

UNRAMIFIED DEGREE THREE INVARIANTS OF REDUCTIVE GROUPS

A. MERKURJEV

ABSTRACT. We prove that if G is a reductive group over an algebraically closed field, then the group of unramified Galois cohomology $H_{\text{nr}}^3(F(BG), \mathbb{Q}/\mathbb{Z}(2))$ is trivial for the classifying space BG of G .

1. INTRODUCTION

The notion of a cohomological invariant of an algebraic group was introduced by J-P. Serre in [6]. Let G be an algebraic group over a field F and M a Galois module over F . A *degree d invariant* of G assigns to every G -torsor over a field extension K over F an element in the Galois cohomology group $H^d(K, M)$, functorially in K . In this paper we consider the cohomology groups $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$, where $\mathbb{Q}/\mathbb{Z}(d-1)$ is defined as the Galois module of $(d-1)$ -twisted roots of unity. The p -part of this module requires special care if $p = \text{char}(F) > 0$. All degree d invariants of G form an abelian group $\text{Inv}^d(G)$. An invariant is *normalized* if it takes a trivial torsor to the trivial cohomology class. The group $\text{Inv}^d(G)$ is the direct sum of the subgroup $\text{Inv}^d(G)_{\text{norm}}$ of normalized invariants and the subgroup of *constant* invariants isomorphic to $H^d(F)$.

The group $\text{Inv}^d(G)_{\text{norm}}$ for small values of d is well understood. The group $\text{Inv}^1(G)_{\text{norm}}$ is trivial if G is connected. There is a canonical isomorphism $\text{Inv}^2(G)_{\text{norm}} \simeq \text{Pic}(G)$ for every reductive group G (see [2, Theorem 2.4]). M. Rost proved (see [6, Part 2]) that if G is simple simply connected then the group $\text{Inv}^3(G)_{\text{norm}}$ is cyclic of finite order with a canonical generator called the *Rost invariant*. The group $\text{Inv}^3(G)_{\text{norm}}$ for an arbitrary semisimple group G was studied in [10].

For a prime integer p , write $H^d(K, p)$ and $\text{Inv}^d(G, p)$ for the p -primary components of $H^d(K)$ and $\text{Inv}^d(G)$ respectively. If v is a discrete valuation of a field extension K/F trivial on F with residue field $F(v)$, then there is defined the *residue* homomorphism $\partial_v : H^d(K, p) \rightarrow H^{d-1}(F(v), p)$ for every $p \neq \text{char}(F)$. An element $a \in H^d(K, p)$ is *unramified* with respect to v if $\partial_v(a) = 0$. We write $H_{\text{nr}}^d(K, p)$ for the subgroup of all elements unramified with respect to every discrete valuation of K over F . An invariant in $\text{Inv}^d(G, p)$

Date: October, 2014.

The work has been supported by the NSF grant DMS #1160206 and the Guggenheim Fellowship.

is called *unramified* if all values of the invariant over every K/F belongs to $H_{\text{nr}}^d(K, p)$. We write $\text{Inv}_{\text{nr}}^d(G, p)$ for the group of all unramified invariants.

Let V be a generically free representation of G . There is a nonempty G -invariant open subscheme $U \subset V$ and a *versal* G -torsor $U \rightarrow X$ for a variety X over F . We think of X as an approximation of the *classifying space* BG of G . The larger the codimension of $V \setminus U$ in V the better X approximates BG . Abusing notation, we will write BG for X . Note that the stable birational type of BG is well defined.

The generic fiber of the versal G -torsor is the *generic* G -torsor over the function field $F(BG)$ of the classifying space. A theorem of Rost and Totaro asserts that the evaluation at the generic G -torsor yields an isomorphism between $\text{Inv}_{\text{nr}}^d(G, p)$ and the subgroup of $H^d(F(BG), p)$ of all elements unramified with respect to the discrete valuations associated with all irreducible divisors of BG . This isomorphism restricts to an isomorphism

$$\text{Inv}_{\text{nr}}^d(G, p) \xrightarrow{\sim} H_{\text{nr}}^d(F(BG), p).$$

A classical question is whether the classifying space BG of an algebraic group G is stably rational. To disprove stable rationality of BG it suffices to show that the map $H^d(F, p) \rightarrow H_{\text{nr}}^d(F(BG), p)$ is not surjective for some d and p or, equivalently, to find a non-constant unramified invariant of G . For example, D. Saltman disproved in [14] the Noether Conjecture (that V/G is stably rational for a faithful representation V of a finite group G over an algebraically closed field) by proving that $H_{\text{nr}}^2(F(BG), p) \neq H^2(F, p)$ for some G and p , i.e., by establishing a non-constant degree 2 invariant of G . E. Peyre found new examples of finite groups with non-constant unramified degree 3 invariants in [12]. Degree 3 unramified invariants of simply connected groups (over arbitrary fields) were studied in [11] (classical groups) and [7] (exceptional groups).

It is still a wide open problem whether there exists a connected algebraic group G over an algebraically closed field F with the classifying space BG that is not stably rational. Connected groups have no non-trivial degree 1 invariants. F. Bogomolov proved in [3, Lemma 5.7] (see also [2, Theorem 5.10]) that connected groups have no non-trivial degree 2 unramified invariants. In [15] and [16], D. Saltman proved that the projective linear group PGL_n has no non-trivial degree 3 unramified invariants.

In the present paper, we study unramified degree 3 invariants of an arbitrary (connected) reductive group G over an algebraically closed field, or equivalently, the unramified elements in $H^3(F(BG))$. The language of invariants seems easier to work with. The main result is the following theorem (see Theorems 8.4 and 11.3):

Theorem. Let G be a split reductive group over an algebraically closed field F and p a prime integer different from $\text{char}(F)$. Then

$$\text{Inv}_{\text{nr}}^3(G, p) = H_{\text{nr}}^3(F(BG)) = 0$$

if p is odd or the commutator subgroup of G is (almost) simple.

Let H be the commutator subgroup of a split reductive group G . We have $\text{Inv}_{\text{nr}}^3(G, p) = \text{Inv}_{\text{nr}}^3(H, p)$ (see Proposition 6.1). If H is a simple group, we compare the group $\text{Inv}^3(H)$ with the group $\text{Inv}^3(\tilde{H}^{\text{gen}})$, where $\tilde{H}^{\text{gen}}$ is the simply connected cover of H twisted by a generic H -torsor, and use our knowledge of the unramified degree 3 invariants in the simply connected case. The key statement is the injectivity of the homomorphism $\text{Inv}^3(H) \longrightarrow \text{Inv}^3(\tilde{H}^{\text{gen}})$ (see Section 8).

In general, when H is semisimple but not necessarily simple, we consider an embedding of H into a reductive group G' as the commutator subgroup. Then $\text{Inv}^3(G')$ is identified with a subgroup of $\text{Inv}^3(H)$. If G' is *strict*, i.e., the center of G' is a torus, this subgroup is the smallest possible and is independent of the choice of G' . We write $\text{Inv}_{\text{red}}^3(H)$ for this subgroup. It satisfies

$$\text{Inv}_{\text{nr}}^3(H, p) \subset \text{Inv}_{\text{red}}^3(H, p) \subset \text{Inv}^3(H, p)$$

for every prime $p \neq \text{char}(F)$. The group $\text{Inv}_{\text{red}}^3(H, p)$ is easier to control than $\text{Inv}_{\text{nr}}^3(H, p)$. We show that $\text{Inv}_{\text{red}}^3(H, p) = 0$ which implies that $\text{Inv}_{\text{nr}}^3(H, p)$ is also trivial.

ACKNOWLEDGEMENTS. The author thanks J-P. Tignol for valuable remarks and the Max Planck Institute (Bonn) for hospitality.

2. BASIC DEFINITIONS AND FACTS

Let F be a field. If $d \geq 1$, we write $H^d(F)$ for the Galois cohomology group $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$, with $\mathbb{Q}/\mathbb{Z}(d-1)$ the direct sum of $\text{colim}_n \mu_n^{\otimes(d-1)}$, where μ_n is the group of roots of unity of degree n , and the p -component if $p = \text{char}(F) > 0$ (see [6, Part 2, Appendix A]). In particular, $H^1(F)$ is the group of (continuous) characters of the absolute Galois group $\text{Gal}(F_{\text{sep}}/F)$ of F and $H^2(F)$ is the Brauer group $\text{Br}(F)$. We view H^d as a functor from the category **Fields** $_F$ of field extensions of F to the category of abelian groups (or the category **Sets** of sets).

Let G be a (linear) algebraic group over a field F . The notion of an *invariant* of G was defined in [6] as follows. Consider the functor

$$\text{Tors}_G : \mathbf{Fields}_F \longrightarrow \mathbf{Sets}$$

taking a field K to the set $\text{Tors}_G(K) := H^1(K, G)$ of isomorphism classes of (right) G -torsors over $\text{Spec } K$. A *degree d cohomological invariant* of G is then a morphism of functors

$$\text{Tors}_G \longrightarrow H^d,$$

i.e., a functorial in K collection of maps of sets $\text{Tors}_G(K) \longrightarrow H^d(K)$ for all field extensions K/F . We denote the group of such invariants by $\text{Inv}^d(G)$.

An invariant $I \in \text{Inv}^d(G)$ is called *normalized* if $I(E) = 0$ for a trivial G -torsor E . The normalized invariants form a subgroup $\text{Inv}^d(G)_{\text{norm}}$ of $\text{Inv}^d(G)$ and there is a natural isomorphism

$$\text{Inv}^d(G) \simeq H^d(F) \oplus \text{Inv}^d(G)_{\text{norm}}.$$

Example 2.1. Let G be a (connected) reductive group over F . It is shown in [2, Theorem 2.4] that there is an isomorphism

$$\beta_G : \text{Pic}(G) \xrightarrow{\sim} \text{Inv}^2(G)_{\text{norm}}.$$

Let G be a split reductive group and H the commutator subgroup of G . Let $\pi : \tilde{H} \rightarrow H$ be a simply connected cover with kernel $\tilde{C}$. There are canonical isomorphisms (see [17, §6])

$$(2.2) \quad \text{Pic}(G) \xrightarrow{\sim} \text{Pic}(H) \simeq \tilde{C}^* := \text{Hom}(\tilde{C}, \mathbb{G}_m).$$

Take any character $\chi \in \tilde{C}^*$ and consider the push-out diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \tilde{C} & \longrightarrow & \tilde{H} & \xrightarrow{\pi} & H \longrightarrow 1 \\ & & \downarrow \chi & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & H' & \longrightarrow & H \longrightarrow 1. \end{array}$$

The isomorphism $\tilde{C}^* \simeq \text{Pic}(H)$ takes a character χ to the class the line bundle L_χ on H given by the $\mathbb{G}_m$ -torsor $H' \rightarrow H$ in the bottom row of the diagram. For a field extension K/F and an H -torsor E over K , the value of the invariant $\beta_H(L_\chi)$ is equal to $\delta([E]) \in H^2(K, \mathbb{G}_m) = \text{Br}(K)$, where $[E]$ is the class of E in $H^1(K, H)$ and $\delta : H^1(K, H) \rightarrow H^2(K, \mathbb{G}_m)$ is the connecting map for the bottom exact sequence in the diagram.

If $f : G_1 \rightarrow G_2$ is a homomorphism of algebraic groups over F and E_1 is a G_1 -torsor over a field extension K/F , then $E_2 := (E_1 \times G_2)/G_1$ is the G_2 -torsor over K which we denote by $f_*(E_1)$. If I is a degree d invariant of G_2 , we define an invariant $f^*(I)$ of G_1 by $f^*(I)(E_1) := I(f_*(E_1))$. Thus, we have a group homomorphism

$$(2.3) \quad f^* : \text{Inv}^d(G_2) \rightarrow \text{Inv}^d(G_1)$$

taking normalized invariants to the normalized ones.

Let G be an algebraic group over a field F and let V be a generically free representation of G . There is a nonempty G -invariant open subscheme $U \subset V$ such that U is a G -torsor over a variety which we denote by U/G (see [18, Remark 1.4]). We think of U/G as an approximation of the “classifying space” BG of G and abusing notation write $U/G = BG$. The space BG is better approximated by U/G if the codimension of $V \setminus U$ in V is large. For our purposes it suffices to assume that this codimension is at least 3 (see [2]).

Note that by the No-name Lemma, the stable rationality type of BG is uniquely determined by G .

The generic fiber $E^{\text{gen}} \rightarrow \text{Spec}(F(BG))$ of the projection $U \rightarrow U/G$ is called the *generic* G -torsor. The value of an invariant of G at the generic torsor E^{gen} yields a homomorphism

$$\text{Inv}^d(G) \rightarrow H^d(F(BG)).$$

Rost proved (see [6, Part 2, Th. 3.3] or [2, Theorem 2.2]) that this map is injective, i.e., every invariant is determined by its value at the generic torsor.

We decompose the group of invariants into a direct sum of primary components:

$$\mathrm{Inv}^d(G) = \coprod_{p \text{ prime}} \mathrm{Inv}^d(G, p).$$

Let K be a field extension of F . For a prime integer p , write $H^d(K, p)$ for the p -primary component of $H^d(K)$. Let v be a discrete valuation of K over F with residue field $F(v)$. If $\mathrm{char}(F) \neq p$, there is the *residue map* (see [6, Chapter 2])

$$\partial_v : H^d(K, p) \longrightarrow H^{d-1}(F(v), p).$$

An element $a \in H^d(K, p)$ is *unramified with respect to v* if $\partial_v(a) = 0$.

A point x of codimension 1 in BG for an algebraic group G yields a discrete valuation v_x on the function field $F(BG)$ over F . Write $A^0(BG, H^d, p)$ for the group of all elements in $H^d(F(BG), p)$ that are unramified with respect to v_x for all points x of codimension 1 in BG . It is proved in [6, Part 1, Theorem 11.7] that the value of every invariant from $\mathrm{Inv}^d(G, p)$ at the generic G -torsor E^{gen} belongs to $A^0(BG, H^d, p)$. Moreover, we have the following theorem (see [6, Part 1, Appendix C]):

Theorem 2.4. *Let G be an algebraic group over F and p a prime different from $\mathrm{char}(F)$. Then the evaluation of an invariant at the generic G -torsor yields an isomorphism*

$$\mathrm{Inv}^d(G, p) \xrightarrow{\sim} A^0(BG, H^d, p).$$

The inverse isomorphism is defined as follows. Let E be a G -torsor over a field extension K/F and $BG = U/G$. We have the following canonical morphisms:

$$\mathrm{Spec} K = E/G \xleftarrow{f} (E \times U)/G \xrightarrow{h} U/G = BG.$$

Note that the groups $H^d(K, p)$ for all d and all field extensions K/F form a *cycle module* in the sense of Rost (see [13]). In particular, we have flat pull-back homomorphisms

$$H^d(K, p) = A^0(\mathrm{Spec} K, H^d, p) \xrightarrow{f^*} A^0((E \times U)/G, H^d, p) \xleftarrow{h^*} A^0(BG, H^d, p).$$

The variety $(E \times U)/G$ is an open subscheme of the vector bundle $(E \times V)/G$ over $\mathrm{Spec} K$. By the homotopy invariance property, the pull-back homomorphism

$$H^d(K, p) = A^0(\mathrm{Spec} K, H^d, p) \longrightarrow A^0((E \times V)/G, H^d, p)$$

is an isomorphism. Since the inclusion of $(E \times U)/G$ into $(E \times V)/G$ is a bijection on points of codimension 1 (by our assumption on the codimension of $V \setminus U$ in V), the restriction homomorphism

$$A^0((E \times V)/G, H^d, p) \longrightarrow A^0((E \times U)/G, H^d, p)$$

is an isomorphism. It follows that f^* is an isomorphism.

Let $a \in A^0(BG, H^d, p)$. The invariant $I \in \mathrm{Inv}^d(G, p)$ defined by $I(E) = (f^*)^{-1}h^*(a)$ is the inverse image of a under the isomorphism in Theorem 2.4.

3. DECOMPOSABLE INVARIANTS

The group of decomposable degree 3 invariants of a semisimple group was defined in [10, §1]. We extend this definition to the class of split reductive groups.

Let G be a split reductive group over F . The $\cup$ -product $H^2(K) \otimes K^\times \longrightarrow H^3(K)$ for any field extension K/F yields a pairing

$$\mathrm{Inv}^2(G)_{\mathrm{norm}} \otimes F^\times \longrightarrow \mathrm{Inv}^3(G)_{\mathrm{norm}}.$$

The subgroup of *decomposable invariants* $\mathrm{Inv}^3(G)_{\mathrm{dec}}$ is the image of the pairing.

Proposition 3.1. *Let G be a split reductive group over F . Then the composition*

$$\mathrm{Pic}(G) \otimes F^\times \xrightarrow{\sim} \mathrm{Inv}^2(G)_{\mathrm{norm}} \otimes F^\times \longrightarrow \mathrm{Inv}^3(G)_{\mathrm{dec}}$$

is an isomorphism.

Proof. The surjectivity of the composition follows from the definition. Let H be the commutator subgroup of G . By [10, Theorem 4.2]), the composition is an isomorphism when G is replaced by H . The injectivity of the composition for G follows then from the fact that the map $\mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(H)$ in (2.2) is an isomorphism. $\square$

It follows from the proposition that $\mathrm{Inv}^3(G)_{\mathrm{dec}} = 0$ if $\mathrm{Pic}(G) = 0$ (for example, G is semisimple simply connected) or F is algebraically closed.

We write

$$\mathrm{Inv}^3(G)_{\mathrm{ind}} := \mathrm{Inv}^3(G)_{\mathrm{norm}} / \mathrm{Inv}^3(G)_{\mathrm{dec}}.$$

4. UNRAMIFIED INVARIANTS

Let K/F be a field extension and p a prime integer different from $\mathrm{char}(F)$. We write $H_{\mathrm{nr}}^d(K/F, p)$ for the subgroup of all elements in $H^d(K, p)$ that are unramified with respect to all discrete valuations of K over F . A field extension L/K yields a natural homomorphism $H^d(K) \longrightarrow H^d(L)$ that takes $H_{\mathrm{nr}}^d(K/F, p)$ into $H_{\mathrm{nr}}^d(L/F, p)$ by [6, Part 1, Proposition 8.2].

Let G be an algebraic group over F . An invariant $I \in \mathrm{Inv}^d(G, p)$ is called *unramified* if for every field extension K/F and every $E \in \mathrm{Tors}_G(K)$, we have $I(E) \in H_{\mathrm{nr}}^d(K/F, p)$. Note that the constant invariants are always unramified. We will write $\mathrm{Inv}_{\mathrm{nr}}^d(G, p)$ for the subgroup of all unramified invariants in $\mathrm{Inv}^d(G, p)$.

If $f : G_1 \longrightarrow G_2$ is a group homomorphism, then the map f^* in (2.3) takes $\mathrm{Inv}_{\mathrm{nr}}^d(G_2, p)$ into $\mathrm{Inv}_{\mathrm{nr}}^d(G_1, p)$.

Proposition 4.1. *Let G be an algebraic group over F . An invariant $I \in \mathrm{Inv}^d(G, p)$ is unramified if and only if the value of I at the generic G -torsor in $H^d(F(BG), p)$ is unramified. In particular, $\mathrm{Inv}_{\mathrm{nr}}^d(G, p) \simeq H_{\mathrm{nr}}^d(F(BG), p)$.*

Proof. It suffices to show that the inverse of the isomorphism in Theorem 2.4 takes unramified elements to unramified invariants. Let $a \in H_{\mathrm{nr}}^d(F(BG), p) \subset$

$A^0(BG, H^d, p)$. The corresponding invariant $I \in \text{Inv}^d(G, p)$ is defined by $I(E) = (f^*)^{-1}h^*(a)$ (see Section 2). Note that h^* takes unramified elements to unramified ones and f^* yields an isomorphism on the unramified elements as the function field of $(E \times U)/G$ is a purely transcendental extension of K . It follows that $I(E)$ is unramified for all E , hence the invariant I is unramified. $\square$

Unramified invariants are constant along rational families of torsors. Precisely, if K/F is a purely transcendental field extension and E is a G -torsor over K , then for every invariant $I \in \text{Inv}_{\text{nr}}^d(G, p)$ we have

$$I(E) \in \text{Im}(H^d(F, p) \longrightarrow H^d(K, p)).$$

Indeed, $I(E) \in H_{\text{nr}}^d(K, p)$ which is the image of $H^d(F, p)$ in $H^d(K, p)$.

5. ABSTRACT CHERN CLASSES

Let A be a lattice (written additively). Consider the symmetric ring $\mathcal{S}^*(A)$ over $\mathbb{Z}$ and the group ring $\mathbb{Z}[A]$ of A . We use the exponential notation for $\mathbb{Z}[A]$: every element can be written as a finite sum $\sum_{a \in A} n_a e^a$ with $n_a \in \mathbb{Z}$. There are the *abstract Chern classes* (see [10, 3c])

$$c_i : \mathbb{Z}[A] \longrightarrow \mathcal{S}^i(A), \quad i \geq 0$$

satisfying in particular,

$$c_1\left(\sum_i e^{a_i}\right) = \sum_i a_i \in A \quad \text{and} \quad c_2\left(\sum_i e^{a_i}\right) = \sum_{i < j} a_i a_j \in \mathcal{S}^2(A).$$

The map c_1 is a homomorphism and

$$c_2(x + y) = c_2(x) + c_2(y) + c_1(x)c_1(y)$$

for all $x, y \in \mathbb{Z}[A]$.

If A is a W -lattice for a group W acting on A , then all the c_i 's are W -equivariant. It follows that c_2 yields a map (not a homomorphism in general) of groups of W -invariant elements:

$$c_2^W : \mathbb{Z}[A]^W \longrightarrow \mathcal{S}^2(A)^W.$$

The group $\mathbb{Z}[A]^W$ is generated by the elements $\sum e_i^a$, where the a_i 's form a W -orbit in A . It follows that the subgroup of $\mathcal{S}^2(A)^W$ generated by the image of c_2^W is generated by $\sum_{i < j} a_i a_j$ with the a_i 's forming a W -orbit in A and aa' for $a, a' \in A^W$. The elements of these two types can be viewed as “obvious” elements in $\mathcal{S}^2(A)^W$ which we call *decomposable*.

Write $\mathcal{S}^2(A)_{\text{dec}}^W$ for the subgroup of $\mathcal{S}^2(A)^W$ generated by the decomposable elements, or equivalently, by the image of c_2^W . Set

$$\mathcal{S}^2(A)_{\text{ind}}^W := \mathcal{S}^2(A)^W / \mathcal{S}^2(A)_{\text{dec}}^W.$$

Note that if $A^W = 0$, the map c_2^W is a homomorphism and $\mathcal{S}^2(A)_{\text{ind}}^W$ is the cokernel of c_2^W .

Lemma 5.1. *Let A_1 and A_2 be W_1 - and W_2 -lattices respectively. Then there is a canonical isomorphism*

$$S^2(A_1 \oplus A_2)_{\text{ind}}^{W_1 \times W_2} \simeq S^2(A_1)_{\text{ind}}^{W_1} \oplus S^2(A_2)_{\text{ind}}^{W_2}.$$

Proof. We have

$$S^2(A_1 \oplus A_2)^{W_1 \times W_2} \simeq S^2(A_1)^{W_1} \oplus S^2(A_2)^{W_2} \oplus (A_1^{W_1} \otimes A_2^{W_2})$$

and

$$\mathbb{Z}[A_1 \oplus A_2]^{W_1 \times W_2} \simeq \mathbb{Z}[A_1]^{W_1} \otimes \mathbb{Z}[A_2]^{W_2}.$$

The standard formulas on the Chern classes show that $c_1(\mathbb{Z}[A_i]^{W_i}) = A_i^{W_i}$ and

$$S^2(A_1 \oplus A_2)_{\text{dec}}^{W_1 \times W_2} \simeq S^2(A_1)_{\text{dec}}^{W_1} \oplus S^2(A_2)_{\text{dec}}^{W_2} \oplus (A_1^{W_1} \otimes A_2^{W_2}),$$

whence the result. $\square$

Lemma 5.2. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of W -lattices. Suppose that W acts trivially on A and $C^W = 0$. Then*

(1) *The sequence*

$$0 \rightarrow S^2(A) \rightarrow S^2(B)^W \rightarrow S^2(C)^W$$

is exact.

(2) *The natural homomorphism $S^2(B)_{\text{ind}}^W \rightarrow S^2(C)_{\text{ind}}^W$ is injective.*

Proof. The first statement is proved in [5, Lemma 4.9]. Since W acts trivially on A , for every subgroup $W' \subset W$, we have $H^1(W', A) = 0$, hence the map $B^{W'} \rightarrow C^{W'}$ is surjective. The group $\mathbb{Z}[C]^W$ is generated by elements of the form $\sum_i e^{c_i}$, where the c_i 's form a W -orbit in C . By the surjectivity above, applied to the stabilizer $W' \subset W$, this orbit can be lifted to a W -orbit in B . Therefore, the map $\mathbb{Z}[B]^W \rightarrow \mathbb{Z}[C]^W$ is surjective. The second statement follows from this, the first statement of the lemma and the fact that $S^2(A) = S^2(A)_{\text{dec}}^W \subset S^2(B)_{\text{dec}}^W$. $\square$

6. DEGREE 3 INVARIANTS OF SPLIT REDUCTIVE GROUPS

Let G be a split reductive group over F and let H be the commutator subgroup of G . Thus, H is a split semisimple group and the factor group $Q := G/H$ is a split torus.

Proposition 6.1. *1. The restriction maps $\text{Inv}^d(G) \rightarrow \text{Inv}^d(H)$ and $\text{Inv}^d(G)_{\text{ind}} \rightarrow \text{Inv}^d(H)_{\text{ind}}$ are injective.*

2. For every prime $p \neq \text{char}(F)$, the restriction map $\text{Inv}_{\text{nr}}^d(G, p) \rightarrow \text{Inv}_{\text{nr}}^d(H, p)$ is an isomorphism.

Proof. For a field extension K/F , the map

$$j : H^1(K, H) \rightarrow H^1(K, G)$$

is surjective as $H^1(K, Q) = 1$ and the group $Q(K)$ acts transitively on the fibers of j . It follows that the restriction map $\text{Inv}^d(G) \rightarrow \text{Inv}^d(H)$ is injective. The injectivity of $\text{Inv}^d(G)_{\text{ind}} \rightarrow \text{Inv}^d(H)_{\text{ind}}$ follows then from Proposition 3.1.

As Q is a rational variety, the fibers of j are rational families of H -torsors. Since an unramified invariant of H must be constant on the fibers, it defines an invariant of G . This proves the second statement. $\square$

Let G be a split reductive group, $T \subset G$ a split maximal torus. By [10, 3d], there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{CH}^2(BG) & \longrightarrow & \overline{H}_{\text{ét}}^{4,2}(BG) & \longrightarrow & \mathrm{Inv}^3(G)_{\text{norm}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{CH}^2(BT) & \longrightarrow & \overline{H}_{\text{ét}}^{4,2}(BT) & \longrightarrow & \mathrm{Inv}^3(T)_{\text{norm}} \longrightarrow 0 \end{array}$$

with the exact rows, where $\overline{H}_{\text{ét}}^{4,2}(BH) = \overline{H}^4(BH, \mathbb{Z}(2))$ for an algebraic group H is the reduced weight two étale motivic cohomology group (see [9, §5]). The group $\mathrm{Inv}^3(T)_{\text{norm}}$ is trivial as T has no nontrivial torsors and $\mathrm{CH}^2(BT) = \mathcal{S}^2(T^*)$ by [2, Example A.5], hence the middle term in the bottom row is isomorphic to $\mathcal{S}^2(T^*)$.

Let N be the normalizer of T in G and $W = N/T$ the Weyl group. The group W acts naturally on BT . Moreover, if $w \in W$, the composition

$$BT \xrightarrow{w} BT \xrightarrow{s} BG,$$

where s is the natural morphism, coincides with s . Therefore, the image of the middle vertical homomorphism in the diagram

$$\overline{H}_{\text{ét}}^{4,2}(BG) \longrightarrow \overline{H}_{\text{ét}}^{4,2}(BT) = \mathcal{S}^2(T^*)$$

is contained in the subgroup $\mathcal{S}^2(T^*)^W$ of W -invariant elements. By [10, Lemma 3.8], the image of $\mathrm{CH}^2(BG)$ under this homomorphism is equal to $\mathcal{S}^2(T^*)_{\text{dec}}^W$. Therefore, by diagram chase, we have a homomorphism $\mathrm{Inv}^3(G)_{\text{norm}} \longrightarrow \mathcal{S}^2(T^*)_{\text{ind}}^W$. The group of decomposable invariants $\mathrm{Inv}^3(G)_{\text{dec}}$ is in the kernel of this map since $\mathrm{Inv}^3(G)_{\text{dec}}$ vanishes over an algebraic closure of F and the group $\mathcal{S}^2(T^*)_{\text{ind}}^W$ does not change. Therefore, we have a well-defined homomorphism

$$\alpha_G : \mathrm{Inv}^3(G)_{\text{ind}} \longrightarrow \mathcal{S}^2(T^*)_{\text{ind}}^W.$$

Theorem 6.2. *Let G be a split reductive group over F . Then the map α_G is injective. If G is semisimple, then α_G is an isomorphism.*

Proof. The second statement is proved in [10, Theorem 3.9]. The first statement follows from Proposition 6.1(1), the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Inv}^3(G)_{\text{ind}} & \xrightarrow{\alpha_G} & \mathcal{S}^2(T^*)_{\text{ind}}^W \\ \downarrow & & \downarrow \\ \mathrm{Inv}^3(H)_{\text{ind}} & \xrightarrow{\alpha_H} & \mathcal{S}^2(S^*)_{\text{ind}}^W, \end{array}$$

where H is the commutator subgroup of G and S is a maximal torus of H , and the second statement applied to H . $\square$

Proposition 3.1 and Lemma 5.1 yield the following additivity property.

Corollary 6.3. *Let H_1 and H_2 be two split semisimple groups. Then there is a canonical isomorphism*

$$\mathrm{Inv}^3(H_1 \times H_2) \simeq \mathrm{Inv}^3(H_1) \oplus \mathrm{Inv}^3(H_2).$$

Let H be a split semisimple group over a field F , $\pi : \tilde{H} \rightarrow H$ a simply connected cover, $\tilde{S}$ the pre-image of a split maximal torus S of H , so $\tilde{S}$ is a split maximal torus of $\tilde{H}$. Then $\mathcal{S}^2(S^*)$ can be viewed with respect to π as a sublattice of $\mathcal{S}^2(\tilde{S}^*)$ of finite index and we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}^3(H)_{\mathrm{ind}} & \xrightarrow[\sim]{\alpha_H} & \mathcal{S}^2(S^*)_{\mathrm{ind}}^W \\ \pi^* \downarrow & & \downarrow \\ \mathrm{Inv}^3(\tilde{H})_{\mathrm{norm}} & \xlongequal{\quad} \mathrm{Inv}^3(\tilde{H})_{\mathrm{ind}} & \xrightarrow[\sim]{\alpha_{\tilde{H}}} \mathcal{S}^2(\tilde{S}^*)_{\mathrm{ind}}^W. \end{array}$$

If H is simple, the group $\mathcal{S}^2(\tilde{S}^*)^W$ is infinite cyclic with a canonical generator q (see [6, Part 2, §7]). It follows that $\mathcal{S}^2(S^*)^W$ is also infinite cyclic with kq a generator for a unique integer $k > 0$. The invariant $R \in \mathrm{Inv}^3(\tilde{H})_{\mathrm{norm}}$ corresponding to the generator q is called the *Rost invariant* of $\tilde{H}$. It is a generator of the cyclic group $\mathrm{Inv}^3(\tilde{H})$.

7. CHANGE OF GROUPS

In this section we prove the following useful property.

Proposition 7.1. *Let p be a prime integer different from $\mathrm{char}(F)$, G an algebraic group over F , $C \subset G$ a finite central diagonalizable subgroup of order not divisible by p , $H = G/C$. Then the natural maps $\mathrm{Inv}^d(H, p) \rightarrow \mathrm{Inv}^d(G, p)$ and $\mathrm{Inv}_{\mathrm{nr}}^d(H, p) \rightarrow \mathrm{Inv}_{\mathrm{nr}}^d(G, p)$ are isomorphisms.*

Proof. Both functors in the definition of an invariant can be naturally extended to the category $\mathcal{C}$ of F -algebras that are finite product of fields, and every invariant extends uniquely to a morphism of extended functors. If $K \rightarrow L$ is a morphism in $\mathcal{C}$ and M is an étale K -algebra, then $L \otimes_K M$ is also an object of the category $\mathcal{C}$.

For any K in $\mathcal{C}$ we have an exact sequence

$$H_{\mathrm{ét}}^1(K, G) \rightarrow H_{\mathrm{ét}}^1(K, H) \xrightarrow{\delta_K} H_{\mathrm{ét}}^2(K, C)$$

and the group $H_{\mathrm{ét}}^1(K, C)$ acts transitively on the fibers of the first map in the sequence.

Proof of injectivity. Let $I \in \mathrm{Inv}^d(H, p)$ be such that $f^*(I) = 0$, where $f : G \rightarrow H$ is the canonical homomorphism. We prove that $I = 0$. Take any K in $\mathcal{C}$ and $E \in \mathrm{Tors}_H(K)$. As an element of the group $H_{\mathrm{ét}}^2(K, C)$ is a tuple of elements in $\mathrm{Br}(K)$ of order prime to p , there is an étale K -algebra L of (constant) finite rank $[L : K]$ prime to p such that $\delta_L(E_L) = 0$. It follows that $E_L = f_*(E')$ for some $E' \in \mathrm{Tors}_G(L)$. We have

$$I(E)_L = I(E_L) = I(f_*(E')) = f^*(I)(E') = 0.$$

Since $[L : K]$ is prime to p , we have $I(E) = 0$, i.e., $I = 0$.

Proof of surjectivity. Let $J \in \text{Inv}^d(G, p)$. We construct an invariant $I \in \text{Inv}^d(H, p)$ such that $J = f^*(I)$. Take any K in $\mathcal{C}$ and $E \in \text{Tors}_H(K)$. As above, choose an étale K -algebra L of finite rank prime to p such that $\delta_L(E_L) = 0$ and an element $E' \in \text{Tors}_G(L)$ with $E_L = f_*(E')$. We set

$$I(E) = \frac{1}{[L : K]} \text{cor}_{L/F}(J(E')).$$

This is independent of the choice of E' . Indeed, if $E_L = f_*(E'')$ for $E'' \in \text{Tors}_G(L)$, then there exists $\nu \in H_{\text{ét}}^1(L, C)$ with $E'' = \nu(E')$. Choose an L -algebra P of constant rank $[P : L]$ prime to p such that $\nu_P = 1$. It follows that $E''_P = E'_P$ and therefore,

$$[P : L] \text{cor}_{L/F}(J(E'')) = \text{cor}_{P/F}(J(E''_P)) = \text{cor}_{P/F}(J(E'_P)) = [P : L] \text{cor}_{L/F}(J(E')).$$

Since $[P : L]$ is prime to p , we have $\text{cor}_{L/F}(J(E'')) = \text{cor}_{L/F}(J(E'))$.

In order to show that the value $I(E)$ is independent of the choice of L , for the two choices L and L' , it suffices to compare the formulas for L and $LL' := L \otimes_F L'$:

$$\frac{1}{[L : K]} \text{cor}_{L/F}(J(E')) = \frac{[L' : K]}{[LL' : K]} \text{cor}_{L/F}(J(E')) = \frac{1}{[LL' : K]} \text{cor}_{LL'/F}(J(E'_{LL'})).$$

We have constructed the invariant $I \in \text{Inv}^d(H, p)$. For any K in $\mathcal{C}$ and $E' \in \text{Tors}_G(K)$, by the definition of I , we have $f^*(I)(E') = I(f_*(E')) = J(E')$, hence $f^*(I) = J$. Note that if J is an unramified invariant, I is also unramified since the corestriction map preserves unramified elements by [6, Part 1, Proposition 8.6]. $\square$

8. DEGREE 3 UNRAMIFIED INVARIANTS OF SIMPLE GROUPS

The following statement was proved in [11] (classical groups) and [7] (exceptional groups).

Proposition 8.1. *Let H be an absolutely simple simply connected group over F and p a prime different from $\text{char}(F)$.*

1. If the Dynkin diagram of H is different from 2A_n , n odd, and 1D_4 , then $\text{Inv}_{\text{nr}}^3(H, p)_{\text{norm}} = 0$.

2. If H is split, then $\text{Inv}_{\text{nr}}^3(H, p)_{\text{norm}} = 0$.

Let H be a semisimple group over F , E an H -torsor over $\text{Spec}(K)$ for a field extension K/F . The *twist* $H^E := \text{Aut}_H(E)$ of H by E is a semisimple group over K . The twisting argument shows that $BH^E = BH_K$ and there is a canonical isomorphism $\text{Inv}^d(H^E) \simeq \text{Inv}^d(H_K)$. If E^{gen} is a generic H -torsor, we write H^{gen} for $H^{E^{\text{gen}}}$. Let $\tilde{H}^{\text{gen}} \rightarrow H^{\text{gen}}$ be a simply connected cover.

Proposition 8.2. *Let H be a split simple group. Then the composition*

$$\text{Inv}^3(H)_{\text{ind}} \longrightarrow \text{Inv}^3(H^{\text{gen}})_{\text{ind}} \longrightarrow \text{Inv}^3(\tilde{H}^{\text{gen}})_{\text{ind}} = \text{Inv}^3(\tilde{H}^{\text{gen}})$$

is injective.

Proof. The statement is clear if H is a simply connected group. The case of an adjoint group H was considered in [10, Theorem 4.10]. Consider the other split semisimple groups type-by-type. It suffices to restrict to the p -component of $\text{Inv}^3(H)$ for a prime p .

Type A_{n-1} , $n \geq 2$. We have $H = \text{SL}_n / \mu_m$ for an integer m dividing n . By Proposition 7.1, we may assume that $m = p^r$ for some r . It is shown in [1, Theorem 4.1] and Theorem 6.2 that

$$\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\sim} \mathcal{S}^2(S^*)_{\text{ind}}^W \hookrightarrow (\mathbb{Z}/m\mathbb{Z})q.$$

On the other hand, an H -torsor yields a central simple algebra of degree n and exponent dividing m . A generic torsor gives an algebra with the exponent exactly m , hence $\text{Inv}^3(\tilde{H}^{\text{gen}}) = (\mathbb{Z}/m\mathbb{Z})R$ by [6, Part 2, Theorem 11.5].

Type D_n , $n \geq 4$. We have $H = \text{O}_{2n}^+$, the special orthogonal group or $H = \text{HSpin}_{2n}$, the half-spin group if n is even. It is shown in [6, Part 1, Chapter VI] in the case $\text{char}(F) \neq 2$ that $\text{Inv}^3(\text{O}_{2n}^+)_{\text{ind}} = 0$. In general, recall that the character group of a maximal split torus S is a free group of rank n . Let $x_1, x_2, \dots, x_n$ be a basis for S^* such that the Weyl group W acts on the x_i 's by permutations and change of signs. The generator of $\mathcal{S}^2(S^*)^W$ is the quadratic form $q = x_1^2 + x_2^2 + \dots + x_n^2$. It is in $\mathcal{S}^2(S^*)_{\text{dec}}^W$ since $c_2(\sum_i e^{x_i} + e^{-x_i}) = -q$. By [10, Theorem 3.9], $\text{Inv}^3(\text{O}_{2n}^+)_{\text{ind}} = 0$.

Finally, assume that n is even and $H = \text{HSpin}_{2n}$, the half-spin group. It follows from [1, Theorem 5.1] and Theorem 6.2 that

$$\text{Inv}^3(H)_{\text{ind}} \xrightarrow{\sim} \mathcal{S}^2(S^*)_{\text{ind}}^W \hookrightarrow (\mathbb{Z}/4\mathbb{Z})q$$

and $\text{Inv}^3(H)_{\text{ind}} = 0$ if $n = 4$. On the other hand, an H -torsor yields a central simple algebra of degree $2n$. A generic torsor gives a nonsplit algebra. By [6, Part 2, Theorem 15.4], $\text{Inv}^3(\tilde{H}^{\text{gen}}) = (\mathbb{Z}/4\mathbb{Z})R$ if $n > 4$. $\square$

Remark 8.3. The statement fails for semisimple groups that are not simple, see Example 11.2.

Theorem 8.4. *Let H be a split simple group over an algebraically closed field F and p a prime integer different from $\text{char}(F)$. Then $\text{Inv}_{\text{nr}}^3(H, p) = 0$.*

Proof. Let $I \in \text{Inv}_{\text{nr}}^3(H, p)$. Note that since F is algebraically closed, every decomposable invariant is trivial.

The pull-back $\tilde{I}$ of I under the composition in Proposition 8.2 is an unramified invariant. As $\tilde{H}^{\text{gen}}$ is an inner form of $\tilde{H}$, by Proposition 8.1, $\tilde{I} = 0$ and hence $I = 0$ by Proposition 8.2 unless the Dynkin diagram of H is D_4 .

If H is a simply connected group of type D_4 , then $I = 0$ by Proposition 8.1. If H is a half-spin group of type D_4 , then $I = 0$ by [1, Theorem 5.1]. Finally assume that H is an adjoint group of type D_4 . By [10, Theorem 4.7], the group $\text{Inv}^3(H)$ is cyclic of order 2.

Assume that $I \neq 0$. The group $\tilde{H}^{\text{gen}}$ is the spinor group of a central simple algebra A of degree 8 with and orthogonal involution σ of trivial discriminant.

Consider the corresponding special orthogonal group $\widehat{H}^{\text{gen}} := \text{O}^+(A, \sigma)$ of (A, σ) . An $\widehat{H}^{\text{gen}}$ -torsor over a field K is given by a pair (a, x) , where a is an invertible σ -symmetric element in A and $x \in K^\times$ such that $\text{Nrd}(a) = x^2$ and Nrd is the reduced norm map (see [8, 29.27]).

The canonical homomorphism $\text{Inv}^3(H^{\text{gen}}) \rightarrow \text{Inv}^3(\widetilde{H}^{\text{gen}})$ factors through $\text{Inv}^3(\widehat{H}^{\text{gen}})$. By [10, §4, type D_n], the pull-back of I in $\text{Inv}^3(\widehat{H}^{\text{gen}})$ is the class of the invariant taking a pair (a, x) to the cup-product $(x) \cup [A] \in H^3(K)$. This invariant is ramified as it is non-constant when a runs over a subfield of A of dimension n fixed by σ element-wise, a contradiction. $\square$

9. STRUCTURE OF REDUCTIVE GROUPS

Let H be a split semisimple group over a field F , $S \subset H$ a split maximal torus. Write $\Lambda_r \subset S^*$ for the root lattice of H . Let $\widetilde{H} \rightarrow H$ be a simply connected cover and let $\widetilde{S}$ for the inverse image of S , a maximal torus in $\widetilde{H}$. Write Λ_w for the character group of $\widetilde{S}$. This is the weight lattice freely generated by the fundamental weights. We have

$$\Lambda_r \subset S^* \subset \Lambda_w.$$

The center C of H is a finite diagonalizable group with $C^* = S^*/\Lambda_r$.

Let G be a split reductive group over a field F with the commutator subgroup H . Choose a split maximal $T \subset G$ such that $T \cap H = S$. The roots of H can be uniquely lifted to T^* (to the roots of G), so the inclusion of Λ_r into S^* is lifted to the inclusion of Λ_r into T^* . The composition $\widetilde{S} \rightarrow S \rightarrow T$ yields a homomorphism $T^* \rightarrow \Lambda_w$ of lattices. Thus, we have the two homomorphisms

$$(9.1) \quad \Lambda_r \hookrightarrow T^* \xrightarrow{f} \Lambda_w$$

with the composition the canonical embedding of Λ_r into Λ_w . The image of f in (9.1) is equal to S^* . The center Z of G is a diagonalizable group with $Z^* = T^*/\Lambda_r$. The factor group $G/H = T/S$ is a torus Q with the character lattice $Q^* = \text{Ker}(f)$.

We would like to study all split reductive groups with the fixed commutator subgroup H .

Let H be a split semisimple group over F . Fix a split maximal torus $S \subset H$ and consider the root system of H relative to S with the root and weight lattices $\Lambda_r \subset \Lambda_w$ respectively.

Consider a category $\mathbf{Red}(H)$ with objects split reductive groups G over F with the commutator subgroup H . A morphism between G_1 and G_2 in this category is a group homomorphism $G_1 \rightarrow G_2$ over F that is the identity on H .

Consider another category $\mathbf{Lat}(H)$ with objects the diagrams of the form

$$(9.2) \quad \Lambda_r \longrightarrow A \xrightarrow{f} \Lambda_w,$$

where A is a lattice, $\text{Im}(f) = S^*$ and the composition is the embedding of Λ_r into Λ_w . A morphism in $\text{Lat}(R)$ is a morphism between the diagrams which is identity on Λ_r and Λ_w .

Let G be an object in $\text{Red}(H)$. Write Z for the center of G . Then $T := S \cdot Z$ is a split maximal torus of G . The diagram (9.1) yields then a contravariant functor

$$\rho : \text{Red}(H) \longrightarrow \text{Lat}(H).$$

Proposition 9.3. *For every split semisimple group H , the functor ρ is an equivalence of categories $\text{Red}(H)$ and $\text{Lat}(H)^{\text{op}}$.*

Proof. We construct a functor $\varepsilon : \text{Lat}(H) \longrightarrow \text{Red}(H)$ as follows. Given the diagram (9.2), let T be a split torus with $T^* = A$ and Z a diagonalizable subgroup of T with $Z^* = A/\Lambda_r$. We view the torus S as a subgroup of T via the dual surjective homomorphism $A \longrightarrow \text{Im}(f) = S^*$.

We embed the center C of H into Z via a homomorphism dual to the surjective composition

$$Z^* = A/\Lambda_r \longrightarrow \text{Im}(f)/\Lambda_r = S^*/\Lambda_r = C^*.$$

The sequence

$$0 \longrightarrow A \xrightarrow{g} S^* \oplus (A/\Lambda_r) \xrightarrow{h} S^*/\Lambda_r \longrightarrow 0,$$

where $g(a) = (f(a), a + \Lambda_r)$ and $h(x, a + \Lambda_r) = (x - f(a)) + \Lambda_r$ is exact. It follows that the product homomorphism $S \times Z \longrightarrow T$ is surjective with the kernel C embedded into $S \times Z$ via $c \mapsto (c, c^{-1})$, i.e., $T \simeq (S \times Z)/C$.

We set $G = (H \times Z)/C$. The group Z is naturally a subgroup of G which coincides with the center of G . The torus T is a subgroup of G generated by S and Z , hence T is a split maximal torus of G . The natural sequence

$$0 \longrightarrow \text{Ker}(f) \longrightarrow A/\Lambda_r \longrightarrow \text{Im}(f)/\Lambda_r \longrightarrow 0$$

is exact. It follows that Z/C is a torus dual to $\text{Ker}(f)$. Since $G/H \simeq Z/C$, G is a (smooth connected) reductive group with H the commutator subgroup. The functor ε , by definition, takes the diagram (9.2) to the group G . By construction, both compositions of ρ and ε are isomorphic to the identity functors. $\square$

Let H be a split semisimple group as above. We consider another category $\text{Mor}(H)$ with objects homomorphisms $h : B \longrightarrow \Lambda_w/\Lambda_r$ with B a finitely generated abelian group, $\text{Im}(h) = S^*/\Lambda_r$ and torsion free $\text{Ker}(h)$. Morphisms are defined in the obvious way. Consider a contravariant functor

$$\nu : \text{Red}(H) \longrightarrow \text{Mor}(H)$$

taking a reductive group G to the composition $Z^* \longrightarrow C^* \hookrightarrow \Lambda_w/\Lambda_r$, where Z is the center of G . The kernel of this homomorphism is the character lattice of the torus $Z/C = G/H$ and hence has no torsion.

Proposition 9.4. *For every split semisimple group H , the functor ν is an equivalence of categories $\text{Red}(H)$ and $\text{Mor}(H)^{\text{op}}$.*

Proof. We construct a functor $\lambda : \mathbf{Mor}(H) \rightarrow \mathbf{Red}(H)$ as follows. Let $h : B \rightarrow \Lambda_w/\Lambda_r$ be an object in $\mathbf{Mor}(H)$ and Z a diagonalizable group with $Z^* = B$. The map h yields an embedding of C into Z and the factor group Z/C is a torus. Set $G = (H \times Z)/C$ as in the proof of Proposition 9.3. The factor group G/H is isomorphic to the torus Z/C , hence G is a reductive group with the commutator subgroup H , i.e., G is an object of $\mathbf{Red}(H)$. Then Z is the center of G as the group $G/Z \simeq H/C$ is adjoint. We set $\lambda(h) = G$. By construction, both compositions of ρ and λ are isomorphic to the identity functors. $\square$

Remark 9.5. It follows from Propositions 9.3 and 9.4 that the categories $\mathbf{Lat}(H)$ and $\mathbf{Mor}(H)$ are equivalent. An equivalence between the categories can be described directly as follows. If $\Lambda_r \rightarrow A \xrightarrow{f} \Lambda_w$ is an object in $\mathbf{Lat}(H)$, then the induced morphism $A/\Lambda_r \rightarrow \Lambda_w/\Lambda_r$ is the corresponding object in $\mathbf{Mor}(H)$. Conversely, let $\mu : B \rightarrow \Lambda_w/\Lambda_r$ be an object in $\mathbf{Mor}(H)$. Write A for the kernel of the homomorphism

$$h : S^* \oplus B \rightarrow S^*/\Lambda_r$$

defined by $h(x, b) = (x + \Lambda_r) - \mu(b)$. The corresponding object

$$\Lambda_r \rightarrow A \xrightarrow{f} \Lambda_w$$

in $\mathbf{Lat}(H)$ is defined as follows. The map f is given by the first projection followed by the inclusion of S^* into Λ_w and the inclusion $\Lambda_r \rightarrow A$ takes x to $(x, 0)$. Note that W acts on $S^* \oplus B$ naturally on S^* and trivially on B .

A split reductive group G is called *strict* if the center Z of G is a torus, i.e., Z^* is a lattice. An object G of $\mathbf{Red}(H)$ is *strict* if G is strict. If $B \rightarrow \Lambda_w/\Lambda_r$ is the object $\nu(G)$ of $\mathbf{Mor}(H)$, then G is strict if and only if B is torsion-free.

A semisimple group is strict if and only if it is adjoint. A *strict envelope* of a split semisimple group H is a strict object in $\mathbf{Red}(H)$.

Example 9.6. The group GL_n is a strict envelope of SL_n .

Example 9.7. The object G in $\mathbf{Red}(H)$ corresponding to the composition $S^* \rightarrow S^*/\Lambda_r \hookrightarrow \Lambda_w/\Lambda_r$, viewed as an object of the category $\mathbf{Mor}(H)$, is strict. We call such G the *standard* strict envelope of H . By Remark 9.5, the lattice T^* is the subgroup in $S^* \oplus S^*$ consisting of all pairs (x, y) such that $x - y \in \Lambda_r$. Note that the Weyl group acts naturally on the first component of $S^* \oplus S^*$ and trivially on the second.

A strict envelope of H behaves like an “injective resolution” of H .

Lemma 9.8. *Let G_1 and G_2 be two objects in $\mathbf{Red}(H)$. If G_2 is strict, then there is a morphism $G_1 \rightarrow G_2$ in $\mathbf{Red}(H)$.*

Proof. Let $h_i : B_i \rightarrow \Lambda_w/\Lambda_r$ be the object $\nu(G_i)$ in $\mathbf{Mor}(H)$ for $i = 1, 2$. By assumption, B_2 is a free $\mathbb{Z}$ -module. Therefore, there is a group homomorphism $g : B_2 \rightarrow B_1$ such that $g \circ h_1 = h_2$, i.e., g is a morphism in $\mathbf{Mor}(H)$. By

Proposition 9.4, there is a morphism $G_1 \longrightarrow G_2$ in $\text{Red}(H)$ corresponding to g . $\square$

10. REDUCTIVE INVARIANTS

Let H be a split semisimple group and G is a reductive group with the commutator subgroup H , i.e., G is an object in $\text{Red}(H)$. By Proposition 6.1, the map $\text{Inv}^d(G) \longrightarrow \text{Inv}^d(H)$ is injective. We view $\text{Inv}^d(G)$ as a subgroup of $\text{Inv}^d(H)$. If G' is a strict envelope of H , then it follows from Lemma 9.8 that $\text{Inv}^d(G') \subset \text{Inv}^d(G)$. Therefore, the subgroup $\text{Inv}^d(G')$ is independent of the choice of the strict resolution G' of G . We write $\text{Inv}_{\text{red}}^d(H)$ for this subgroup and call the invariants in this group the *reductive* invariants. By Proposition 6.1, for any prime $p \neq \text{char}(F)$ we have

$$(10.1) \quad \text{Inv}_{\text{nr}}^d(H, p) \subset \text{Inv}_{\text{red}}^d(H, p) \subset \text{Inv}^d(H, p).$$

Let A be a lattice and $q \in \mathcal{S}^2(A)$. We can view q as an integral quadratic form on the lattice $\widehat{A}$ dual to A . The *polar* bilinear form h of q is the image of q under the polar map $\text{pol} : \mathcal{S}^2(A) \longrightarrow A \otimes A$, $aa' \mapsto a \otimes a' + a' \otimes a$. The polar form h is symmetric and *even*, i.e., $h(x, x) \in 2\mathbb{Z}$ for all $x \in \widehat{A}$. Conversely, if $h \in A \otimes A$ is a symmetric even bilinear form, then $q(x) = \frac{1}{2}h(x, x)$ is an integral quadratic form with the polar form h .

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a set of simple roots of an irreducible root system, $\{w_1, w_2, \dots, w_n\}$ the corresponding fundamental weights generating the weight lattice Λ_w and W the Weyl group. Let d_i be the square of the length of the co-root $\alpha_i^\vee$. (We assume that the length of the shortest co-root is 1.) Consider the bilinear form

$$h = \sum_{i=1}^n w_i \otimes d_i \alpha_i = \sum_{i,j} w_i \otimes d_i c_{ij} w_j \in \Lambda_w \otimes \Lambda_w,$$

where (c_{ij}) is the Cartan matrix (see [4, Chapitre VI]). The matrix $(d_i c_{ij})$ is symmetric with even diagonal terms, hence h is a symmetric even bilinear form. The corresponding quadratic form

$$q = \frac{1}{2} \sum_{i=1}^n d_i w_i \alpha_i \in \mathcal{S}^2(\Lambda_w)$$

is W -invariant by [10, Lemma 3.2]. It follows that the polar form h of q is also W -invariant.

Consider the three embeddings $i_1, i_2, j = i_1 + i_2 : \Lambda_w \longrightarrow \Lambda_w^2 := \Lambda_w \oplus \Lambda_w$ given by $x \mapsto (x, 0), (0, x), (x, x)$ respectively, and the two quadratic forms $q^{(1)}, q^{(2)}$ that are the images of q under the maps $\mathcal{S}^2(i_1), \mathcal{S}^2(i_2) : \mathcal{S}^2(\Lambda_w) \longrightarrow \mathcal{S}^2(\Lambda_w^2)$ respectively. We let W act on Λ_w^2 naturally on the first summand and trivially on the second.

Let A be the sublattice of Λ_w^2 of all pairs (x, y) such that $x - y \in \Lambda_r$. Note that $\text{Im}(j) \subset A$. In particular, $\mathcal{S}^2(j)(q) \in \mathcal{S}^2(A)$. Moreover, since

$h \in (\Lambda_r \otimes \Lambda_w) \cap (\Lambda_w \otimes \Lambda_r)$ by [9, Lemma 2.1], we have $(i_k \otimes j)(h) \in A \otimes A$ and $(j \otimes i_k)(h) \in A \otimes A$ for $k = 1, 2$.

Write $m : \Lambda_w^2 \otimes \Lambda_w^2 \rightarrow \mathcal{S}^2(\Lambda_w^2)$ for the canonical homomorphism. We have $m(i_k \otimes j)(h) \in \mathcal{S}^2(A)$ and $m(j \otimes i_k)(h) \in \mathcal{S}^2(A)$ for $k = 1, 2$.

Proposition 10.2. *We have $q^{(1)} - q^{(2)} \in \mathcal{S}^2(A)^W$ with the polar form $h^{(1)} - h^{(2)} = (j \otimes i_1)(h) - (i_2 \otimes j)(h) \in A \otimes A$.*

Proof. By construction, $q^{(1)} - q^{(2)}$ is W -invariant. We have

$$\begin{aligned} q^{(1)} - q^{(2)} &= (q^{(1)} + q^{(2)}) - 2q^{(2)} \\ &= q^{(1)} + q^{(2)} - m(i_2 \otimes i_2)(h) \\ &= q^{(1)} + q^{(2)} + m(i_1 \otimes i_2)(h) - m(j \otimes i_2)(h) \\ &= \mathcal{S}^2(j)(q) - m(j \otimes i_2)(h) \in \mathcal{S}^2(A). \end{aligned}$$

The second statement follows from the equality $j = i_1 + i_2$. $\square$

Corollary 10.3. *The image of $h^{(1)} - h^{(2)}$ under the map*

$$A \otimes A \xrightarrow{p_1 \otimes 1} \Lambda_w \otimes A,$$

where p_1 is the first projection, coincides with the image of h under the natural map

$$\Lambda_w \otimes \Lambda_r \xrightarrow{1 \otimes i_1} \Lambda_w \otimes A.$$

Proof. The statement follows from Proposition 10.2 and the equalities $p_1 \circ j = p_1 \circ i_1 = 1$ and $p_1 \circ i_2 = 0$. $\square$

Let $\tilde{H}$ be a split simply connected cover of H with a split maximal torus $\tilde{S}$, thus $\tilde{S}^* = \Lambda_w$. Consider the standard strict envelope $\tilde{G}$ of $\tilde{H}$ (see Example 9.7). The character group $\tilde{T}^*$ of the maximal torus $\tilde{T}$ of $\tilde{G}$ coincides with the group A as above. If $\tilde{H}$ is simple, by Proposition 9.7, $\bar{q} := q^{(1)} - q^{(2)} \in \mathcal{S}^2(\tilde{T}^*)^W$. The form $\bar{q}$ maps to q under the natural map $\mathcal{S}^2(\tilde{T}^*)^W \rightarrow \mathcal{S}^2(\tilde{S}^*)^W = \mathcal{S}^2(\Lambda_w)^W$.

In the general case,

$$\tilde{H} = \tilde{H}_1 \times \tilde{H}_2 \times \cdots \times \tilde{H}_s,$$

with $\tilde{H}_j$ the simple simply connected components of $\tilde{H}$. The components define a basis $q_1, q_2, \dots, q_s$ of $\mathcal{S}^2(\tilde{H}^*)^W$. Every q_j has a lift $\bar{q}_j \in \mathcal{S}^2(\tilde{T}^*)^W$ as above. Lemma 5.2 then yields the following statement.

Corollary 10.4. *The map $\mathcal{S}^2(\tilde{T}^*)^W \rightarrow \mathcal{S}^2(\tilde{S}^*)^W$ is surjective and $\mathcal{S}^2(\tilde{T}^*)_{\text{ind}}^W \rightarrow \mathcal{S}^2(\tilde{S}^*)_{\text{ind}}^W$ is an isomorphism. In particular, $\mathcal{S}^2(\tilde{T}^*)_{\text{ind}}^W$ is generated by the classes of the forms $\bar{q}_j$.*

We will write α_{ij} for the simple roots of the j -th component and w_{ij} for the corresponding fundamental weights, etc.

Let $\tilde{C} \subset \tilde{H}$ be a central subgroup and set $G := \tilde{G}/\tilde{C}$ and $T := \tilde{T}/\tilde{C}$. The character group $\tilde{C}^*$ is a factor group of Λ_w/Λ_r . Consider the composition

$$(10.5) \quad \mathcal{S}^2(\tilde{T}^*) \xrightarrow{\text{pol}} \tilde{T}^* \otimes \tilde{T}^* \xrightarrow{p_1} \Lambda_w \otimes \tilde{T}^* \longrightarrow \tilde{C}^* \otimes \tilde{T}^*.$$

Note that $\mathcal{S}^2(T^*)$ is contained in the kernel of the composition.

By Corollary 10.3, the image of q_j under this composition is equal to

$$\sum_{i,j} d_{ij} \bar{w}_{ij} \otimes (\alpha_{ij}, 0),$$

where $\bar{x}$ denotes the image of an $x \in \tilde{T}^*$ in $\tilde{C}^*$.

Let $\tilde{T}_j \subset \tilde{T}$ be a maximal torus of the j -th simple component of $\tilde{G}$, so that $\tilde{T} = \tilde{T}_1 \times \cdots \times \tilde{T}_s$. Let $\tilde{C}_j$ be the image of the projection $\tilde{C} \longrightarrow \tilde{T}_j$. Then $\tilde{C}_j^*$ can be viewed as a subgroup of $\tilde{C}^*$ and $\bar{w}_{ij} \in \tilde{C}_j^*$.

Proposition 10.6. *Let $q := \sum_{j=1}^s k_j \bar{q}_j \in \mathcal{S}^2(\tilde{T})$ be a linear combination with integer coefficients k_j . If q has trivial image under the composition (10.5) (for example, if $q \in \mathcal{S}^2(T^*)$), then the order of $\bar{w}_{ij}$ in $\tilde{C}_j^*$ divides $k_j d_{ij}$ for all i and j .*

Proof. We have $\sum_{i,j} k_j d_{ij} \bar{w}_{ij} \otimes (\alpha_{ij}, 0) = 0$ in $\tilde{C}^* \otimes \tilde{T}^*$. Note that the elements $(\alpha_{ij}, 0)$ form part of a basis of $\tilde{T}^*$ (with the complement (w_{ij}, w_{ij})). It follows that $k_j d_{ij} \bar{w}_{ij} = 0$ in $\tilde{C}_j^*$ for all i and j , whence the result. $\square$

11. DEGREE 3 UNRAMIFIED INVARIANTS OF REDUCTIVE GROUPS

We assume that the base field F is algebraically closed.

Proposition 11.1. *Let H be a (split) semisimple group over F with the components of the Dynkin diagram of types A_m for some m or E_6 . Suppose that E_6^{sc} does not split off H as a direct factor. Then $\text{Inv}_{\text{red}}^3(H, p) = \text{Inv}_{\text{nr}}^3(H, p) = 0$ for all odd primes $p \neq \text{char}(F)$.*

Proof. Let $\tilde{H} \longrightarrow H$ be a simply connected cover with kernel $\tilde{C}$ and $\tilde{G}$ be the standard strict envelope of $\tilde{H}$. By Proposition 7.1, replacing $\tilde{C}$ if necessary, we may assume that $\tilde{C}^*$ is a p -group. Set $G := \tilde{G}/\tilde{C}$. We choose split maximal tori $S \subset H$, $\tilde{S} \subset \tilde{H}$, $T \subset G$, $\tilde{T} \subset \tilde{G}$ as in Section 10. The group $Q := G/H = \tilde{G}/\tilde{H}$ is a torus.

By Proposition 6.1, it suffices to prove that $\text{Inv}^3(G, p) = 0$. By Theorem 6.2, we are reduced to proving that $\mathcal{S}^2(T^*)_{\text{ind}}^W \{p\} = 0$.

By Lemma 5.2(1) and Corollary 10.4, the rows of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}^2(Q^*) & \longrightarrow & \mathcal{S}^2(T^*)^W & \longrightarrow & \mathcal{S}^2(S^*)^W \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{S}^2(Q^*) & \longrightarrow & \mathcal{S}^2(\tilde{T}^*)^W & \longrightarrow & \mathcal{S}^2(\tilde{S}^*)^W \longrightarrow 0 \end{array}$$

are exact.

Let $\alpha \in S^2(T^*)_{\text{ind}}^W \{p\}$. Since p is odd, it is sufficient to show that $2\alpha = 0$. The element α lifts to a form $q \in S^2(T^*)^W$. Recall that $S^2(\tilde{S}^*)^W$ is a free abelian group with basis $\{q_j\}$. Hence the image of q in $S^2(\tilde{S}^*)^W$ is equal to $\sum_{j=1}^s k_j q_j$ for some $k_j \in \mathbb{Z}$. Write $\bar{q}$ for $\sum_{j=1}^s k_j \bar{q}_j \in S^2(\tilde{T}^*)^W$. Therefore, in $S^2(\tilde{T}^*)^W$ we have $q = \bar{q} + t$ for some $t \in S^2(Q^*)$.

Note that since the Dynkin diagram of H is simply laced all the integer d_{ij} are equal to 1 for all i and j . The images of q and t are trivial under (10.5), hence so is $\bar{q}$. By Proposition 10.6, the order of $\bar{w}_{ij}$ in $\tilde{C}_j^*$ divides k_j for all i and j .

We claim that the class of $2k_j q_j$ is contained in $S^2(S^*)_{\text{dec}}^W$ for all j .

Case 1: The j -th simple component $\tilde{G}_j$ is of type A_m for some m , i.e., $\tilde{H}_j = \text{SL}_{m+1}$. The center of $\tilde{H}_j$ is μ_{m+1} , hence $\tilde{C}_j = \mu_{p^r}$ for some r . The element $\bar{w}_{1j}$ is a generator of $\tilde{C}^* = \mathbb{Z}/p^r\mathbb{Z}$, hence the order of $\bar{w}_{1j}$ is equal to p^r . Therefore, k_j is divisible by p^k . As p is odd, by [1, 4.2], the form $p^k q_j$ and hence $k_j q_j$ belongs to $S^2(S_j^*)_{\text{dec}}^{W_j}$. Taking the image of $k_j q_j$ under the homomorphism $S_j^* \rightarrow S^*$, we see that $k_j q_j \in S^2(S^*)_{\text{dec}}^W$.

Case 2: The j -th simple component $\tilde{H}_j$ is of type E_6^{sc} . The center of $\tilde{H}_j$ is μ_3 , hence $\tilde{C}_j$ is a subgroup of μ_3 . If $\tilde{C}_j = 1$, then $\tilde{H}_j$ is a direct factor of H and hence E_6^{sc} is a direct factor of H . This is impossible by the assumption. Therefore, $\tilde{C}_j = \mu_3$ (and hence $p = 3$). The element $\bar{w}_{1j}$ is a generator of $\tilde{C}^* = \mathbb{Z}/3\mathbb{Z}$, hence k_j is divisible by 3. By [10, §4, type E_6], the form $6q_j$ and hence $2k_j q_j$ belongs to $S^2(S_j^*)_{\text{dec}}^{W_j}$. Taking the image of $2k_j q_j$ under the homomorphism $S_j^* \rightarrow S^*$, we see that $2k_j q_j \in S^2(S^*)_{\text{dec}}^W$. The claim is proved.

It follows from the claim that 2α belongs to the kernel of the map $S^2(T^*)_{\text{ind}}^W \rightarrow S^2(S^*)_{\text{ind}}^W$. By Lemma 5.2, this map is injective, hence $2\alpha = 0$. $\square$

Example 11.2. The statement of the proposition is wrong if $p = 2$. Consider the group $H := (\text{SL}_2)^n / \tilde{C}$, where $\tilde{C} \subset (\mu_2)^n$ consists of all n -tuples with trivial product. Then the group $G := (\text{GL}_2)^n / \tilde{C}$ is a strict envelope of H . A G -torsor over a field K is a tuple $(Q_1, Q_2, \dots, Q_n)$ of quaternion algebras over K such that $[Q_1] + [Q_2] + \dots + [Q_n] = 0$ in $\text{Br}(K)$. Let φ_i be the reduced norm quadratic form of Q_i . The sum φ of the forms φ_i in the Witt ring $W(K)$ of K belongs to the cube of the fundamental ideal of $W(K)$. The Arason invariant of φ in $H^3(K)$ yields a degree 3 invariant I of G (see [8, page 431]). The restriction J of I to H belongs to $\text{Inv}_{\text{red}}^3(H) = \text{Im}(\text{Inv}^3(G) \rightarrow \text{Inv}^3(H))$, and I and J are nontrivial if $n \geq 3$. Note that the invariants I and J are ramified. Moreover, the map $\text{Inv}^3(G) \rightarrow \text{Inv}^3(\tilde{H}^{\text{gen}})$ factors through $\text{Inv}^3(\tilde{G}^{\text{gen}})$, where $\tilde{G}^{\text{gen}}$ is the product of $\text{GL}_1(Q_i^{\text{gen}})$. The group $\text{Inv}^3(\tilde{G}^{\text{gen}})$ is trivial since $\text{GL}_1(Q_i^{\text{gen}})$ have only trivial torsors. It follows that J belong to the kernel of

$$\text{Inv}^3(H) \rightarrow \text{Inv}^3(\tilde{H}^{\text{gen}}),$$

hence the map in Proposition 8.2 is not injective.

Theorem 11.3. *Let G be a (split) reductive group over an algebraically closed field F . Then $\text{Inv}_{\text{nr}}^3(G, p) = 0$ for every odd prime $p \neq \text{char}(F)$.*

Proof. Let H be the commutator subgroup of G . By Proposition 6.1(2), it suffices to prove that $\text{Inv}_{\text{nr}}^3(H, p) = 0$. Let $\tilde{H} \rightarrow H$ be a simply connected cover with kernel $\tilde{C}$. Let $\tilde{C}' \subset \tilde{C}$ be a subgroup such that $(\tilde{C}/\tilde{C}')^*$ is the 2-component of $\tilde{C}^*$. Since p is odd, by Proposition 7.1, $\text{Inv}_{\text{nr}}^3(H, p) = \text{Inv}_{\text{nr}}^3(\tilde{H}/\tilde{C}', p)$. Replacing H by $\tilde{H}/\tilde{C}'$, we may assume that $\tilde{C}^*$ has odd order.

Write $\tilde{H}$ as a product of simple simply connected groups $\tilde{H}_j$ and let $\tilde{C}_j$ be the center of $\tilde{H}_j$. If the order of $\tilde{C}_j^*$ is a power of 2, the projection $\tilde{C} \rightarrow \tilde{C}_j$ is trivial and therefore, the simply connected group $\tilde{H}_j$ splits off H as a direct factor. Thus, the simply connected simple groups of types B_n , C_n , D_n , E_7 , E_8 , F_4 and G_2 split off H , i.e., $H = H_1 \times H_2$, where H_1 is simply connected and H_2 satisfies the conditions of Proposition 11.1. By the additivity property Corollary 6.3, Propositions 8.1(2) and 11.1, we have $\text{Inv}_{\text{nr}}^3(H, p) = 0$. $\square$

REFERENCES

- [1] H. Bermudez and A. Ruozzi, *Degree 3 cohomological invariants of groups that are neither simply connected nor adjoint*, Preprint (2013).
- [2] S. Blinstein and A. Merkurjev, *Cohomological invariants of algebraic tori*, Algebra Number Theory **7** (2013), no. 7, 1643–1684.
- [3] F. A. Bogomolov, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 485–516, 688.
- [4] N. Bourbaki, *Éléments de mathématique*, Masson, Paris, 1981, Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6].
- [5] H. Esnault, B. Kahn, M. Levine, and E. Viehweg, *The Arason invariant and mod 2 algebraic cycles*, J. Amer. Math. Soc. **11** (1998), no. 1, 73–118.
- [6] R. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, American Mathematical Society, Providence, RI, 2003.
- [7] S. Garibaldi, *Unramified cohomology of classifying varieties for exceptional simply connected groups*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 359–371 (electronic).
- [8] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.
- [9] A. Merkurjev, *Weight two motivic cohomology of classifying spaces for semisimple groups*, Preprint, <http://www.math.ucla.edu/~merkurev/papers/ssinv.pdf> (2013).
- [10] A. Merkurjev, *Degree three cohomological invariants of semisimple groups*, To appear in JEMS.
- [11] A. Merkurjev, *Unramified cohomology of classifying varieties for classical simply connected groups*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 445–476.
- [12] E. Peyre, *Unramified cohomology of degree 3 and Noether’s problem*, Invent. Math. **171** (2008), no. 1, 191–225.
- [13] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).
- [14] D. J. Saltman, *Noether’s problem over an algebraically closed field*, Invent. Math. **77** (1984), no. 1, 71–84.

- [15] D. J. Saltman, *Brauer groups of invariant fields, geometrically negligible classes, an equivariant Chow group, and unramified H^3* , *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 189–246.
- [16] D. J. Saltman, *H^3 and generic matrices*, *J. Algebra* **195** (1997), no. 2, 387–422.
- [17] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, *J. Reine Angew. Math.* **327** (1981), 12–80.
- [18] B. Totaro, *The Chow ring of a classifying space*, *Algebraic K-theory* (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095-1555, USA

E-mail address: merkurev@math.ucla.edu