UNRAMIFIED DEGREE THREE INVARIANTS FOR REDUCTIVE GROUPS OF TYPE A

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1. Introduction

Let G be a (linear) algebraic group over a field F. Choose an embedding of G into $\mathbf{GL}_N$ as a (closed) subgroup for some N. The factor variety $\mathbf{GL}_N/G$ "classifies" principal homogeneous spaces (G-torsors) of G over field extensions K of F. More precisely, there is a naturally bijection of pointed sets:

$$\boxed{\begin{array}{c}\text{The set } H^1(K,G) \text{ of isomorphism}\\ \text{classes of } G\text{-torsors over } K\end{array}} \simeq \boxed{\begin{array}{c}\text{The set of } \mathrm{GL}_N(K)\text{-orbits in}\\ \text{the set of } K\text{-points of } \mathbf{GL}_N/G\end{array}}$$

We write $\mathrm{B}G$ for $\mathbf{GL}_N/G$ and call it a *classifying space of* G. The stable birational type of $\mathrm{B}G$ is independent of the choice of the embedding of G.

The following question is wide open.

Question 1.1. *Let G be a connected group over an algebraically closed field. Is $\mathrm{B}G$ stably rational?*

In what follows, we assume that F is an algebraically closed field of characteristic zero. For a field extension K/F, write $H^d(K)$ for the Galois cohomology group

$$H^d(K,\mathbb{Q}/\mathbb{Z}) := H^d(\Gamma_K,\mathbb{Q}/\mathbb{Z}(d-1)),$$

where $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ is the absolute Galois group of K. If v is a discrete valuation on K that is trivial on F, we have the *residue* homomorphism

$$\partial_v : H^d(K) \to H^{d-1}(L),$$

where L is the residue field of v. The completion $\widehat{K}$ of K with respect to v is isomorphic to the power series field $L((t))$. The map ∂_v factors as the composition

$$H^d(K) \to H^d(\widehat{K}) \xrightarrow{\sim} H^d(L((t))) \to H^{d-1}(L),$$

where the last map is the residue homomorphism with respect to the canonical discrete valuation on $L((t))$.

The subgroup of *unramified* elements $H^d_{\mathrm{nr}}(K) \subset H^d(K)$ is the intersection of $\mathrm{Ker}(\partial_v)$ for all discrete valuations v on K/F.

A key observation is that if X is a stably rational integral variety over F, then $H^d_{\mathrm{nr}}(F(X)) = 0$. In particular, to answer Question 1.1 in the negative, it suffices to prove that $H^d_{\mathrm{nr}}(F(\mathrm{B}G)) \neq 0$ for some d.

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We will use the language of cohomological invariants. Let

$$\mathcal{A} : \mathsf{Fields}_F \longrightarrow \mathsf{PSets}$$

be a functor from the category of field extensions of F to the category of pointed sets. There are two main examples: the functor $K \mapsto H^1(K, G)$ for an algebraic group G over F, which we will denote by $\mathrm{B}G$ and the functor $K \mapsto H^d(K)$ for every d.

A *degree d (normalized cohomological) invariant α of a functor $\mathcal{A}$* is a collection of maps of pointed sets

$$\alpha_K : \mathcal{A}(K) \to H^d(K)$$

for all field extensions K/F, natural in K. In other words, α is a morphism of functors $\mathcal{A} \to H^d$. All degree d invariants of $\mathcal{A}$ form an abelian group $\mathrm{Inv}^d(\mathcal{A})$.

An invariant $\alpha \in \mathrm{Inv}^d(\mathcal{A})$ is called *unramified* if for every field extension K/F and every element $a \in \mathcal{A}(K)$, we have $\alpha(a) \in H^d_{\mathrm{nr}}(K)$. Write $\mathrm{Inv}^d_{\mathrm{nr}}(\mathcal{A})$ for the subgroup of unramified invariants in $\mathrm{Inv}^d(\mathcal{A})$.

The passage to the completion yields the following observation. An invariant $\alpha \in \mathrm{Inv}^d(\mathcal{A})$ is unramified if and only if for every field extension K/F and every $a \in \mathcal{A}(K((t)))$, we have $\partial(\alpha(a)) = 0$, where $\partial : H^d(K((t))) \to H^{d-1}(K)$ is the residue homomorphism.

The generic fiber of the versal G-torsor $\mathbf{GL}_N \to \mathbf{GL}_N / G = \mathrm{B}G$ is a G-torsor over $\operatorname{Spec} F(\mathrm{B}G)$. Evaluating an invariant from $\mathrm{Inv}^d(\mathrm{B}G)$ at this generic fiber yields a homomorphism

$$\mathrm{Inv}^d(\mathrm{B}G) \to H^d\big(F(\mathrm{B}G)\big).$$

By Rost's theorem, this homomorphism is injective, thus, identifying $\mathrm{Inv}^d(\mathrm{B}G)$ with a subgroup of $H^d\big(F(\mathrm{B}G)\big)$. Under this identification, we have an equality

$$\mathrm{Inv}^d_{\mathrm{nr}}(\mathrm{B}G) = H^d_{\mathrm{nr}}\big(F(\mathrm{B}G)\big).$$

We propose the following steps to compute the group $H^d_{\mathrm{nr}}\big(F(\mathrm{B}G)\big)$. First, we compute the group of invariants of $\mathrm{B}G$, i.e, we determine the subgroup $\mathrm{Inv}^d(\mathrm{B}G)$ of $H^d\big(F(\mathrm{B}G)\big)$. Next, we determine which invariants are unramified, i.e, we determine the groups $\mathrm{Inv}^d_{\mathrm{nr}}(\mathrm{B}G) = H^d_{\mathrm{nr}}\big(F(\mathrm{B}G)\big)$.

Let G be a (connected) reductive group over F. Every degree 1 invariant of $\mathrm{B}G$ is trivial, i.e., $\mathrm{Inv}^1(\mathrm{B}G) = 0$. The group of degree 2 invariants $\mathrm{Inv}^2(\mathrm{B}G)$ is canonically isomorphic to $\mathrm{Pic}(G)$ by [1, Theorem 2.4], but $\mathrm{Inv}^2_{\mathrm{nr}}(\mathrm{B}G) = 0$ (see [2, Lemma 5.7]).

The group $\mathrm{Inv}^3(\mathrm{B}G)$ was determined in [4]. It is known that the group $\mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G)$ is 2-torsion and it is trivial if the semisimple part of G is either (almost) simple or simply connected, or adjoint (see [5]). It is not yet clear whether $\mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G)$ is trivial for all reductive G.

In the present paper, we prove the following theorem.

Theorem 1.2. *Let G be a reductive group over an algebraically closed field F of characteristic zero. Suppose that the Dynkin diagram of G is the sum of diagrams of type A. Then every unramified degree 3 cohomological invariant of G is trivial, i.e.,* $\mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G) = 0$. *Equivalently,* $H^3_{\mathrm{nr}}\big(F(\mathrm{B}G)\big) = 0$.

2. Preliminaries

2.1. **Cohomology.** Let K be a field extension of an algebraically closed field F of characteristic zero. Write $H^d(K)$ for $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$. If $d = 1$,

$$\mathrm{Ch}(K) := H^1(K) = H^1(K, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma_K, \mathbb{Q}/\mathbb{Z})$$

is the character group of Γ_K. The kernel of an element $x \in \mathrm{Ch}(K)$ is an open subgroup $\Gamma_x \subset \Gamma_K$. The field of Γ_x-invariants $K(x) := (K_{\mathrm{sep}})^{\Gamma_x}$ is a cyclic extension of K of order $\mathrm{ord}(x)$, the order of x in $\mathrm{Ch}(K)$.

If $d = 2$, the group $H^2(K) = H^2(K, K^\times_{\mathrm{sep}})$ is naturally isomorphic to the Brauer group $\mathrm{Br}(K)$ of K.

There is a cup-product pairing

$$H^d(K) \otimes K^\times \to H^{d+1}(K), \quad x \otimes a \mapsto x \cup a.$$

In particular, we have a pairing

$$\mathrm{Ch}(K) \otimes K^\times \to \mathrm{Br}(K).$$

For every $a \in K^\times$, write (a) for the image of a under the composition

$$K^\times \to H^1(K, \mathbb{Z}/2\mathbb{Z}) \to H^1(K).$$

In other words, (a) is the character of Γ_K giver by the quadratic extension $K(\sqrt{a})/K$.

If $a_1, a_2, \dots, a_d \in K^\times$, we write $(a_1, a_2, \dots, a_d)$ for the image in $H^d(K)$ of the product of the classes of the a_i's in the ring $H^*(K, \mathbb{Z}/2\mathbb{Z})$.

The *cohomological class* $e_d(\varphi)$ of a d-fold quadratic Pfister form

$$\varphi = \langle\langle a_1, a_2, \dots, a_d\rangle\rangle$$

is the class $(a_1, a_2, \dots, a_d)$ in $H^d(K)$.

2.2. **Central simple algebras.** Let A be a central simple algebra over a field K. Then $\dim_K(A)$ is the square of a positive integer $\deg(A)$ that is called the *degree of* A. By Wedderburn's theorem, $A \simeq M_k(D)$ for a division algebra D over K and some $k > 0$. The *index* of A is the integer $\mathrm{ind}(A) := \deg(D)$. The index $\mathrm{ind}(A)$ divides $\deg(A)$.

Example 2.1. Let $a, b \in K^\times$. We write $Q = (a, b)$ for the (generalized) quaternion central simple algebra of degree 2 over K generated by two elements u and v subject to the relations $u^2 = a$, $v^2 = b$ and $uv = -vu$. The reduced norm quadratic form of Q is the 2-fold Pfister form $\langle\langle a, b\rangle\rangle = \langle 1, a\rangle \otimes \langle 1, b\rangle$.

Write $\mathrm{Br}(K)$ for the *Brauer group* of Brauer equivalence classes of central simple algebras over K. If A is a central simple algebra over K, we write $[A]$ for its class in $\mathrm{Br}(K)$. There are canonical isomorphisms

$$\mathrm{Br}(K) \simeq H^2(K) \simeq H^2(K, K_{\mathrm{sep}}^{\times}).$$

If $a \in \mathrm{Br}(K)$, then the *index of* a is the integer $\mathrm{ind}(a) := \mathrm{ind}(A)$, where A is such that $[A] = a$. The *exponent* $\exp(a)$ *of* a is the order of a in $\mathrm{Br}(K)$. The integer $\exp(a)$ always divides $\mathrm{ind}(a)$.

For every integer $n > 0$, the map $A \mapsto [A]$ yields a bijection

$$\boxed{\begin{array}{c}\text{Isomorphism classes of central}\\ \text{simple } K\text{-algebras of degree } n\end{array}} \simeq \boxed{\begin{array}{c}\text{Elements } a \in \mathrm{Br}(K)\\ \text{such that } \mathrm{ind}(a) \text{ divides } n\end{array}}$$

Let $K((t))$ be the power series field over K. The homomorphism

$$\mathrm{Br}(K) \oplus \mathrm{Ch}(K) \to \mathrm{Br}\, K((t))$$

taking (b, x) to $b_{K((t))} + (x_{K((t))} \cup t)$ is an isomorphism (see [3, Prop. 7.11]).

By [8, Prop. 2.4],

$$\mathrm{ind}\big(b_{K((t))} + (x_{K((t))} \cup t)\big) = \mathrm{ind}(b_{K(x)}) \cdot \mathrm{ord}(x). \tag{1}$$

2.3. **Severi-Brauer varieties.** Let A be a central simple algebra of degree n over K. For an integer k dividing n, write $\mathrm{SB}(k, A)$ for the *generalized Severi-Brauer variety* of left ideals in A of reduced dimension k. Then $\mathrm{SB}(k, A)$ has a point over a field extension L/K if and only if $\mathrm{ind}(A_L)$ divides k.

The *Severi-Brauer variety* $\mathrm{SB}(A) := \mathrm{SB}(1, A)$ satisfies

$$\mathrm{Ker}\big(\mathrm{Br}(K) \to \mathrm{Br}\, K(\mathrm{SB}(A)\big) = \mathrm{span}([A])$$

by a theorem of Amitsur. An immediate corollary in the following lemma.

Lemma 2.2. *Let $B \subset \mathrm{Br}(F)$ be a subgroup generated by the classes $[A_1]$, $[A_2]$,..., $[A_n]$ of central simple algebras. Let X be the product of Severi-Brauer varieties $\mathrm{SB}(A_i)$ for all i. Then*

$$B = \mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\, F(X)\big).$$

Lemma 2.3. *Let $n_1, n_2, \dots, n_r$ be positive integers and $b_1, b_2, \dots, b_r \in \mathrm{Br}(K)$ such that $\exp(b_i)$ divides n_i for all i. Let $L_1, L_2, \dots, L_r$ be finite field extensions of K of degrees $s_1, s_2, \dots, s_r$, respectively. Suppose that s_i divides n_i for all i. Then there is a field extension K'/K such that*

(1) *The map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ is injective.*
(2) *$\mathrm{ind}(b_i)_{K'L_i}$ divides $\frac{n_i}{s_i}$ for all i.*

Proof. Let B_i be a central simple K-algebra of degree m_i representing the Brauer class b_i. Let X_i be the corestriction

$$R_{L_i/K}\big(\mathrm{SB}\big(n_i/s_i, (B_i)_{L_i}\big)\big)$$

of the generalized Severi-Brauer variety and let K' be the function field $K(X)$ of the product X of all X_i. Every variety X_i has an K'-point, hence $\mathrm{ind}(B_i)_{K'L_i}$ divides $\frac{n_i}{s_i}$ for all i.

It suffices to show that the map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ is injective. The variety X is a projective homogeneous variety of the product G of the groups $R_{L_i/K}(\mathbf{SL}_1(B_i))$. The Dynkin diagram D of G is the disjoint sum of D_i's over all i, where D_i is the sum of s_i copies of A_{m_i-1}. Let Π be the set of vertices of D. For every i, choose the $\frac{n_i}{s_i}$th vertex on every component of D_i. Write Π' for the set of all chosen vertices over all i. Then the type of the parabolic subgroup of G corresponding to X is the set $\Pi \setminus \Pi'$. Note that the absolute Galois group of K acts transitively on the set of irreducible components of D_i for all i.

The Tits class of a chosen vertex in D_i is equal to $\frac{n_i}{s_i} b_i$. By [6], the kernel of $\mathrm{Br}(K) \to \mathrm{Br}\, K(X)$ is generated by $s_i(\frac{n_i}{s_i}) b_i = n_i b_i$. But $n_i b_i = 0$ since $\exp(b_i)$ divides n_i for all i. $\square$

3. Invariants of functors given by central simple algebras

Let I be a finite set and let $n_I = (n_i)_{i\in I}$ be a family of positive integers. For every $i \in I$, write C_i for a cyclic group of order n_i with a generator e_i and set $C(n_I) := \coprod_{i\in I} C_i$. Let D be a subgroup of $C(n_I)$.

Consider a functor

$$\mathcal{A}(n_I, D) : \mathit{Fields}_F \longrightarrow \mathit{PSets}$$

taking a field extension K/F to the set

$$\Big\{\varphi \in \mathrm{Hom}\big(C(n_I)/D, \mathrm{Br}(K)\big) \text{ such that } \operatorname{ind}\varphi(e_i) \mid n_i \text{ for all } i \in I\Big\}.$$

There are other equivalent description of the functor $\mathcal{A}(n_I, D)$:

- $\mathcal{A}(n_I, D)(K)$ is the set of families $a_I := (a_i)_{i\in I}$ of elements of $\mathrm{Br}(K)$ such that $\operatorname{ind} a_i$ divides n_i for all $i \in I$ and satisfying $\sum_{i\in I} d_i a_i = 0$ in $\mathrm{Br}(K)$ for all $d = (d_i)_{i\in I}$ in D.
- $\mathcal{A}(n_I, D)(K)$ is the set of isomorphism classes of families $A_I := (A_i)_{i\in I}$ of central simple algebras over K such that $\deg A_i = n_i$ for all $i \in I$ and satisfying $\sum_{i\in I} d_i [A_i] = 0$ in $\mathrm{Br}(K)$ for all $d = (d_i)_{i\in I}$ in D.

We call elements $d \in D$ the *relations*.

Example 3.1. The group $C(n_I)$ is the character group $\boldsymbol{\mu}^*$ of $\boldsymbol{\mu} := \prod_{i\in I} \boldsymbol{\mu}_{n_i}$. Let $Z \subset \boldsymbol{\mu}$ be a subgroup such that $Z^* = \boldsymbol{\mu}^*/D$. Let

$$G := \big(\prod_{i\in I} \mathbf{GL}_{n_i}\big)/Z.$$

Thus, G is a reductive group with Dynkin diagram $\coprod_{i\in I} \mathsf{A}_{n_i-1}$.

For a field extension K/F, there is a natural bijection (see [7])

$$H^1(K, G) \simeq \mathcal{A}(n_I, D)(K).$$

In particular,

$$\mathrm{Inv}^d(\mathrm{B}G) \simeq \mathrm{Inv}^d(\mathcal{A}(n_I, D))$$

for every d.

3.1. **Arason invariant of a tuple of quaternion algebras.** Let $Q_I = (Q_i)_{i\in I}$ be a finite family of quaternion K-algebras such that $\sum_{i\in I}[Q_i] = 0$ in $\mathrm{Br}(K)$. Let φ_j be the reduced norm quadratic form of Q_j. The form $\varphi =_{i\in I} \varphi_i$ in the Witt group $W(K)$ of K belongs to the cube of the fundamental ideal of $W(K)$, i.e., φ is the sum of general 3-fold Prister forms $\rho_1, \rho_2, \dots, \rho_s$. Write $\mathrm{Ar}(Q_I)$ for the Arason invariant $\sum_{j=1}^{s} e_3(\rho_j)$ of φ in $H^3(K)$, where $e_3(\rho_j)$ is the class of ρ_j in $H^3(K)$.

Example 3.2. Let $Q_i = (a_i, b_i)$, $Q'_i = (a_i, b'_i)$ and $Q''_i = (a_i, b_i b'_i)$ for some $a_i, b_i, b'_i \in K^\times$ such that $\sum_{i\in I}[Q_i] = 0$ and $\sum_{i\in I}[Q'_i] = 0$. Then $\sum_{i\in I}[Q''_i] = 0$. Since

$$\langle\langle a_i, b_i\rangle\rangle + \langle\langle a_i, b'_i\rangle\rangle = \langle\langle a_i, b_i b'_i\rangle\rangle + \langle\langle a_i, b_i, b'_i\rangle\rangle$$

in $W(K)$, we have

$$\mathrm{Ar}(Q_I) + \mathrm{Ar}(Q'_I) = \mathrm{Ar}(Q''_I) + \sum_{i\in I}(a_i, b_i, b'_i)$$

in $H^3(K)$.

3.2. **The invariants** $\mathrm{Ar}(n_I, D, d)$**.** Let $d \in D$ be an element of exponent 2. Then $2d_i$ is divisible by n_i for every $i \in I$. Let $a_I = (a_i)_{i\in I}$ be a family of elements of $\mathrm{Br}(K)$ in $\mathcal{A}(n_I, D)(K)$. In particular, $\operatorname{ind} a_i$ divides n_i for all $i \in I$ and $\sum_{i\in I} d_i a_i = 0$. Then for every i, the class $d_i a_i$ in $\mathrm{Br}(K)$ is represented by a quaternion algebra Q_i and $\sum_{i\in I}[Q_i] = 0$. Thus, the relation d yields a degree 3 invariant $\mathrm{Ar}(n_I, D, d)$ of the functor $\mathcal{A}(n_I, D)$:

$$\mathrm{Ar}(n_I, D, d)(a_I) := \mathrm{Ar}(Q_I).$$

4. A key proposition

In this section we prove the following key proposition.

Proposition 4.1. *The group of unramified invariants* $\mathrm{Inv}^3_{\mathrm{nr}}\big(\mathcal{A}(n_I, D)\big)$ *is trivial.*

By [7], every invariant in $\mathrm{Inv}^3\big(\mathcal{A}(n_I, D)\big)$ is of the form $\mathrm{Ar}(n_I, D, d)$ for some $d \in D$ of exponent 2. Therefore, it suffices to show that if the invariant $\mathrm{Ar}(n_I, D, d)$ is nontrivial, it is ramified. In this section we reduce to the case when all $n_i = 2$.

Let $I' = I'(d) \subset I$ be the subset of all $i \in I$ such that $d_i \neq 0$. If $i \in I'$, the integer n_i is even. Let m_i be the constant family $m_i = 2$ for all $i \in I'$ and let $C(m_{I'})$ be the direct sum over all $i \in I'$ of cyclic groups of order 2. We have a unique natural embedding $C(m_{I'}) \hookrightarrow C(n_I)$. Write D' for the intersection of D with $C(m_{I'})$. We have then a natural morphism of functors

$$\mathcal{A}(n_I, D) \to \mathcal{A}(m_{I'}, D')$$

and therefore, a homomorphism

$$\mathrm{Inv}^3\big(\mathcal{A}(m_{I'}, D')\big) \to \mathrm{Inv}^3\big(\mathcal{A}(n_I, D)\big).$$

Note that $d \in D'$ and the latter homomorphism takes the invariant $\mathrm{Ar}(m_{I'}, D', d)$ to the invariant $\mathrm{Ar}(n_I, D, d)$.

Lemma 4.2. *If the invariant* $\mathrm{Ar}(m_{I'}, D', d)$ *is ramified, then so is* $\mathrm{Ar}(n_I, D, d)$.

Proof. By assumption, there is a family $q_I = (q_i)_{i\in I'}$ of classes of quaternion algebras over $K((t))$, where K is a field extension of F, such that the value of $\mathrm{Ar}(m_{I'}, D', d)$ at q_I is ramified. This value is a homomorphism

$$C(m_{I'})/D' \to \mathrm{Br}\, K((t)) \simeq \mathrm{Br}(K) \oplus \mathrm{Ch}(K).$$

Since K contains all roots of unity, the group $\mathrm{Ch}(K)$ is divisible. Every element in $\mathrm{Br}(K)$ is the sum of classes of cyclic algebras. Therefore, the group $\mathrm{Br}(K)$ is also divisible. It follows that the homomorphism is extended to a homomorphism

$$C(n_I)/D \to \mathrm{Br}\, K((t)) \simeq \mathrm{Br}(K) \oplus \mathrm{Ch}(K).$$

Let $(b_i, x_i) \in \mathrm{Br}(K) \oplus \mathrm{Ch}(K)$ be the image of e_i under the latter homomorphism. Then $\exp(b_i)$ divides n_i. Denote $L_i := K(x_i)$. Note that $s_i := [L_i : K] = \mathrm{ord}(x_i)$ divides n_i.

By Lemma 2.3, there is a field extension K'/K such that the map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ is injective and $\mathrm{ind}(b_i)_{K'L_i}$ divides $\frac{n_i}{s_i}$ for all i. Write

$$a_i := (b_i)_{K((t))} + ((x_i)_{K((t))} \cup t) \in \mathrm{Br}\, K((t)).$$

It follows from (1) that

$$\mathrm{ind}(a_i)_{K'((t))} = \mathrm{ind}(b_i)_{K'L_i} \cdot \exp(x_i)_{L_i} \quad \text{divides} \quad \frac{n_i}{s_i} \cdot s_i = n_i.$$

It follows that the family $a_I = (a_i)_{K'((t))}$ represents an element of $\mathcal{A}(n_I, D)(K'((t)))$. The residue $r \in \mathrm{Br}(K')$ of the value of the invariant $\mathrm{Ar}(n_I, D, d)$ at $(a_I)_{K'((t))}$ is the image of the nonzero residue in $\mathrm{Br}(K)$ of $\mathrm{Ar}(m_{I'}, D', d)$ at $(q_{I'})$. As the map $\mathrm{Br}(K) \to \mathrm{Br}(K')$ is injective, we have $r_{K'} \neq 0$, i.e., the invariant $\mathrm{Ar}(n_I, D, d)$ is ramified. $\square$

By Lemma 4.2, we may assume that $n_i = 2$ for all $i \in I$. For every element $d \in C(n_I)$ write $\mathrm{supp}(d)$ for the set of all $i \in I$ such that $d_i \neq 0$. Equivalently, $d = \sum e_i$, where i runs over $\mathrm{supp}(d)$. We have $a_I \in \mathcal{A}(n_I, D)$ if and only if $\sum_{i\in\mathrm{supp}(d)} a_i = 0$ for all $d \in D$.

Lemma 4.3. *Suppose that* $n_i = 2$ *for all* i *and for every nonzero* $d \in D$, *the set* $\mathrm{supp}(d)$ *has at least* 3 *elements. Then for every nonzero* $d \in D$, *the invariant* $\mathrm{Ar}(n_I, D, d)$ *is ramified.*

Proof. Let $d' \in D$ be a nonzero element. Set $S := \mathrm{supp}(d')$. Choose an element $s \in S$ and consider the set $J := I\setminus\{s\}$. We claim that there is a field extension K/F and elements $a_i, b_i \in K^\times$ for $i \in I$ such that

- $\sum_{i\in\mathrm{supp}(d)\setminus\{s\}}(a_i) = 0$ in $H^1(K)$ for all $d \in D$,
- $\sum_{i\in\mathrm{supp}(d)}(a_i, b_i) = 0$ in $H^2(K)$ for all $d \in D$,
- $(a_i, b_i) \neq 0$ in $H^2(K)$ for all $i \in I$.

To prove the claim, let F_1 be a field extension of F such that there are elements $a_i \in F_1^\times$ for $i \in I$ such that the elements (a_i) in $H^1(F_1)$ are linearly independent. The classes $\sum_{i\in \mathrm{supp}(d)\setminus\{s\}}(a_i)$ in $H^1(F_1)$ for all $d \in D$ form a subgroup $H \subset H^1(F_1) = F_1^\times/F_1^{\times 2}$. Since by assumption, for every nonzero $d \in D$, the set $\mathrm{supp}(d)$ has at least 3 elements, then $\mathrm{card}\big(\mathrm{supp}(d) \setminus \{s\}\big) \geq 2$. Hence $(a_i) \notin H$ for all $i \in I$.

By Kummer theory, there is a finite field extension F_2/F_1 such that

$$H = \mathrm{Ker}\big(H^1(F_1) \to H^1(F_2)\big).$$

It follows that $(a_i) \neq 0$ in $H^1(F_2)$, i.e., $a_i \notin (F_2)^{\times 2}$, for all $i \in I$.

Let F_3 be the rational function field over F_2 with variables b_i for $i \in I$. As every a_i is not a square in F_2, the quaternion algebras $Q_i := (a_i, b_i)$ are linearly independent in ${}_2\mathrm{Br}(F_3)$. For every $d \in D$, consider the tensor product A_d of all Q_i such that $i \in \mathrm{supp}(d)$. The classes of A_d in ${}_2\mathrm{Br}(F_3)$ for all $d \in D$ form a subgroup B.

Let X be the product of Severi-Brauer varieties of the algebras A_d over F_3 for all $d \in D$. By Lemma 2.2, the subgroup B coincides with the kernel of the natural homomorphism $\mathrm{Br}(F_3) \to \mathrm{Br}(K)$, where K is the function field of X over F_3. Since the classes of Q_i are not in B and F_3 is algebraically closed in K, the field K satisfies the conclusion of the claim. The claim is proved.

Consider the following families of elements in $\mathrm{Br}\, K((t))$, where K is the field in the claim:

$$q_i = \begin{cases} (a_i, b_i), & \text{if } i = s; \\ (a_i, b_i t), & \text{if } i \neq s, \end{cases} \qquad q_i' = \begin{cases} 0, & \text{if } i = s; \\ (a_i, t), & \text{if } i \neq s. \end{cases}$$

By the claim,

$$\sum_{i\in \mathrm{supp}(d)} q_i = \sum_{i\in \mathrm{supp}(d)} (a_i, b_i) + \sum_{i\in \mathrm{supp}(d)\setminus\{s\}} (a_i, t) = 0 \quad \text{in} \quad \mathrm{Br}\, K((t)),$$

i.e., the family $q_I := (q_i)$ represent an element of $\mathcal{A}(n_I, D)(K((t)))$. Similarly, $q_I' := (q_i')$ and $q_I'' := (q_i + q_i')$ belong to $\mathcal{A}(n_I, D)(K((t)))$.

Consider the family $\widetilde{Q}_I$ of quaternion algebras over $K((t))$ such that $[\widetilde{Q}_i] = q_i$ if $i \in S$ and $[\widetilde{Q}_i] = 0$ otherwise. We define the families $\widetilde{Q}_I'$ and $\widetilde{Q}_I''$ similarly. By definition,

$$\mathrm{Ar}(n_I, D, d')(q_I) = \mathrm{Ar}(\widetilde{Q}_I), \quad \mathrm{Ar}(n_I, D, d')(q_I') = \mathrm{Ar}(\widetilde{Q}_I') \qquad \text{and}$$

$$\mathrm{Ar}(n_I, D, d')(q_I'') = \mathrm{Ar}(\widetilde{Q}_I'').$$

By example 3.2,

$$\mathrm{Ar}(\widetilde{Q}_I) + \mathrm{Ar}(\widetilde{Q}_I') = \mathrm{Ar}(\widetilde{Q}_I'') + \sum_{i\in S\setminus\{s\}} (a_i, b_i, t) = \mathrm{Ar}(\widetilde{Q}_I'') + (a_s, b_s, t)$$

since

$$\sum_{i\in S} (a_i, b_i) = \sum_{i\in \mathrm{supp}(d')} (a_i, b_i) = 0.$$

As the residue (a_s, b_s) of (a_s, b_s, t) is nontrivial, the invariant $\mathrm{Ar}(n_I, D, d')$ is ramified. $\square$

Suppose now that $n_i = 2$ for all i and $D \subset C(n_I)$ an arbitrary subgroup. If $e_i \in D$ for some i, then $a_i = 0$ for every $a_I \in \mathcal{A}(n_I, D)(K)$. If I' is the subset of all i with $e_i \notin D$ and $D' := D \cap C(n_{I'})$, then $\mathcal{A}(n_I, D) = \mathcal{A}(n_{I'}, D')$. Replacing I by I' and D by D', we may assume that $e_i \notin D$ for all $i \in I$.

Consider the following equivalence relation on I: we write $i \sim i'$ if $e_i + e_{i'} \in D$. Write $\overline{I}$ for the set of equivalence classes in I and $\bar{i}$ for the equivalence class of i. The (only) isomorphism $C_i \xrightarrow{\sim} C_{\bar{i}}$ for every $i \in I$ yields a homomorphism $C_I \to C_{\overline{I}}$. Let $\overline{D} \subset C_{\overline{I}}$ be the image of D.

Lemma 4.4. *The homomorphism $C_I/D \to C_{\overline{I}}/\overline{D}$ is an isomorphism. In particular, the induced morphism $\mathcal{A}(n_{\overline{I}}, \overline{D}) \to \mathcal{A}(n_I, D)$ is an isomorphism.*

Proof. The kernel of $C_I \to C_{\overline{I}}$ is generated by elements of the form $c = e_i + e_{i'}$, where $i \sim i'$ in I. Since $c \in D$, the pre-image of $\overline{D}$ coincides with D. $\square$

Lemma 4.5. *The support of every nonzero element in $\overline{D}$ has at least* 3 *elements.*

Proof. Consider a nonzero element $\bar{u} = \sum_{\bar{i} \in \overline{J}} e_{\bar{i}}$ in $\overline{D}$, where $\overline{J} = \mathrm{supp}(\bar{u})$ is a nonempty subset of $\overline{I}$. Let $J \subset I$ be a subset that is bijective to $\overline{J}$ under the map $I \to \overline{I}$ and let $u = \sum_{i \in J} e_i \in C_I$. By Lemma 4.4, $u \in D$. As we assumed that $e_i \neq D$ for every i, the set J has at least two element. If $\mathrm{card}(J) = 2$, we have $u = e_i + e_{i'}$. But then $i \sim i'$, a contradiction since $\bar{i} \neq \bar{i}'$. It follows that $\mathrm{card}(\overline{J}) = \mathrm{card}(J) \geq 3$. $\square$

Now we finish the proof of Proposition 4.1. We prove that the invariant $\mathrm{Ar}(n_I, D, d)$ is ramified if nontrivial. By Lemma 4.2, we may assume that $n_i = 2$ for all i. By Lemmas 4.4 and 4.5, we may assume that the support of every nonzero element in D has at least 3 elements. Finally, the statement follows from Lemma 4.3.

5. Proof of the main theorem

We prove Theorem 1.2. Let G be a reductive group over an algebraically closed field F of characteristic zero with the Dynkin diagram the sum of diagrams of type A. Then the semisimple part H of G is isomorphic to

$$\Big(\prod_{i \in I} \mathbf{SL}_{n_i}\Big)/Z$$

for some family of integers $(n_i)_{i \in I}$ and a subgroup

$$Z \subset \boldsymbol{\mu} := \prod_{i \in I} \boldsymbol{\mu}_{n_i}.$$

Let D be the kernel of the restriction homomorphism

$$C(n_I) := \coprod_{i \in I} \mathbb{Z}/n_i\mathbb{Z} = \boldsymbol{\mu}^* \to Z^*.$$

By Example 3.1,

$$\mathrm{Inv}^3(\mathrm{B}G') \simeq \mathrm{Inv}^3(\mathcal{A}(n_I, D)),$$

where

$$G' = \Big(\prod_{i\in I} \mathbf{GL}_{n_i}\Big)/Z.$$

Therefore, by Proposition 4.1,

$$H^3_{\mathrm{nr}}\big(F(\mathrm{B}G')\big) = \mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G') = \mathrm{Inv}^3(\mathcal{A}(n_I, D)) = 0.$$

Note that the natural morphism $f : \mathrm{B}H \to \mathrm{B}G$ is a T-torsor, where T is the (split) torus G/H. Hence f is split generically and $\mathrm{B}H$ is stably birationally isomorphic to $\mathrm{B}G$. As H is the semisimple part of both reductive groups G and G', the spaces $\mathrm{B}H$, $\mathrm{B}G$ and $\mathrm{B}G'$ are stably birationally isomorphic. Therefore,

$$\mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G) = H^3_{\mathrm{nr}}\big(F(\mathrm{B}G)\big) = \mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}H) = \mathrm{Inv}^3_{\mathrm{nr}}(\mathrm{B}G') = 0.$$

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