let G be an algebraic group over F. An invariant $I \in \mathrm{Inv}^d(G, p)$ is unramified if and only if the value of I at the generic G-torsor in $H^d(F(BG), p)$ is unramified. In particular,* $\mathrm{Inv}^d_{\mathrm{nr}}(G, p) \simeq H^d_{\mathrm{nr}}(F(BG), p)$.

Proof. It suffices to show that the inverse of the isomorphism in Theorem 2.4 takes unramified elements to unramified invariants. Let $a \in H^d_{\mathrm{nr}}(F(BG), p) \subset A^0(BG, H^d, p)$. The corresponding invariant $I \in \mathrm{Inv}^d(G, p)$ is defined by $I(E) = (f^*)^{-1}h^*(a)$ (see Section 2). Note that h^* takes unramified elements to unramified ones and f^* yields an isomorphism on the unramified elements as the function field of $(E \times U)/G$ is a purely transcendental extension of K. It follows that $I(E)$ is unramified for all E, hence the invariant I is unramified. □

Unramified invariants are constant along rational families of torsors. Precisely, if K/F is a purely transcendental field extension and E is a G-torsor over K, then for every invariant $I \in \mathrm{Inv}^d_{\mathrm{nr}}(G, p)$ we have

$$I(E) \in \mathrm{Im}\big(H^d(F, p) \longrightarrow H^d(K, p)\big).$$

Indeed, $I(E) \in H^d_{\mathrm{nr}}(K, p)$ which is the image of $H^d(F, p)$ in $H^d(K, p)$.

5. Abstract Chern classes

Let A be a lattice (written additively). Consider the symmetric ring $S^*(A)$ over $\mathbb{Z}$ and the group ring $\mathbb{Z}[A]$ of A. We use the exponential notation for $\mathbb{Z}[A]$: every element can be written as a finite sum $\sum_{a\in A} n_a e^a$ with $n_a \in \mathbb{Z}$. There are the *abstract Chern classes* (see [10, 3c])

$$c_i : \mathbb{Z}[A] \longrightarrow S^i(A), \quad i \geq 0$$

satisfying in particular,

$$c_1\Big(\sum_i e^{a_i}\Big) = \sum_i a_i \in A \qquad \text{and} \qquad c_2\Big(\sum_i e^{a_i}\Big) = \sum_{i<j} a_i a_j \in S^2(A).$$

The map c_1 is a homomorphism and

$$c_2(x + y) = c_2(x) + c_2(y) + c_1(x)c_1(y)$$

for all $x, y \in \mathbb{Z}[A]$.

If A is a W-lattice for a group W acting on A, then all the c_i's are W-equivariant. It follows that c_2 yields a map (not a homomorphism in general) of groups of W-invariant elements:

$$c_2^W : \mathbb{Z}[A]^W \longrightarrow S^2(A)^W.$$

The group $\mathbb{Z}[A]^W$ is generated by the elements $\sum e_i^a$, where the a_i's form a W-orbit in A. It follows that the subgroup of $S^2(A)^W$ generated by the image of c_2^W is generated by $\sum_{i<j} a_i a_j$ with the a_i's forming a W-orbit in A and aa'

for $a, a' \in A^W$. The elements of these two types can be viewed as "obvious" elements in $S^2(A)^W$ which we call *decomposable.*

Write $S^2(A)^W_{\mathrm{dec}}$ for the subgroup of $S^2(A)^W$ generated by the decomposable elements, or equivalently, by the image of c_2^W. Set

$$S^2(A)^W_{\mathrm{ind}} := S^2(A)^W / S^2(A)^W_{\mathrm{dec}}.$$

Note that if $A^W = 0$, the map c_2^W is a homomorphism and $S^2(A)^W_{\mathrm{ind}}$ is the cokernel of c_2^W.

Lemma 5.1. *Let A_1 and A_2 be W_1- and W_2-lattices respectively. Then there is a canonical isomorphism*

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2}_{\mathrm{ind}} \simeq S^2(A_1)^{W_1}_{\mathrm{ind}} \oplus S^2(A_2)^{W_2}_{\mathrm{ind}}.$$

Proof. We have

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2} \simeq S^2(A_1)^{W_1} \oplus S^2(A_2)^{W_2} \oplus (A_1^{W_1} \otimes A_2^{W_2})$$

and

$$\mathbb{Z}[A_1 \oplus A_2]^{W_1 \times W_2} \simeq \mathbb{Z}[A_1]^{W_1} \otimes \mathbb{Z}[A_2]^{W_2}.$$

The standard formulas on the Chern classes show that $c_1(\mathbb{Z}[A_i]^{W_i}) = A_i^{W_i}$ and

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2}_{\mathrm{dec}} \simeq S^2(A_1)^{W_1}_{\mathrm{dec}} \oplus S^2(A_2)^{W_2}_{\mathrm{dec}} \oplus (A_1^{W_1} \otimes A_2^{W_2}),$$

whence the result. □

Lemma 5.2. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of W-lattices. Suppose that W acts trivially on A and $C^W = 0$. Then*

(1) *The sequence*

$$0 \longrightarrow S^2(A) \longrightarrow S^2(B)^W \longrightarrow S^2(C)^W$$

is exact.

(2) *The natural homomorphism $S^2(B)^W_{\mathrm{ind}} \longrightarrow S^2(C)^W_{\mathrm{ind}}$ is injective.*

Proof. The first statement is proved in [5, Lemma 4.9]. Since W acts trivially on A, for every subgroup $W' \subset W$, we have $H^1(W', A) = 0$, hence the map $B^{W'} \longrightarrow C^{W'}$ is surjective. The group $\mathbb{Z}[C]^W$ is generated by elements of the form $\sum_i e^{c_i}$, where the c_i's form a W-orbit in C. By the surjectivity above, applied to the stabilizer $W' \subset W$, this orbit can be lifted to a W-orbit in B. Therefore, the map $\mathbb{Z}[B]^W \longrightarrow \mathbb{Z}[C]^W$ is surjective. The second statement follows from this, the first statement of the lemma and the fact that $S^2(A) = S^2(A)^W_{\mathrm{dec}} \subset S^2(B)^W_{\mathrm{dec}}$. □

6. Degree 3 invariants of split reductive groups

Let G be a split reductive group over F and let H be the commutator subgroup of G. Thus, H is a split semisimple group and the factor group $Q := G/H$ is a split torus.

Proposition 6.1. *1. The restriction maps* $\mathrm{Inv}^d(G) \longrightarrow \mathrm{Inv}^d(H)$ *and* $\mathrm{Inv}^d(G)_{\mathrm{ind}} \longrightarrow \mathrm{Inv}^d(H)_{\mathrm{ind}}$ *are injective.*

2. For every prime $p \neq \mathrm{char}(F)$*, the restriction map* $\mathrm{Inv}^d_{\mathrm{nr}}(G,p) \longrightarrow \mathrm{Inv}^d_{\mathrm{nr}}(H,p)$ *is an isomorphism.*

Proof. For a field extension K/F, the map

$$j : H^1(K,H) \longrightarrow H^1(K,G)$$

is surjective as $H^1(K,Q) = 1$ and the group $Q(K)$ acts transitively on the fibers of j. It follows that the restriction map $\mathrm{Inv}^d(G) \longrightarrow \mathrm{Inv}^d(H)$ is injective. The injectivity of $\mathrm{Inv}^d(G)_{\mathrm{ind}} \longrightarrow \mathrm{Inv}^d(H)_{\mathrm{ind}}$ follows then from Proposition 3.1.

As Q is a rational variety, the fibers of j are rational families of H-torsors. Since an unramified invariant of H must be constant on the fibers, it defines an invariant of G. This proves the second statement. □

Let G be a split reductive group, $T \subset G$ a split maximal torus. By [10, 3d], there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^2(BG) \arrow[r] \arrow[d] & \overline{H}^{4,2}_{\text{ét}}(BG) \arrow[r] \arrow[d] & \mathrm{Inv}^3(G)_{\mathrm{norm}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^2(BT) \arrow[r] & \overline{H}^{4,2}_{\text{ét}}(BT) \arrow[r] & \mathrm{Inv}^3(T)_{\mathrm{norm}} \arrow[r] & 0
\end{tikzcd}$$

with the exact rows, where $\overline{H}^{4,2}_{\text{ét}}(BH) = \overline{H}^4(BH, \mathbb{Z}(2))$ for an algebraic group H is the reduced weight two étale motivic cohomology group (see [9, §5]). The group $\mathrm{Inv}^3(T)_{\mathrm{norm}}$ is trivial as T has no nontrivial torsors and $\mathrm{CH}^2(BT) = S^2(T^*)$ by [2, Example A.5], hence the middle term in the bottom row is isomorphic to $S^2(T^*)$.

Let N be the normalizer of T in G and $W = N/T$ the Weyl group. The group W acts naturally on BT. Moreover, if $w \in W$, the composition

$$BT \xrightarrow{w} BT \xrightarrow{s} BG,$$

where s is the natural morphism, coincides with s. Therefore, the image of the middle vertical homomorphism in the diagram

$$\overline{H}^{4,2}_{\text{ét}}(BG) \longrightarrow \overline{H}^{4,2}_{\text{ét}}(BT) = S^2(T^*)$$

is contained in the subgroup $S^2(T^*)^W$ of W-invariant elements. By [10, Lemma 3.8], the image of $\mathrm{CH}^2(BG)$ under this homomorphism is equal to $S^2(T^*)^W_{\mathrm{dec}}$. Therefore, by diagram chase, we have a homomorphism $\mathrm{Inv}^3(G)_{\mathrm{norm}} \longrightarrow S^2(T^*)^W_{\mathrm{ind}}$. The group of decomposable invariants $\mathrm{Inv}^3(G)_{\mathrm{dec}}$ is in the kernel of this map since $\mathrm{Inv}^3(G)_{\mathrm{dec}}$ vanishes over an algebraic closure of F and the group $S^2(T^*)^W_{\mathrm{ind}}$ does not change. Therefore, we have a well-defined homomorphism

$$\alpha_G : \mathrm{Inv}^3(G)_{\mathrm{ind}} \longrightarrow S^2(T^*)^W_{\mathrm{ind}}.$$

Theorem 6.2. *Let G be a split reductive group over F. Then the map α_G is injective. If G is semisimple, then α_G is an isomorphism.*

Proof. The second statement is proved in [10, Theorem 3.9]. The first statement follows from Proposition 6.1(1), the commutativity of the diagram

$$\begin{tikzcd} \operatorname{Inv}^3(G)_{\mathrm{ind}} \arrow[r, "\alpha_G"] \arrow[d] & S^2(T^*)^W_{\mathrm{ind}} \arrow[d] \\ \operatorname{Inv}^3(H)_{\mathrm{ind}} \arrow[r, "\alpha_H"] & S^2(S^*)^W_{\mathrm{ind}}, \end{tikzcd}$$

where H is the commutator subgroup of G and S is a maximal torus of H, and the second statement applied to H. □

Proposition 3.1 and Lemma 5.1 yield the following additivity property.

Corollary 6.3. *Let H_1 and H_2 be two split semisimple groups. Then there is a canonical isomorphism*

$$\operatorname{Inv}^3(H_1 \times H_2) \simeq \operatorname{Inv}^3(H_1) \oplus \operatorname{Inv}^3(H_2).$$

Let H be a split semisimple group over a field F, $\pi : \widetilde{H} \longrightarrow H$ a simply connected cover, $\widetilde{S}$ the pre-image of a split maximal torus S of H, so $\widetilde{S}$ is a split maximal torus of $\widetilde{H}$. Then $S^2(S^*)$ can be viewed with respect to π as a sublattice of $S^2(\widetilde{S}^*)$ of finite index and we have the following commutative diagram

$$\begin{tikzcd} & \operatorname{Inv}^3(H)_{\mathrm{ind}} \arrow[r, "\alpha_H", "\sim"'] \arrow[d, "\pi^*"'] & S^2(S^*)^W_{\mathrm{ind}} \arrow[d] \\ \operatorname{Inv}^3(\widetilde{H})_{\mathrm{norm}} \arrow[r, equal] & \operatorname{Inv}^3(\widetilde{H})_{\mathrm{ind}} \arrow[r, "\alpha_{\tilde{H}}", "\sim"'] & S^2(\widetilde{S}^*)^W_{\mathrm{ind}}. \end{tikzcd}$$

If H is simple, the group $S^2(\widetilde{S}^*)^W$ is infinite cyclic with a canonical generator q (see [6, Part 2, §7]). It follows that $S^2(S^*)^W$ is also infinite cyclic with kq a generator for a unique integer $k > 0$. The invariant $R \in \operatorname{Inv}^3(\widetilde{H})_{\mathrm{norm}}$ corresponding to the generator q is called the *Rost invariant* of $\widetilde{H}$. It is a generator of the cyclic group $\operatorname{Inv}^3(\widetilde{H})$.

7. Change of groups

In this section we prove the following useful property.

Proposition 7.1. *Let p be a prime integer, G an algebraic group over F, $C \subset G$ a finite central diagonalizable subgroup of order not divisible by p, $H = G/C$. Then the natural map $\operatorname{Inv}^d(H, p) \longrightarrow \operatorname{Inv}^d(G, p)$ is an isomorphism. If $\operatorname{char}(F) \neq p$, then the map $\operatorname{Inv}^d_{\mathrm{nr}}(H, p) \longrightarrow \operatorname{Inv}^d_{\mathrm{nr}}(G, p)$ is also an isomorphism.*

Proof. Both functors in the definition of an invariant can be naturally extended to the category $\mathcal{C}$ of F-algebras that are finite product of fields, and every invariant extends uniquely to a morphism of extended functors. If $K \longrightarrow L$ is a morphism in $\mathcal{C}$ and M is an étale K-algebra, then $L \otimes_K M$ is also an object of the category $\mathcal{C}$.

For any K in $\mathcal{C}$ we have an exact sequence

$$H^1_{fppf}(K,G) \longrightarrow H^1_{fppf}(K,H) \xrightarrow{\delta_K} H^2_{fppf}(K,C)$$

and the group $H^1_{fppf}(K,C)$ acts transitively on the fibers of the first map in the sequence.

Proof of injectivity. Let $I \in \mathrm{Inv}^d(H,p)$ be such that $f^*(I) = 0$, where $f : G \longrightarrow H$ is the canonical homomorphism. We prove that $I = 0$. Take any K in $\mathcal{C}$ and $E \in \mathrm{Tors}_H(K)$. As an element of the group $H^2_{fppf}(K,C)$ is a tuple of elements in $\mathrm{Br}(K)$ of order prime to p, there is an étale K-algebra L of (constant) finite rank $[L : K]$ prime to p such that $\delta_L(E_L) = 0$. It follows that $E_L = f_*(E')$ for some $E' \in \mathrm{Tors}_G(L)$. We have

$$I(E)_L = I(E_L) = I(f_*(E')) = f^*(I)(E') = 0.$$

Since $[L : K]$ is prime to p, we have $I(E) = 0$, i.e., $I = 0$.

Proof of surjectivity. Let $J \in \mathrm{Inv}^d(G,p)$. We construct an invariant $I \in \mathrm{Inv}^d(H,p)$ such that $J = f^*(I)$. Take any K in $\mathcal{C}$ and $E \in \mathrm{Tors}_H(K)$. As above, choose an étale K-algebra L of finite rank prime to p such that $\delta_L(E_L) = 0$ and an element $E' \in \mathrm{Tors}_G(L)$ with $E_L = f_*(E')$. We set

$$I(E) = \frac{1}{[L:K]}\,\mathrm{cor}_{L/F}(J(E')).$$

This is independent of the choice of E'. Indeed, if $E_L = f_*(E'')$ for $E'' \in \mathrm{Tors}_G(L)$, then there exists $\nu \in H^1_{fppf}(L,C)$ with $E'' = \nu(E')$. Choose an L-algebra P of constant rank $[P : L]$ prime to p such that $\nu_P = 1$. It follows that $E''_P = E'_P$ and therefore,

$$[P:L]\,\mathrm{cor}_{L/F}(J(E'')) = \mathrm{cor}_{P/F}(J(E''_P)) = \mathrm{cor}_{P/F}(J(E'_P)) = [P:L]\,\mathrm{cor}_{L/F}(J(E')).$$

Since $[P : L]$ is prime to p, we have $\mathrm{cor}_{L/F}(J(E'')) = \mathrm{cor}_{L/F}(J(E'))$.

In order to show that the value $I(E)$ is independent of the choice of L, for the two choices L and L', it suffices to compare the formulas for L and $LL' := L \otimes_F L'$:

$$\frac{1}{[L:K]}\,\mathrm{cor}_{L/F}(J(E')) = \frac{[L':K]}{[LL':K]}\,\mathrm{cor}_{L/F}(J(E')) = \frac{1}{[LL':K]}\,\mathrm{cor}_{LL'/F}(J(E'_{LL'})).$$

We have constructed the invariant $I \in \mathrm{Inv}^d(H,p)$. For any K in $\mathcal{C}$ and $E' \in \mathrm{Tors}_G(K)$, by the definition of I, we have $f^*(I)(E') = I(f_*(E)) = J(E')$, hence $f^*(I) = J$. Note that if J is an unramified invariant, I is also unramified since the corestriction map preserves unramified elements by [6, Part 1, Proposition 8.6]. □

8. Degree 3 unramified invariants of simple groups

The following statement was proved in [11] (classical groups) and [7] (exceptional groups).

Proposition 8.1. *Let H be an absolutely simple simply connected group over F and p a prime different from* $\mathrm{char}(F)$.

1. If the Dynkin diagram of H is different from ${}^2\mathsf{A}_n$, n odd, and ${}^1\mathsf{D}_4$, then $\mathrm{Inv}^3_{\mathrm{nr}}(H,p)_{\mathrm{norm}} = 0$.

2. If H is split, then $\mathrm{Inv}^3_{\mathrm{nr}}(H,p)_{\mathrm{norm}} = 0$.

Let H be a semisimple group over F, E an H-torsor over $\mathrm{Spec}(K)$ for a field extension K/F. The *twist* $H^E := \mathrm{Aut}_H(E)$ of H by E is a semisimple group over K. The twisting argument shows that $BH^E = BH_K$ and there is a canonical isomorphism $\mathrm{Inv}^d(H^E) \simeq \mathrm{Inv}^d(H_K)$. If E^{gen} is a generic H-torsor, we write H^{gen} for $H^{E^{\mathrm{gen}}}$. Let $\widetilde{H}^{\mathrm{gen}} \longrightarrow H^{\mathrm{gen}}$ be a simply connected cover.

Proposition 8.2. *Let H be a split simple group. Then the composition*

$$\mathrm{Inv}^3(H)_{\mathrm{ind}} \longrightarrow \mathrm{Inv}^3(H^{\mathrm{gen}})_{\mathrm{ind}} \longrightarrow \mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}})_{\mathrm{ind}} = \mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}})$$

is injective.

Proof. The statement is clear if H is a simply connected group. The case of an adjoint group H was considered in [10, Theorem 4.10]. Consider the other split semisimple groups type-by-type. It suffices to restrict to the p-component of $\mathrm{Inv}^3(H)$ for a prime p.

Type A_{n-1}, $n \geq 2$. We have $H = \mathrm{SL}_n/\mu_m$ for an integer m dividing n. By Proposition 7.1, we may assume that $m = p^r$ for some r. It is shown in [1, Theorem 4.1] and Theorem 6.2 that

$$\mathrm{Inv}^3(H)_{\mathrm{ind}} \xrightarrow{\sim} S^2(S^*)^W_{\mathrm{ind}} \hookrightarrow (\mathbb{Z}/m\mathbb{Z})q.$$

On the other hand, an H-torsor yields a central simple algebra of degree n and exponent dividing m. A generic torsor gives an algebra with the exponent exactly m, hence $\mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}}) = (\mathbb{Z}/m\mathbb{Z})R$ by [6, Part 2, Theorem 11.5].

Type D_n, $n \geq 4$. We have $H = \mathrm{O}^+_{2n}$, the special orthogonal group or $H = \mathrm{HSpin_{2n}}$, the half-spin group if n is even. It is shown in [6, Part 1,Chapter VI] in the case $\mathrm{char}(F) \neq 2$ that $\mathrm{Inv}^3(\mathrm{O}^+_{2n})_{\mathrm{ind}} = 0$. In general, recall that the character group of a maximal split torus S is a free group of rank n. Let $x_1, x_2, \dots, x_n$ be a basis for S^* such that the Weyl group W acts on the x_i's by permutations and change of signs. The generator of $S^2(S^*)^W$ is the quadratic form $q = x_1^2 + x_2^2 + \cdots + x_n^2$. It is in $S^2(S^*)^W_{\mathrm{dec}}$ since $c_2(\sum_i e^{x_i} + e^{-x_i}) = -q$. By [10, Theorem 3.9], $\mathrm{Inv}^3(\mathrm{O}^+_{2n})_{\mathrm{ind}} = 0$.

Finally, assume that n is even and $H = \mathrm{HSpin_{2n}}$, the half-spin group. It follows from [1, Theorem 5.1] and Theorem 6.2 that

$$\mathrm{Inv}^3(H)_{\mathrm{ind}} \xrightarrow{\sim} S^2(S^*)^W_{\mathrm{ind}} \hookrightarrow (\mathbb{Z}/4\mathbb{Z})q$$

and $\mathrm{Inv}^3(H)_{\mathrm{ind}} = 0$ if $n = 4$. On the other hand, an H-torsor yields a central simple algebra of degree $2n$. A generic torsor gives a nonsplit algebra. By [6, Part 2, Theorem 15.4], $\mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}}) = (\mathbb{Z}/4\mathbb{Z})R$ if $n > 4$. □

Remark 8.3. The statement fails for semisimple groups that are not simple, see Example 11.2.

Theorem 8.4. *Let H be a split simple group over an algebraically closed field F and p a prime integer different from* $\mathrm{char}(F)$. *Then* $\mathrm{Inv}^3_{\mathrm{nr}}(H,p)=0$.

Proof. Let $I\in \mathrm{Inv}^3_{\mathrm{nr}}(H,p)$. Note that since F is algebraically closed, every decomposable invariant is trivial.

The pull-back $\tilde I$ of I under the composition in Proposition 8.2 is an unramified invariant. As $\widetilde{H}^{\mathrm{gen}}$ is an inner form of $\widetilde{H}$, by Proposition 8.1, $\tilde I=0$ and hence $I=0$ by Proposition 8.2 unless the Dynkin diagram of H is D_4.

If H is a simply connected group of type D_4, then $I=0$ by Proposition 8.1. If H is a half-spin group of type D_4, then $I=0$ by [1, Theorem 5.1]. Finally assume that H is an adjoint group of type D_4. By [10, Theorem 4.7], the group $\mathrm{Inv}^3(H)$ is cyclic of order 2.

Assume that $I\neq 0$. The group $\widetilde{H}^{\mathrm{gen}}$ is the spinor group of a central simple algebra A of degree 8 with and orthogonal involution σ of trivial discriminant. Consider the corresponding special orthogonal group $\widehat{H}^{\mathrm{gen}}:=\mathrm{O}^+(A,\sigma)$ of (A,σ). An $\widehat{H}^{\mathrm{gen}}$-torsor over a field K is given by a pair (a,x), where a is an invertible σ-symmetric element in A and $x\in K^\times$ such that $\mathrm{Nrd}(a)=x^2$ and Nrd is the reduced norm map (see [8, 29.27]).

The canonical homomorphism $\mathrm{Inv}^3(H^{\mathrm{gen}})\longrightarrow \mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}})$ factors through $\mathrm{Inv}^3(\widehat{H}^{\mathrm{gen}})$. By [10, §4, type D_n], the pull-back of I in $\mathrm{Inv}^3(\widehat{H}^{\mathrm{gen}})$ is the class of the invariant taking a pair (a,x) to the cup-product $(x)\cup[A]\in H^3(K)$. This invariant is ramified as it is non-constant when a runs over a subfield of A of dimension n fixed by σ element-wise, a contradiction. □

9. Structure of reductive groups

Let H be a split semisimple group over a field F, $S\subset H$ a split maximal torus. Write $\Lambda_r\subset S^*$ for the root lattice of H. Let $\widetilde{H}\longrightarrow H$ be a simply connected cover and let $\widetilde{S}$ for the inverse image of S, a maximal torus in $\widetilde{H}$. Write Λ_w for the character group of $\widetilde{S}$. This is the weight lattice freely generated by the fundamental weights. We have

$$\Lambda_r\subset S^*\subset\Lambda_w.$$

The center C of H is a finite diagonalizable group with $C^*=S^*/\Lambda_r$.

Let G be a split reductive group over a field F with the commutator subgroup H. Choose a split maximal $T\subset G$ such that $T\cap H=S$. The roots of H can be uniquely lifted to T^* (to the roots of G), so the inclusion of Λ_r into S^* is lifted to the inclusion of Λ_r into T^*. The composition $\widetilde{S}\longrightarrow S\longrightarrow T$ yields a homomorphism $T^*\longrightarrow\Lambda_w$ of lattices. Thus, we have the two homomorphisms

$$\Lambda_r\hookrightarrow T^*\xrightarrow{f}\Lambda_w \tag{9.1}$$

with the composition the canonical embedding of Λ_r into Λ_w. The image of f in (9.1) is equal to S^*. The center Z of G is a diagonalizable group with $Z^*=T^*/\Lambda_r$. The factor group $G/H=T/S$ is a torus Q with the character lattice $Q^*=\mathrm{Ker}(f)$.

We would like to study all split reductive groups with the fixed commutator subgroup H.

Let H be a split semisimple group over F. Fix a split maximal torus $S \subset H$ and consider the root system of H relative to S with the root and weight lattices $\Lambda_r \subset \Lambda_w$ respectively.

Consider a category $\mathit{Red}(H)$ with objects split reductive groups G over F with the commutator subgroup H. A morphism between G_1 and G_2 in this category is a group homomorphism $G_1 \longrightarrow G_2$ over F that is the identity on H.

Consider another category $\mathit{Lat}(H)$ with objects the diagrams of the form

$$\Lambda_r \longrightarrow A \xrightarrow{f} \Lambda_w, \tag{9.2}$$

where A is a lattice, $\mathrm{Im}(f) = S^*$ and the composition is the embedding of Λ_r into Λ_w. A morphism in $\mathit{Lat}(R)$ is a morphism between the diagrams which is identity on Λ_r and Λ_w.

Let G be an object in $\mathit{Red}(H)$. Write Z for the center of G. Then $T := S \cdot Z$ is a split maximal torus of G. The diagram (9.1) yields then a contravariant functor

$$\rho : \mathit{Red}(H) \longrightarrow \mathit{Lat}(H).$$

Proposition 9.3. *For every split semisimple group H, the functor ρ is an equivalence of categories $\mathit{Red}(H)$ and $\mathit{Lat}(H)^{op}$.*

Proof. We construct a functor $\varepsilon : \mathit{Lat}(H) \longrightarrow \mathit{Red}(H)$ as follows. Given the diagram (9.2), let T be a split torus with $T^* = A$ and Z a diagonalizable subgroup of T with $Z^* = A/\Lambda_r$. We view the torus S as a subgroup of T via the dual surjective homomorphism $A \longrightarrow \mathrm{Im}(f) = S^*$.

We embed the center C of H into Z via a homomorphism dual to the surjective composition

$$Z^* = A/\Lambda_r \longrightarrow \mathrm{Im}(f)/\Lambda_r = S^*/\Lambda_r = C^*.$$

The sequence

$$0 \longrightarrow A \xrightarrow{g} S^* \oplus (A/\Lambda_r) \xrightarrow{h} S^*/\Lambda_r \longrightarrow 0,$$

where $g(a) = (f(a), a + \Lambda_r)$ and $h(x, a + \Lambda_r) = (x - f(a)) + \Lambda_r$ is exact. It follows that the product homomorphism $S \times Z \longrightarrow T$ is surjective with the kernel C embedded into $S \times Z$ via $c \mapsto (c, c^{-1})$, i.e., $T \simeq (S \times Z)/C$.

We set $G = (H \times Z)/C$. The group Z is naturally a subgroup of G which coincides with the center of G. The torus T is a subgroup of G generated by S and Z, hence T is a split maximal torus of G. The natural sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A/\Lambda_r \longrightarrow \mathrm{Im}(f)/\Lambda_r \longrightarrow 0$$

is exact. It follows that Z/C is a torus dual to $\mathrm{Ker}(f)$. Since $G/H \simeq Z/C$, G is a (smooth connected) reductive group with H the commutator subgroup. The functor ε, by definition, takes the diagram (9.2) to the group G. By construction, both compositions of ρ and ε are isomorphic to the identity functors. $\square$

Let H be a split semisimple group as above. We consider another category $\mathsf{Mor}(H)$ with objects homomorphisms $h : B \longrightarrow \Lambda_w/\Lambda_r$ with B a finitely generated abelian group, $\mathrm{Im}(h) = S^*/\Lambda_r$ and torsion free $\mathrm{Ker}(h)$. Morphisms are defined in the obvious way. Consider a contravariant functor

$$\nu : \mathsf{Red}(H) \longrightarrow \mathsf{Mor}(H)$$

taking a reductive group G to the composition $Z^* \longrightarrow C^* \hookrightarrow \Lambda_w/\Lambda_r$, where Z is the center of G. The kernel of this homomorphism is the character lattice of the torus $Z/C = G/H$ and hence has no torsion.

Proposition 9.4. *For every split semisimple group H, the functor ν is an equivalence of categories $\mathsf{Red}(H)$ and $\mathsf{Mor}(H)^{op}$.*

Proof. We construct a functor $\lambda : \mathsf{Mor}(H) \longrightarrow \mathsf{Red}(H)$ as follows. Let $h : B \longrightarrow \Lambda_w/\Lambda_r$ be an object in $\mathsf{Mor}(H)$ and Z a diagonalizable group with $Z^* = B$. The map h yields an embedding of C into Z and the factor group Z/C is a torus. Set $G = (H \times Z)/C$ as in the proof of Proposition 9.3. The factor group G/H is isomorphic to the torus Z/C, hence G is a reductive group with the commutator subgroup H, i.e., G is an object of $\mathsf{Red}(H)$. Then Z is the center of G as the group $G/Z \simeq H/C$ is adjoint. We set $\lambda(h) = G$. By construction, both compositions of ρ and λ are isomorphic to the identity functors. $\square$

Remark 9.5. It follows from Propositions 9.3 and 9.4 that the categories $\mathsf{Lat}(H)$ and $\mathsf{Mor}(H)$ are equivalent. An equivalence between the categories can be described directly as follows. If $\Lambda_r \longrightarrow A \xrightarrow{f} \Lambda_w$ is an object in $\mathsf{Lat}(H)$, then the induced morphism $A/\Lambda_r \longrightarrow \Lambda_w/\Lambda_r$ is the corresponding object in $\mathsf{Mor}(H)$. Conversely, let $\mu : B \longrightarrow \Lambda_w/\Lambda_r$ be an object in $\mathsf{Mor}(H)$. Write A for the kernel of the homomorphism

$$h : S^* \oplus B \longrightarrow S^*/\Lambda_r$$

defined by $h(x, b) = (x + \Lambda_r) - \mu(b)$. The corresponding object

$$\Lambda_r \longrightarrow A \xrightarrow{f} \Lambda_w$$

in $\mathsf{Lat}(H)$ is defined as follows. The map f is given by the first projection followed by the inclusion of S^* into Λ_w and the inclusion $\Lambda_r \longrightarrow A$ takes x to $(x, 0)$. Note that W acts on $S^* \oplus B$ naturally on S^* and trivially on B.

A split reductive group G is called *strict* if the center Z of G is a torus, i.e., Z^* is a lattice. An object G of $\mathsf{Red}(H)$ is *strict* if G is strict. If $B \longrightarrow \Lambda_w/\Lambda_r$ is the object $\nu(G)$ of $\mathsf{Mor}(H)$, then G is strict if and only if B is torsion-free.

A semisimple group is strict if and only if it is adjoint. A *strict envelope* of a split semisimple group H is a strict object in $\mathsf{Red}(H)$.

Example 9.6. The group GL_n is a strict envelope of SL_n.

Example 9.7. The object G in $\mathsf{Red}(H)$ corresponding to the composition $S^* \longrightarrow S^*/\Lambda_r \hookrightarrow \Lambda_w/\Lambda_r$, viewed as an object of the category $\mathsf{Mor}(H)$, is strict. We call such G the *standard* strict envelope of H. By Remark 9.5, the lattice T^* is the subgroup in $S^* \oplus S^*$ consisting of all pairs (x, y) such that $x - y \in \Lambda_r$. Note that the Weyl group acts naturally on the first component of $S^* \oplus S^*$ and trivially on the second.

A strict envelope of H behaves like an "injective resolution" of H.

Lemma 9.8. *Let G_1 and G_2 be two objects in $\mathsf{Red}(H)$. If G_2 is strict, then there is a morphism $G_1 \longrightarrow G_2$ in $\mathsf{Red}(H)$.*

Proof. Let $h_i : B_i \longrightarrow \Lambda_w/\Lambda_r$ be the object $\nu(G_i)$ in $\mathsf{Mor}(H)$ for $i = 1, 2$. By assumption, B_2 is a free $\mathbb{Z}$-module. Therefore, there is a group homomorphism $g : B_2 \longrightarrow B_1$ such that $g \circ h_1 = h_2$, i.e., g is a morphism in $\mathsf{Mor}(H)$. By Proposition 9.4, there is a morphism $G_1 \longrightarrow G_2$ in $\mathsf{Red}(H)$ corresponding to g. □

10. Reductive invariants

Let H be a split semisimple group and G is a reductive group with the commutator subgroup H, i.e., G is an object in $\mathsf{Red}(H)$. By Proposition 6.1, the map $\operatorname{Inv}^d(G) \longrightarrow \operatorname{Inv}^d(H)$ is injective. We view $\operatorname{Inv}^d(G)$ as a subgroup of $\operatorname{Inv}^d(H)$. If G' is a strict envelope of H, then it follows from Lemma 9.8 that $\operatorname{Inv}^d(G') \subset \operatorname{Inv}^d(G)$. Therefore, the subgroup $\operatorname{Inv}^d(G')$ is independent of the choice of the strict resolution G' of G. We write $\operatorname{Inv}^d_{\mathrm{red}}(H)$ for this subgroup and call the invariants in this group the *reductive* invariants. By Proposition 6.1, for any prime $p \neq \operatorname{char}(F)$ we have

$$\operatorname{Inv}^d_{\mathrm{nr}}(H, p) \subset \operatorname{Inv}^d_{\mathrm{red}}(H, p) \subset \operatorname{Inv}^d(H, p). \tag{10.1}$$

Let A be a lattice and $q \in \mathsf{S}^2(A)$. We can view q as an integral quadratic form on the lattice $\widehat{A}$ dual to A. The *polar* bilinear form h of q is the image of q under the polar map $\operatorname{pol} : \mathsf{S}^2(A) \longrightarrow A \otimes A$, $aa' \mapsto a \otimes a' + a' \otimes a$. The polar form h is symmetric and *even*, i.e., $h(x, x) \in 2\mathbb{Z}$ for all $x \in \widehat{A}$. Conversely, if $h \in A \otimes A$ is a symmetric even bilinear form, then $q(x) = \frac{1}{2}h(x, x)$ is an integral quadratic form with the polar form h.

Let $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a set of simple roots of an irreducible root system, $\{w_1, w_2, \dots, w_n\}$ the corresponding fundamental weights generating the weight lattice Λ_w and W the Weyl group. Let d_i be the square of the length of the co-root $\alpha_i^\vee$. (We assume that the length of the shortest co-root is 1.) Consider the bilinear form

$$h = \sum_{i=1}^{n} w_i \otimes d_i \alpha_i = \sum_{i,j} w_i \otimes d_i c_{ij} w_j \in \Lambda_w \otimes \Lambda_w,$$

where (c_{ij}) is the Cartan matrix (see [4, Chapitre VI]). The matrix $(d_i c_{ij})$ is symmetric with even diagonal terms, hence h is a symmetric even bilinear form. The corresponding quadratic form

$$q = \frac{1}{2}\sum_{i=1}^{n} d_i w_i \alpha_i \in S^2(\Lambda_w)$$

is W-invariant by [10, Lemma 3.2]. It follows that the polar form h of q is also W-invariant.

Consider the three embeddings $i_1, i_2, j = i_1 + i_2 : \Lambda_w \longrightarrow \Lambda_w^2 := \Lambda_w \oplus \Lambda_w$ given by $x \mapsto (x,0), (0,x), (x,x)$ respectively, and the two quadratic forms $q^{(1)}, q^{(2)}$ that are the images of q under the maps $S^2(i_1), S^2(i_2) : S^2(\Lambda_w) \longrightarrow S^2(\Lambda_w^2)$ respectively. We let W act on Λ_w^2 naturally on the first summand and trivially on the second.

Let A be the sublattice of Λ_w^2 of all pairs (x,y) such that $x - y \in \Lambda_r$. Note that $\mathrm{Im}(j) \subset A$. In particular, $S^2(j)(q) \in S^2(A)$. Moreover, since $h \in (\Lambda_r \otimes \Lambda_w) \cap (\Lambda_w \otimes \Lambda_r)$ by [9, Lemma 2.1], we have $(i_k \otimes j)(h) \in A \otimes A$ and $(j \otimes i_k)(h) \in A \otimes A$ for $k = 1, 2$.

Write $m : \Lambda_w^2 \otimes \Lambda_w^2 \longrightarrow S^2(\Lambda_w^2)$ for the canonical homomorphism. We have $m(i_k \otimes j)(h) \in S^2(A)$ and $m(j \otimes i_k)(h) \in S^2(A)$ for $k = 1, 2$.

Proposition 10.2. *We have $q^{(1)} - q^{(2)} \in S^2(A)^W$ with the polar form $h^{(1)} - h^{(2)} = (j \otimes i_1)(h) - (i_2 \otimes j)(h) \in A \otimes A$.*

Proof. By construction, $q^{(1)} - q^{(2)}$ is W-invariant. We have

$$\begin{aligned} q^{(1)} - q^{(2)} &= (q^{(1)} + q^{(2)}) - 2q^{(2)} \\ &= q^{(1)} + q^{(2)} - m(i_2 \otimes i_2)(h) \\ &= q^{(1)} + q^{(2)} + m(i_1 \otimes i_2)(h) - m(j \otimes i_2)(h) \\ &= S^2(j)(q) - m(j \otimes i_2)(h) \in S^2(A). \end{aligned}$$

The second statement follows from the equality $j = i_1 + i_2$. □

Corollary 10.3. *The image of $h^{(1)} - h^{(2)}$ under the map*

$$A \otimes A \xrightarrow{p_1 \otimes 1} \Lambda_w \otimes A,$$

where p_1 is the first projection, coincides with the image of h under the natural map

$$\Lambda_w \otimes \Lambda_r \xrightarrow{1 \otimes i_1} \Lambda_w \otimes A.$$

Proof. The statement follows from Proposition 10.2 and the equalities $p_1 \circ j = p_1 \circ i_1 = 1$ and $p_1 \circ i_2 = 0$. □

Let $\widetilde{H}$ be a split simply connected cover of H with a split maximal torus $\widetilde{S}$, thus $\widetilde{S}^* = \Lambda_w$. Consider the standard strict envelope $\widetilde{G}$ of $\widetilde{H}$ (see Example 9.7). The character group $\widetilde{T}^*$ of the maximal torus $\widetilde{T}$ of $\widetilde{G}$ coincides with the group A as above. If $\widetilde{H}$ is simple, by Proposition 9.7, $\bar{q} := q^{(1)} - q^{(2)} \in S^2(\widetilde{T}^*)^W$. The form $\bar{q}$ maps to q under the natural map $S^2(\widetilde{T}^*)^W \longrightarrow S^2(\widetilde{S}^*)^W = S^2(\Lambda_w)^W$.

In the general case,

$$\widetilde{H} = \widetilde{H}_1 \times \widetilde{H}_2 \times \cdots \times \widetilde{H}_s,$$

with $\widetilde{H}_j$ the simple simply connected components of $\widetilde{H}$. The components define a basis $q_1, q_2, \ldots, q_s$ of $S^2(\widetilde{H}^*)^W$. Every q_j has a lift $\bar{q}_j \in S^2(\widetilde{T}^*)^W$ as above. Lemma 5.2 then yields the following statement.

Corollary 10.4. *The map $S^2(\widetilde{T}^*)^W \longrightarrow S^2(\widetilde{S}^*)^W$ is surjective and $S^2(\widetilde{T}^*)^W_{\mathrm{ind}} \longrightarrow S^2(\widetilde{S}^*)^W_{\mathrm{ind}}$ is an isomorphism. In particular, $S^2(\widetilde{T}^*)^W_{\mathrm{ind}}$ is generated by the classes of the forms $\bar{q}_j$.*

We will write α_{ij} for the simple roots of the j-th component and w_{ij} for the corresponding fundamental weights, etc.

Let $\widetilde{C} \subset \widetilde{H}$ be a central subgroup and set $G := \widetilde{G}/\widetilde{C}$ and $T := \widetilde{T}/\widetilde{C}$. The character group $\widetilde{C}^*$ is a factor group of Λ_w/Λ_r. Consider the composition

$$S^2(\widetilde{T}^*) \xrightarrow{\mathrm{pol}} \widetilde{T}^* \otimes \widetilde{T}^* \xrightarrow{p_1} \Lambda_w \otimes \widetilde{T}^* \longrightarrow \widetilde{C}^* \otimes \widetilde{T}^*. \tag{10.5}$$

Note that $S^2(T^*)$ is contained in the kernel of the composition.

By Corollary 10.3, the image of q_j under this composition is equal to

$$\sum_{i,j} d_{ij}\overline{w}_{ij} \otimes (\alpha_{ij}, 0),$$

where $\overline{x}$ denotes the image of an $x \in \widetilde{T}^*$ in $\widetilde{C}^*$.

Let $\widetilde{T}_j \subset \widetilde{T}$ be a maximal torus of of the j-th simple component of $\widetilde{G}$, so that $\widetilde{T} = \widetilde{T}_1 \times \cdots \times \widetilde{T}_s$. Let $\widetilde{C}_j$ be the image of the projection $\widetilde{C} \longrightarrow \widetilde{T}_j$. Then $\widetilde{C}_j^*$ can be viewed as a subgroup of $\widetilde{C}^*$ and $\overline{w}_{ij} \in \widetilde{C}_j^*$.

Proposition 10.6. *Let $q := \sum_{j=1}^s k_j\bar{q}_j \in S^2(\widetilde{T})$ be a linear combination with integer coefficients k_j. If q has trivial image under the composition (10.5) (for example, if $q \in S^2(T^*)$), then the order of $\overline{w}_{ij}$ in $\widetilde{C}_j^*$ divides k_jd_{ij} for all i and j.*

Proof. We have $\sum_{i,j} k_jd_{ij}\overline{w}_{ij} \otimes (\alpha_{ij}, 0) = 0$ in $\widetilde{C}^* \otimes \widetilde{T}^*$. Note that the elements $(\alpha_{ij}, 0)$ form part of a basis of $\widetilde{T}^*$ (with the complement (w_{ij}, w_{ij})). It follows that $k_jd_{ij}\overline{w}_{ij} = 0$ in $\widetilde{C}_j^*$ for all i and j, whence the result. □

11. Degree 3 unramified invariants of reductive groups

We assume that the base field F is algebraically closed.

Proposition 11.1. *Let H be a (split) semisimple group over F with the components of the Dynkin diagram of types A_m for some m or E_6. Suppose that E_6^{sc} does not split off H as a direct factor. Then $\mathrm{Inv}^3_{\mathrm{red}}(H, p) = \mathrm{Inv}^3_{\mathrm{nr}}(H, p) = 0$ for all odd primes $p \neq \mathrm{char}(F)$.*

Proof. Let $\widetilde{H} \longrightarrow H$ be a simply connected cover with kernel $\widetilde{C}$ and $\widetilde{G}$ be the standard strict envelope of $\widetilde{H}$. By Proposition 7.1, replacing $\widetilde{C}$ if necessary, we may assume that $\widetilde{C}^*$ is a p-group. Set $G := \widetilde{G}/\widetilde{C}$. We choose split maximal tori $S \subset H$, $\widetilde{S} \subset \widetilde{H}$, $T \subset G$, $\widetilde{T} \subset \widetilde{G}$ as in Section 10. The group $Q := G/H = \widetilde{G}/\widetilde{H}$ is a torus.

By Proposition 6.1, it suffices to prove that $\mathrm{Inv}^3(G, p) = 0$. By Theorem 6.2, we are reduced to proving that $S^2(T^*)^W_{\mathrm{ind}}\{p\} = 0$.

By Lemma 5.2(1) and Corollary 10.4, the rows of the diagram

$$\begin{tikzcd}
0 \arrow[r] & S^2(Q^*) \arrow[r] \arrow[d, equal] & S^2(T^*)^W \arrow[r] \arrow[d, hook] & S^2(S^*)^W \arrow[d, hook] & \\
0 \arrow[r] & S^2(Q^*) \arrow[r] & S^2(\widetilde{T}^*)^W \arrow[r] & S^2(\widetilde{S}^*)^W \arrow[r] & 0
\end{tikzcd}$$

are exact.

Let $\alpha \in S^2(T^*)^W_{\mathrm{ind}}\{p\}$. Since p is odd, it sufficient to show that $2\alpha = 0$. The element α lifts to a form $q \in S^2(T^*)^W$. Recall that $S^2(\widetilde{S}^*)^W$ is a free abelian group with basis $\{q_j\}$. Hence the image of q in $S^2(\widetilde{S}^*)^W$ is equal to $\sum_{j=1}^s k_j q_j$ for some $k_j \in \mathbb{Z}$. Write $\bar{q}$ for $\sum_{j=1}^s k_j \bar{q}_j \in S^2(\widetilde{T}^*)^W$. Therefore, in $S^2(\widetilde{T}^*)^W$ we have $q = \bar{q} + t$ for some $t \in S^2(Q^*)$.

Note that since the Dynkin diagram of H is simply laced all the integer d_{ij} are equal to 1 for all i and j. The images of q and t are trivial under (10.5), hence so is $\bar{q}$. By Proposition 10.6, the order of $\overline{w}_{ij}$ in $\widetilde{C}_j^*$ divides k_j for all i and j.

We claim that the class of $2k_j q_j$ is contained in $S^2(S^*)^W_{\mathrm{dec}}$ for all j.

Case 1: The j-th simple component $\widetilde{G}_j$ is of type A_m for some m, i.e, $\widetilde{H}_j = \mathrm{SL}_{m+1}$. The center of $\widetilde{H}_j$ is μ_{m+1}, hence $\widetilde{C}_j = \mu_{p^r}$ for some r. The element $\overline{w}_{1j}$ is a generator of $\widetilde{C}^* = \mathbb{Z}/p^r\mathbb{Z}$, hence the order of $\overline{w}_{1j}$ is equal to p^r. Therefore, k_j is divisible by p^k. As p is odd, by [1, 4.2], the form $p^k q_j$ and hence $k_j q_j$ belongs to $S^2(S_j^*)^{W_j}_{\mathrm{dec}}$. Taking the image of $k_j q_j$ under the homomorphism $S_j^* \longrightarrow S^*$, we see that $k_j q_j \in S^2(S^*)^W_{\mathrm{dec}}$.

Case 2: The j-th simple component $\widetilde{H}_j$ is of type E_6^{sc}. The center of $\widetilde{H}_j$ is μ_3, hence $\widetilde{C}_j$ is a subgroup of μ_3. If $\widetilde{C}_j = 1$, then $\widetilde{H}_j$ is a direct factor of H and hence E_6^{sc} is a direct factor of H. This is impossible by the assumption. Therefore, $\widetilde{C}_j = \mu_3$ (and hence $p = 3$). The element $\overline{w}_{1j}$ is a generator of $\widetilde{C}^* = \mathbb{Z}/3\mathbb{Z}$, hence k_j is divisible by 3. By [10, §4, type E_6], the form $6q_j$ and hence $2k_j q_j$ belongs to $S^2(S_j^*)^{W_j}_{\mathrm{dec}}$. Taking the image of $2k_j q_j$ under the homomorphism $S_j^* \longrightarrow S^*$, we see that $2k_j q_j \in S^2(S^*)^W_{\mathrm{dec}}$. The claim is proved.

It follows from the claim that 2α belongs to the kernel of the map $S^2(T^*)^W_{\mathrm{ind}} \longrightarrow S^2(S^*)^W_{\mathrm{ind}}$. By Lemma 5.2, this map is injective, hence $2\alpha = 0$. □

Example 11.2. The statement of the proposition is wrong if $p = 2$. Consider the group $H := (\mathrm{SL}_2)^n/\widetilde{C}$, where $\widetilde{C} \subset (\mu_2)^n$ consists of all n-tuples with trivial product. Then the group $G := (\mathrm{GL}_2)^n/\widetilde{C}$ is a strict envelope of H. A G-torsor over a field K is a tuple $(Q_1, Q_2, \dots, Q_n)$ of quaternion algebras over K such that $[Q_1]+[Q_2]+\dots+[Q_n] = 0$ in $\mathrm{Br}(K)$. Let φ_i be the reduced norm quadratic form of Q_i. The sum φ of the forms φ_i in the Witt ring $W(K)$ of K belongs to the cube of the fundamental ideal of $W(K)$. The Arason invariant of φ in $H^3(K)$ yields a degree 3 invariant I of G (see [8, page 431]). The restriction J of I to H belongs to $\mathrm{Inv}^3_{\mathrm{red}}(H) = \mathrm{Im}\big(\mathrm{Inv}^3(G) \longrightarrow \mathrm{Inv}^3(H)\big)$, and I and J are nontrivial if $n \geq 3$. Note that the invariants I and J are ramified. Moreover, the map $\mathrm{Inv}^3(G) \longrightarrow \mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}})$ factors through $\mathrm{Inv}^3(\widetilde{G}^{\mathrm{gen}})$, where $\widetilde{G}^{\mathrm{gen}}$ is the product of $\mathrm{GL}_1(Q_i^{\mathrm{gen}})$. The group $\mathrm{Inv}^3(\widetilde{G}^{\mathrm{gen}})$ is trivial since $\mathrm{GL}_1(Q_i^{\mathrm{gen}})$ have only trivial torsors. It follows that J belong to the kernel of

$$\mathrm{Inv}^3(H) \longrightarrow \mathrm{Inv}^3(\widetilde{H}^{\mathrm{gen}}),$$

hence the map in Proposition 8.2 is not injective.

Theorem 11.3. *Let G be a (split) reductive group over an algebraically closed field F. Then $\mathrm{Inv}^3_{\mathrm{nr}}(G, p) = 0$ for every odd prime $p \neq \mathrm{char}(F)$.*

Proof. Let H be the commutator subgroup of G. By Proposition 6.1(2), it suffices to prove that $\mathrm{Inv}^3_{\mathrm{nr}}(H, p) = 0$. Let $\widetilde{H} \longrightarrow H$ be a simply connected cover with kernel $\widetilde{C}$. Let $\widetilde{C}' \subset \widetilde{C}$ be a subgroup such that $(\widetilde{C}/\widetilde{C}')^*$ is the 2-component of $\widetilde{C}^*$. Since p is odd, by Proposition 7.1, $\mathrm{Inv}^3_{\mathrm{nr}}(H, p) = \mathrm{Inv}^3_{\mathrm{nr}}(\widetilde{H}/C', p)$. Replacing H by $\widetilde{H}/\widetilde{C}'$, we may assume that $\widetilde{C}^*$ has odd order.

Write $\widetilde{H}$ as a product of simple simply connected groups $\widetilde{H}_j$ and let $\widetilde{C}_j$ be the center of $\widetilde{H}_j$. If the order of $\widetilde{C}_j^*$ is a power of 2, the projection $\widetilde{C} \longrightarrow \widetilde{C}_j$ is trivial and therefore, the simply connected group $\widetilde{H}_j$ splits off H as a direct factor. Thus, the simply connected simple groups of types B_n, C_n, D_n, E_7, E_8, F_4 and G_2 split off H, i.e., $H = H_1 \times H_2$, where H_1 is simply connected and H_2 satisfies the conditions of Proposition 11.1. By the additivity property Corollary 6.3, Propositions 8.1(2) and 11.1, we have $\mathrm{Inv}^3_{\mathrm{nr}}(H, p) = 0$. □

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