R-EQUIVALENCE ON 3-DIMENSIONAL TORI AND ZERO-CYCLES

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Let T be an algebraic torus over a field F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Let $A_0(X)$ be the subgroup of the Chow group $\mathrm{CH}_0(X)$ of classes of zero-dimensional cycles on X consisting of classes of degree zero. The map $T(F) \to A_0(X)$ taking a rational point t in $T(F)$ to $[t]-[1]$ factors through the R-equivalence on $T(F)$ (cf. §2.3):

$$\varphi : T(F)/R \to A_0(X).$$

One can ask the following questions:

1. Is φ a homomorphism?
2. Is φ an isomorphism?

Note that φ is a homomorphism if and only if $[ts]-[t]=[s]-[1]$ for any two rational points $s, t \in T(F)$. If the translation action of T on itself extends to an action on X, the latter means that the natural action of $T(F)$ on $A_0(X)$ is trivial.

In the present paper we prove that φ is an isomorphism for all algebraic tori of dimension at most 3 (Theorem 4.4). All tori of dimension 1 and 2 are rational (cf. [22, §4.9]), therefore, φ is an isomorphism of trivial groups. Birational classification of 3-dimensional tori was given in [14].

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We use the following notation in the paper:

The word "*variety*" will mean a separated scheme of finite type over a field,
F is a field,
F_{sep} is a separable closure of F,
Γ is the Galois group of F_{sep}/F,
$X_L := X \times_F \operatorname{Spec} L$ for a scheme X over F and a field extension L/F,
X_{sep} is $X \times_F \operatorname{Spec} F_{\mathrm{sep}}$,
T^* is the character group of an algebraic torus T over F_{sep} with Γ-action,
$T_* = \mathrm{Hom}(T^*, \mathbb{Z})$ is the co-character group of a torus T,
T° is the dual torus, $(T^\circ)^* = T_*$,
$K_*(X)$ is Quillen's K-group of a scheme X,
$H^*(X, K_*)$ is the K-cohomology group,
$\mathrm{CH}^i(X)$ is the Chow groups of cycles of codimension i on X,
$\mathrm{CH}_i(X)$ is the Chow groups of cycles of dimension i on X,

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Fields$/F$ is the category of field extensions of F,
Ab is the category of abelian groups,
Sets is the category of sets,
$\mathbb{G}_m = \mathbb{G}_{m,F}$.

1. Preliminaries

1.1. **R-equivalence.** Let F be a field. For a field extension L/F, we write H_L for the semilocal ring of all rational functions $f(t)/g(t) \in L(t)$ such that $g(0)$ and $g(1)$ are nonzero. Let A be a functor from the category of semi-simple commutative F-algebras to the category *Sets*. If $i = 0$ or 1, we have a map $A(H_L) \to A(L)$, $a \mapsto a(i)$, induced by the L-algebra homomorphism $H_L \to L$ taking a function h to $h(i)$.

Two points $a_0, a_1 \in A(L)$ are called *strictly R-equivalent* if there is an $a \in A(H_L)$ with $a(0) = a_0$ and $a(1) = a_1$. The strict R-equivalence generates an equivalence relation R on $A(L)$, called the *R-equivalence relation*. The set of R-equivalence classes is denoted by $A(L)/R$.

Example 1.1. A scheme X over F defines the functor

$$X(A) := \mathrm{Mor}_F(\mathrm{Spec}\, A, X).$$

The notion of R-equivalence in $X(L)$ is classical and was introduced in [16, Ch. 2, §4]. If G is an algebraic group over F, then $G(L)/R = G(L)/RG(L)$, where $RG(L)$ is the subgroup of $G(L)$ consisting of all elements that are R-equivalent to the identity.

Example 1.2. Let G be an algebraic group over F. We can define the functor taking a commutative F-algebra A to the set of isomorphism classes $H^1_{\text{ét}}(A, G)$ of G-torsors over $\mathrm{Spec}\, A$.

Example 1.3. Let $1 \to S \to P \to T \to 1$ be an exact sequence of algebraic tori over F with P a quasi-trivial torus, i.e., $P \simeq R_{K/F}(\mathbb{G}_{m,K})$ for an étale F-algebra K. As $H^1_{\text{ét}}(A, P) = H^1_{\text{ét}}(A \otimes_F K, \mathbb{G}_m) = 0$ for any semilocal commutative F-algebra A by Shapiro-Faddeev Lemma and Grothendieck's Hilbert Theorem 90, the sequence

$$P(A) \to T(A) \to H^1_{\text{ét}}(A, S) \to 0$$

is exact. Since P is an open subset in the affine space of K, we have $P(L)/R = 1$ for any field extension L/F. Hence the image of $P(L) \to T(L)$ consists of R-trivial elements in $T(L)$ and therefore,

$$T(L)/R \simeq H^1(L, S)/R.$$

If in addition S is a flasque torus (cf. [22, §4.6]) then by [5, Th. 2],

$$T(L)/R \simeq H^1(L, S).$$

1.2. **Category of Chow motives.** Let $\mathcal{CM}(F)$ be the category of Chow motives over F (cf. [15]). Recall that $\mathcal{CM}(F)$ is an additive category with objects formal finite direct sums $\coprod_k (X_k, i_k)$ (called *Chow motives*) where X_k are smooth proper varieties over F and $i_k \in \mathbb{Z}$. For a smooth proper variety X we write $M(X)(i)$ for the object (X, i) of $\mathcal{CM}(F)$ and shortly $M(X)$ for $M(X)(0)$. If $M(X)$ and $M(Y)$ are objects in $\mathcal{CM}(F)$ and X is irreducible of dimension d then

$$\mathrm{Mor}_{\mathcal{CM}(F)}\big(M(X)(i), M(Y)(j)\big) = \mathrm{CH}_{d+i-j}(X \times Y).$$

We have the functor from the category $\mathcal{SP}(F)$ of smooth proper varieties over F to $\mathcal{CM}(F)$ taking a variety X to $M(X)$ and a morphism $f : X \to Y$ to the cycle of the graph of f.

We write $\mathbb{Z}(i)$ for $M(\operatorname{Spec} F)(i)$. A motive is called *split* if it is isomorphic to a motive of the form $\coprod_{i=1}^{r} \mathbb{Z}(d_i)$.

The functor taking an X to the K-cohomology groups $H^*(X, K_*)$ (cf. [18]) from the category $\mathcal{SP}(F)$ to the category of (bi-graded) abelian groups factors through the category $\mathcal{CM}(F)$ as follows. Let $\alpha \in \mathrm{CH}(X \times Y)$ be a morphism $M(X)(i) \to M(Y)(j)$ in $\mathcal{CM}(F)$. Then the functor takes α to the homomorphism $H^*(X, K_*) \to H^*(Y, K_*)$ defined by $\beta \mapsto (p_2)_*\big(\alpha \cdot p_1^*(\beta)\big)$ where p_1^* and $(p_2)_*$ are the pull-back and the push-forward homomorphisms for the first and the second projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ respectively.

Recall that $H^p(X, K_p) = \mathrm{CH}^p(X)$ for a smooth X and every $p \geq 0$ by [18, §7, Prop. 5.14].

Lemma 1.4. *Let M be a split motive. Then the product map*

$$\mathrm{CH}^p(M) \otimes K_q(F) \to H^p(M, K_{p+q})$$

is an isomorphism.

Proof. The statement is obviously true for the motive $M = \mathbb{Z}(i)$. □

Let X be a smooth proper irreducible variety over F. The push-forward homomorphism

$$\deg : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} F) = \mathbb{Z}$$

with respect to the the structure morphism $X \to \operatorname{Spec} F$ is called the *degree homomorphism*. For every $i \geq 0$, we have the intersection pairing

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}_p(X) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta). \tag{1}$$

Proposition 1.5. *Let X be a smooth proper irreducible variety over F. Then the Chow motive of X is split if and only if*

1. *The Chow group $\mathrm{CH}(X)$ is free abelian of finite rank and the map $\mathrm{CH}(X) \to \mathrm{CH}(X_L)$ is an isomorphism for every field extension L/F and*
2. *The pairing (1) is a perfect duality for every p.*

Proof. Suppose that the motive of X is split. Mutually inverse isomorphisms between $M(X)$ and a split motive $\coprod_{i=1}^{r} \mathbb{Z}(d_i)$ are given by two r-tuples of elements $u_i \in \mathrm{CH}_{d_i}(X)$ and $v_i \in \mathrm{CH}^{d_i}(X)$ such that the tuple u (and also v) form a $\mathbb{Z}$-basis of $\mathrm{CH}(X)$ and $\deg(u_i v_j) = \delta_{ij}$ over any field extension of F.

Conversely, suppose that (1) and (2) hold. Choose dual bases u_i and v_j of $\mathrm{CH}(X)$. They define morphisms α and β from a split motive N to $M(X)$ and back respectively so that $\beta \circ \alpha$ is the identity of N. By Yoneda Lemma, it suffices to prove that for every variety Y over F the morphism

$$u \otimes 1_Y : \mathrm{CH}\big(N \otimes M(Y)\big) \to \mathrm{CH}(X \times Y)$$

is an isomorphism. The injectivity follows from the fact that $\beta \circ \alpha = \mathrm{id}$. The surjectivity follows by induction on the dimension of Y using the localization and the fact that the map $u \otimes 1_Y$ is an isomorphism if Y is the spectrum of a field extension of F. $\square$

1.3. ***K*-theory, *K*-cohomology and the Brown-Gersten-Quillen spectral sequence.** Let X be a smooth variety over F. Let $K_*(X)^{(i)}$ denote the i-th term of the topological filtration on $K_*(X)$. Consider the Brown-Gersten-Quillen (BGQ) spectral sequence (cf. [18, §7, Th. 5.4])

$$E_2^{p,q} = H^p(X, K_{-q}) \Rightarrow K_{-p-q}(X) \tag{2}$$

converging to the K-groups of X with the topological filtration. The K-cohomology groups $H^*(X, K_*)$ can be computed via Gersten complexes (cf. [18, §7.5]).

We have $E_2^{p,q} = 0$ if $p < 0$ or $p + q > 0$, or $p > \dim X$ and $E_2^{p,-p} = \mathrm{CH}^p(X)$. The E_2-term is as follows:

$$\begin{tikzcd}
\mathrm{CH}^0(X) & 0 & & \\
H^0(X, K_1) \arrow[drr] & \mathrm{CH}^1(X) & 0 & \\
& H^1(X, K_2) \arrow[drr] & \mathrm{CH}^2(X) & 0 \\
& & H^2(X, K_3) & \mathrm{CH}^3(X)
\end{tikzcd}$$

If in addition X is geometrically irreducible proper, we have $H^0(X, K_1) = F^\times$. The composition of the pull-back homomorphism $F^\times = K_1(F) \to K_1(X)$ for the structure morphism of X with the edge homomorphism $K_1(X) \to H^0(X, K_1)$ is the identity. Hence all the differentials starting at $E_*^{0,-1}$ are trivial. If in addition $\dim X = 3$, the spectral sequence yields an exact sequence

$$K_1(X)^{(1)} \to H^1(X, K_2) \to \mathrm{CH}^3(X) \xrightarrow{g} K_0(X), \tag{3}$$

where g is the edge homomorphism.

2. Zero cycles on toric models

2.1. **K-theory of toric models.** Let T be an algebraic torus over a field F. Let X be a geometrically irreducible variety containing T as an open subset. We say that X is a *toric model of* T if the translation action of T on itself extends to an action on X. Every torus admits a smooth proper toric model (cf. [1] and [3]).

Let X be a smooth proper toric model of T. It follows from [13, Prop. 3, Cor. 2] that X_{sep} satisfies the conditions (1) and (2) of Proposition 1.5. Thus by Proposition 1.5, we have:

Proposition 2.1. *Let X be a smooth proper toric model of T. Then the Chow motive of X_{sep} is split.*

The proposition and Lemma 1.4 yield:

Corollary 2.2. *Let X be a smooth proper toric model of an algebraic torus T. Then the product map*

$$\mathrm{CH}^p(X_{\mathrm{sep}}) \otimes K_q(F_{\mathrm{sep}}) \to H^p(X_{\mathrm{sep}}, K_{p+q})$$

is an isomorphism.

The absolute Galois group Γ acts naturally on $K_0(X_{\mathrm{sep}})$ leaving each term $K_0(X_{\mathrm{sep}})^{(i)}$ invariant.

The following theorem was proven in [17].

Theorem 2.3. *Let X be a smooth proper toric model of an algebraic torus of dimension d over F. Then*

1. *$K_0(X_{\mathrm{sep}})$ is a direct summand of a permutation Γ-module.*
2. *The subgroup $K_0(X_{\mathrm{sep}})^{(d)}$ is infinite cyclic generated by the class of a rational point of X.*
3. *The natural map $K_i(X) \to K_i(X_{\mathrm{sep}})^{\Gamma}$ is an isomorphism for $i \leq 1$.*
4. *The product map $K_0(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^{\times} \to K_1(X_{\mathrm{sep}})$ is an isomorphism.*

Corollary 2.4. *Let X be a smooth proper toric model of a torus of dimension d over F. We have the following natural isomorphisms:*

1. *$K_i(X)^{(1)} \xrightarrow{\sim} \big(K_i(X_{\mathrm{sep}})^{(1)}\big)^{\Gamma}$ for $i \leq 1$.*
2. *$K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^{\times} \xrightarrow{\sim} K_1(X_{\mathrm{sep}})^{(1)}$.*

Proof. (1): The group $K_i(X)^{(1)}$ is the kernel of the restriction to the generic point $K_i(X) \to K_iF(X)$. The image of this map is equal to $H^0(X, K_i) = K_i(F)$ for $i = 0, 1$. Statement (1) follows from Theorem 2.3(3) applied to the exact sequence

$$0 \to \big(K_i(X_{\mathrm{sep}})^{(1)}\big)^{\Gamma} \to K_i(X_{\mathrm{sep}})^{\Gamma} \to K_i(F_{\mathrm{sep}})^{\Gamma}$$

for $i = 0, 1$.

(2): Tensoring with $F_{\mathrm{sep}}^{\times}$ the split exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}}) \to \mathbb{Z} \to 0$$

we get (2) by Theorem 2.3(4). □

Corollary 2.5. *Let X be a smooth proper toric model of a torus of dimension d over F. Then*

(1) $K_0(X_{\mathrm{sep}})^{(1)}$ *is a direct summand of a permutation* Γ*-module.*
(2) $K_0(X_{\mathrm{sep}})^{(d)}$ *is a direct summand of the* Γ*-module* $K_0(X_{\mathrm{sep}})$.

Proof. (1): We have the canonical decomposition of Γ-modules via the structure sheaf $\mathcal{O}_X$:

$$K_0(X_{\mathrm{sep}}) = K_0(X_{\mathrm{sep}})^{(1)} \oplus \mathbb{Z} \cdot 1,$$

hence $K_0(X_{\mathrm{sep}})^{(1)}$ is a direct summand of a permutation Γ-module by Theorem 2.3(1).
(2): For a rational point $x \in X(F)$, the composition of the push-forward homomorphism $K_0(F_{\mathrm{sep}}) = K_0\big(F_{\mathrm{sep}}(x)\big) \to K_0(X_{\mathrm{sep}})$ with the push-forward map $p_* : K_0(X_{\mathrm{sep}}) \to K_0(F_{\mathrm{sep}})$ induced by the structure morphism p of X_{sep} is the identity. It follows from Theorem 2.3(2) that the inclusion

$$K_0(X_{\mathrm{sep}})^{(d)} \to K_0(X_{\mathrm{sep}})$$

is split by p_* as a homomorphism of Γ-modules. □

We shall need the following property of K-cohomology groups of smooth proper toric models.

Proposition 2.6. *Let X be a smooth proper toric model of a torus of dimension d over F. Then the natural morphism $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)^{\Gamma}$ is an isomorphism.*

Proof. As X is geometrically rational and has a rational point, the statement follows from [4, Prop. 4.3] (if $\mathrm{char}(F) = 0$) and [12, Th. 1(a)] or [8, Th. 8.9] (in general). □

2.2. **The group $A_0(X)$ of 3-dimensional toric models.** Let T be an algebraic torus and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Let P and S be algebraic tori over F such that P^* is the permutation Γ-module with $\mathbb{Z}$-basis the set of irreducible components of $(X \setminus T)_{\mathrm{sep}}$ and $S^* = \mathrm{CH}^1(X_{\mathrm{sep}})$. We have natural Γ-homomorphisms $T^* \to P^*$ taking a character χ to $\mathrm{div}(\chi)$ (we consider χ as a rational function on X_{sep}) and $P^* \to S^*$ taking a component of $(X \setminus T)_{\mathrm{sep}}$ to its class in the Chow group. The sequence

$$0 \to T^* \to P^* \to S^* \to 0 \tag{4}$$

is a flasque resolution of T^* (cf. [5, Prop. 6], [22, §4.6]). Thus we have an exact sequence of algebraic tori

$$1 \to S \to P \to T \to 1, \tag{5}$$

a flasque resolution of T.

By [5, Th. 2] (cf. Example 1.3),

$$T(L)/R \simeq H^1(L, S) \tag{6}$$

for any field extension L/F.

The spectral sequence (2) for X_{sep} yields isomorphisms of Γ-modules

$$K_0(X_{\mathrm{sep}})^{(1/2)} \simeq \mathrm{CH}^1(X_{\mathrm{sep}}) = S^*$$

and

$$K_0(X_{\mathrm{sep}})^{(2/3)} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}).$$

Let T be a 3-dimensional torus and X a smooth proper toric model of T. By [13, Prop. 3, Cor. 2], the pairing

$$\mathrm{CH}^1(X_{\mathrm{sep}}) \otimes \mathrm{CH}^2(X_{\mathrm{sep}}) \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \deg(\alpha\beta)$$

is a perfect duality of Γ-lattices. It follows that $\mathrm{CH}^2(X_{\mathrm{sep}}) \simeq S_*$. Thus, the exact sequence

$$0 \to K_0(X_{\mathrm{sep}})^{(2)} \to K_0(X_{\mathrm{sep}})^{(1)} \to K_0(X_{\mathrm{sep}})^{(1/2)} \to 0$$

yields an exact sequence of algebraic tori

$$1 \to S' \xrightarrow{\tau} Q \to S^\circ \to 1 \tag{7}$$

with $S'_* = K_0(X_{\mathrm{sep}})^{(2)}$ and $Q_* = K_0(X_{\mathrm{sep}})^{(1)}$ a direct summand of a permutation Γ-module by Corollary 2.5(1). By Theorem 2.3(2) and Corollary 2.5(2), we have isomorphisms of Γ-modules

$$S'_* = K_0(X_{\mathrm{sep}})^{(2)} \simeq K_0(X_{\mathrm{sep}})^{(2/3)} \oplus \mathbb{Z} \simeq \mathrm{CH}^2(X_{\mathrm{sep}}) \oplus \mathbb{Z} \simeq S_* \oplus \mathbb{Z}.$$

Hence $S' \simeq S \times \mathbb{G}_m$ is a flasque torus. Let $\widetilde{Q}$ be a torus such that $Q \times \widetilde{Q}$ is a quasisplit torus. Then the exact sequence

$$1 \to S' \times \widetilde{Q} \xrightarrow{\tau \times 1_{\widetilde{Q}}} Q \times \widetilde{Q} \to S^\circ \to 1$$

is a flasque resolution of S°. By [5, Th. 2] (cf. Example 1.3) and (6), we have

$$S^\circ(L)/R \simeq H^1(L, S' \times \widetilde{Q}) \simeq H^1(L, S') \simeq H^1(L, S) \simeq T(L)/R \tag{8}$$

for any field extension L/F, and hence it follows from (7) that

$$\mathrm{Coker}\big(Q(F) \to S^\circ(F)\big) = S^\circ(F)/R. \tag{9}$$

As $K_0(X)$ injects into $K_0(X_{\mathrm{sep}})$ and $K_0(X_{\mathrm{sep}})^{(3)}$ is infinite cyclic group generated by the class of a rational point by Theorem 2.3, the kernel of the homomorphism g in (3) coincides with the kernel of the composition

$$\mathrm{CH}^3(X) \to \mathrm{CH}^3(X_{\mathrm{sep}}) \to K_0(X_{\mathrm{sep}})^{(3)} \simeq \mathbb{Z},$$

which is the degree map. Recall that we write $A_0(X)$ for the kernel of $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$. We then have

$$\mathrm{Ker}(g) = A_0(X). \tag{10}$$

Note that the group $A_0(X)$ is 2-torsion by [17, Cor. 5.11(4)].

By Corollary 2.4, we have isomorphisms

$$\begin{aligned} K_1(X)^{(1)} &\simeq \big(K_1(X_{\mathrm{sep}})^{(1)}\big)^\Gamma \\ &\simeq \big(K_0(X_{\mathrm{sep}})^{(1)} \otimes F_{\mathrm{sep}}^\times\big)^\Gamma = \big(Q_* \otimes F_{\mathrm{sep}}^\times\big)^\Gamma = Q(F). \end{aligned} \tag{11}$$

It follows from Corollary 2.2 and Proposition 2.6 that

$$\begin{aligned}(12)\quad H^1(X,K_2) &\simeq H^1(X_{\mathrm{sep}},K_2)^{\Gamma}\\ &\simeq \big(\mathrm{CH}^1(X_{\mathrm{sep}})\otimes F_{\mathrm{sep}}^{\times}\big)^{\Gamma} = \big(S^*\otimes F_{\mathrm{sep}}^{\times}\big)^{\Gamma} = S^{\circ}(F).\end{aligned}$$

Remark 2.7. The referee has pointed out that using results of [4] one can deduce that $\mathrm{CH}^1(X)\otimes F^{\times} \simeq H^1(X,K_2)$ for a smooth projective rational variety X over an algebraically closed field F of characteristic zero.

Under the identifications (11) and (12), and the fact that the BGQ spectral sequence is compatible with products (cf. [11, §7]), the map $K_1(X)^{(1)} \to H^1(X,K_2)$ in (3) coincides with the homomorphism $Q(F)\to S^{\circ}(F)$ given by (7). It follows from (3), (9) and (10) that

$$\begin{aligned}(13)\quad S^{\circ}(F)/R &= \mathrm{Coker}\big(Q(F)\to S^{\circ}(F)\big)\\ &\simeq \mathrm{Coker}\big(K_1(X)^{(1)}\to H^1(X,K_2)\big)\simeq \mathrm{Ker}(g) = A_0(F).\end{aligned}$$

By (8), there are natural isomorphisms

$$T(F)/R \simeq S^{\circ}(F)/R \simeq A_0(X). \tag{14}$$

Similarly, over any field extension L/F we have an isomorphism

$$\boxed{\rho_L : T(L)/R \simeq A_0(X_L).} \tag{15}$$

We shall view ρ as an isomorphism of functors $L\mapsto T(L)/R$ and $L\mapsto A_0(X_L)$ from *Fields*$/F$ to *Ab*.

The following remark was suggested by J.-L. Colliot-Thélène.

Remark 2.8. The isomorphism (14) yields finiteness of $A_0(X)$ in all cases when $T(F)/R$ is known to be finite, e.g. F a finitely generated over the prime subfield, over the complex field, over a p-adic field (cf. [5, Th. 1 and Prop. 14] and [2, Th. 3.4]).

2.3. **The map $\varphi_L : T(L)/R \to A_0(X_L)$.** Let T be an algebraic torus over F, X a smooth proper geometrically irreducible variety over F containing T as an open subset, and L/F a field extension. By [5, Prop. 12, Cor.], the map

$$\boxed{\varphi_L : T(L)/R \to A_0(X_L)} \tag{16}$$

taking the R-equivalence class of an L-point $t\in T(L)$ to the class of the zero cycle $[t]-[1]$, is well defined. We view φ as a morphism of functors from *Fields*$/F$ to *Sets*.

Proposition 2.9. *The map φ_L does not depend (up to canonical isomorphism) of the choice of X.*

Proof. We may assume that $L=F$. Let X and X' be two smooth proper geometrically irreducible varieties containing T as an open subset. The closure of the graph of a birational isomorphism between X and X' that is identical on T yields morphisms between the motives $M(X)$ and $M(X')$ in *CM*(F). These morphisms induce mutually inverse isomorphisms between $A_0(X)$ and $A_0(X')$ (cf. [7, 16.1.11]). □

Let X be a smooth proper toric model of T. Consider the flasque resolution (5). The S-torsor P_L over T_L can be extended to an S-torsor $q : U \to X_L$ (cf. [5, Prop. 9] or [17, Prop. 5.4]). For any point $x \in X_L$, the fiber U_x of q over x is an S-torsor over $\operatorname{Spec} L(x)$. Denote by $[U_x]$ its class in $H^1\big(L(x), S\big)$. By [5, Prop. 12], the map

$$\psi_L : \mathrm{CH}_0(X_L) \to H^1(L, S) = T(L)/R, \tag{17}$$

taking the class $[x]$ of a closed point $x \in X_L$ to $N_{L(x)/L}([U_x])$ extends to a well defined group homomorphism. The composition $\psi|_{A_0(X_L)} \circ \varphi$ is the identity. It follows that the map φ_L is injective.

3. Functors from *Fields*/F to *Sets*

We consider functors from the category *Fields*/F to the category *Sets*.

All functors we are considering take values in *Ab*, but some of the morphisms between such functors (namely, φ) may not be given by group homomorphisms.

In this section, we study compatibility properties for morphisms between functors with respect to norm and specialization maps.

3.1. **Functors with norm maps.** Let A : *Fields*/$F \to$ *Sets* be a functor. We say that A is a *functor with norms* if for any finite field extension E/F, there is given a *norm map* $N_{E/F} : A(E) \to A(F)$.

Example 3.1. Let T be an algebraic torus over F and E/F a finite field extension. There is an obvious norm map

$$N_{E/F} : T(E) = H^0(E, T_* \otimes E^\times_{\mathrm{sep}}) \to H^0(F, T_* \otimes F^\times_{\mathrm{sep}}) = T(F).$$

Thus the functor $L \mapsto T(L)$ is equipped with norms. Similarly, the functors $L \mapsto T(L)/R$, $L \mapsto H^1(L, T)$, and $L \mapsto A_0(X_L)$ also have norms.

A morphism $\alpha : A \to B$ of functors with norms from *Fields*/F to *Sets* *commutes with norms* if for any field extension E/F, the diagram

$$\begin{array}{ccc} A(E) & \xrightarrow{\alpha_E} & B(E) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ A(F) & \xrightarrow{\alpha_F} & B(F) \end{array}$$

is commutative.

Example 3.2. Let T be a torus of dimension 3. The sequence (5) yields an isomorphism of functors $T(L)/R \xrightarrow{\sim} H^1(L, S)$ that commutes with norms. It follows that the isomorphism $T(L)/R \simeq S^\circ(L)/R$ in (8) commutes with norms.

Example 3.3. Let T be an arbitrary torus and $1 \to S \to P \to T \to 1$ a flasque resolution. Let $\mathrm{End}_F(S) = \mathrm{Hom}_\Gamma(S^*, S^*)$ be the endomorphism ring of S. For a field extension L/F, the group $T(L)/R = H^1(L, S)$ has a natural structure of an $\mathrm{End}_F(S)$-module. For any $\alpha \in \mathrm{End}_F(S)$, the endomorphism of the functor $L \mapsto T(L)/R$ taking a t to $\alpha(t)$ commutes with norms.

Proposition 3.4. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the morphism ψ in* (17) *commutes with norms.*

Proof. Let E/F be a finite field extension, $x \in X_E$ a closed point and x' the image of x under the natural morphism $X_E \to X$. We have $N_{E/F}([x]) = m[x']$ in $\mathrm{CH}_0(X)$, where $m = [E(x) : F(x')]$. The torsor U_x in the definition of ψ is the restriction of $U_{x'}$ to $E(x)$. By [7, Example 1.7.4], we have

$$N_{E(x)/F(x')}\big([U_{x'}]_{E(x)}\big) = m[U_{x'}],$$

hence

$$\begin{aligned} N_{E/F}\big(\psi_E([x])\big) = N_{E(x)/F}\big([U_x]\big) &= N_{F(x')/F} N_{E(x)/F(x')}\big([U_{x'}]_{E(x)}\big) \\ &= m N_{F(x')/F}\big([U_{x'}]\big) = \psi_F\big(N_{E/F}([x])\big). \quad \square \end{aligned}$$

Proposition 3.5. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the map $\varphi_F : T(F)/R \to A_0(X)$ in* (16) *is an isomorphism of groups if and only if the morphism φ commutes with norms.*

Proof. Suppose that φ commutes with norms. We show that φ is surjective. Every closed point in X is rationally equivalent to a zero-divisor with support in T. Let $x \in T$ be a closed point of degree n. It is sufficient to prove that $[x] - n[1]$ belongs to the image of φ_F. Let $E = F(x)$ and $x' \in T_E$ the canonical rational point over x. We have $\varphi_E(x') = [x'] - [1]$ and as φ commutes with norms,

$$[x] - n[1] = N_{E/F}([x'] - [1]) = N_{E/F} \circ \varphi_E(x') = \varphi_F\big(N_{E/F}(x')\big).$$

Thus, φ is a bijection. The inverse map given by (17) is a group homomorphism, hence φ is a group isomorphism.

Conversely, if φ is an isomorphism, then φ commutes with norms as ψ does by Proposition 3.4. $\square$

Proposition 3.6. *Let T be an algebraic torus of dimension* 3 *over F and X a smooth proper toric model of T. Then the morphism of functors ρ in* (15) *commutes with norms.*

Proof. By Example 3.2, it suffices to prove that the morphism $S^\circ(L)/R \to A_0(X_L)$ given by (13) commutes with norms. Let E/F be a finite field extension. The statement follows from the commutativity of the diagram

$$\begin{array}{ccccc}
S^\circ(E)/R & \longrightarrow & H^1(X_E, K_2) & \longrightarrow & \mathrm{CH}^3(X_E) \\
\Big\downarrow{\scriptstyle N_{E/F}} & & \Big\downarrow{\scriptstyle N_{E/F}} & & \Big\downarrow{\scriptstyle N_{E/F}} \\
S^\circ(F)/R & \longrightarrow & H^1(X, K_2) & \longrightarrow & \mathrm{CH}^3(X).
\end{array}$$

The exact direct image functor f_* takes the category $M^p(X_E)$ of coherent sheaves on X_E supported in codimension at least p to $M^p(X)$. Therefore, f_* yields a map of the BGQ spectral sequences for X_E and X. Hence the right square of the diagram is commutative.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, it suffices to prove commutativity of the left square in the split case. The left square coincides with

$$\begin{array}{ccc}
S^* \otimes E^\times & \longrightarrow & H^1(X_E, K_2) \\
\Big\downarrow{\scriptstyle 1\otimes N_{E/F}} & & \Big\downarrow{\scriptstyle N_{E/F}} \\
S^* \otimes F^\times & \longrightarrow & H^1(X, K_2),
\end{array}$$

where the horizontal maps are product maps after the identification of S^* with $\mathrm{CH}^1(X)$. The commutativity follows from the projection formula in K-cohomology (cf. [19, §14.5]). □

3.2. **Functors with specializations.** Let $A : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. We say that A is a *functor with specializations* if for any DVR over F of geometric type (a localization of an F-algebra of finite type) with quotient field L and residue field K there is given a map $s_A : A(L) \to A(F)$ called a *specialization map.*

Example 3.7. Let O be a DVR over F with quotient field L and residue field K and X a variety over F. The specialization homomorphism

$$s : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(X_K)$$

is defined as follows. Let $\alpha \in \mathrm{CH}_0(X_L)$. As the restriction map $\mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_L)$ is surjective, we can choose $\alpha' \in \mathrm{CH}_1(X_O)$ such that $\alpha'_L = \alpha$. Then set $s(\alpha) = i^*(\alpha')$, the image of α' under the Gysin homomorphism $i^* : \mathrm{CH}_1(X_O) \to \mathrm{CH}_0(X_K)$, where $i : X_K \to X_O$ is the regular closed embedding of codimension one (cf. [7, §2.6]). The map s is well defined as $i^* \circ i_* = 0$ for the principal divisor X_K in X_O by [7, Prop. 2.6(c)].

Example 3.8. (cf. [10, Prop. 2.2]) Let T be a torus over F and O a DVR over F with quotient field L and residue field K. Let $1 \to S \to P \to T \to 1$ be a flasque resolution of T. The homomorphism

$$H^1_{\text{ét}}(O, S) \to H^1(L, S)$$

is an isomorphism by [6, Cor. 4.2]. The composition

$$s : T(L)/R \simeq H^1(L,S) \simeq H^1_{\text{ét}}(O,S) \to H^1(K,S) \simeq T(K)/R$$

is called the *specialization homomorphism with respect to* O. One can easily see that the specialization homomorphism does not depend on the choice of a flasque resolution of T. It follows from the triviality of $H^1_{\text{ét}}(O,P)$ that the composition $T(O) \to T(L) \to T(L)/R$ is surjective.

$$\begin{tikzcd}
T(L)/R \arrow[d, "\wr"'] & T(L) \arrow[l, two heads] & T(O) \arrow[l] \arrow[r] \arrow[d] & T(K) \arrow[r, two heads] & T(K)/R \arrow[d, "\wr"] \\
H^1(L,S) & & H^1_{\text{ét}}(O,S) \arrow[ll, "\sim"'] \arrow[rr] & & H^1(K,S)
\end{tikzcd}$$

Let $p \in T(L)/R$ and $q \in T(O)$ be a lift of p. Then it readily follows from the definition that $s(p)$ is the image of q under the composition $T(O) \to T(K) \to T(K)/R$.

Lemma 3.9. *Let T be an algebraic torus over F. Let $t, t' \in T$ be two points such that t belongs to the closure of t' and the local ring $O_{t',t}$ is a DVR. Let $s : T\big(F(t')\big)/R \to T\big(F(t)\big)/R$ be the specialization homomorphism with respect to $O_{t',t}$. Then $s(t') = t$.*

Proof. In the ring $A := F[T]$ let P and P' be the prime ideals of y and y' respectively. Then O is the ring $A_P/P'A_P$. Let $\tilde{t} \in T(O) = \operatorname{Mor}(\operatorname{Spec} O, T)$ be the point given by the natural homomorphism of $A \to O$. Then the images of $\tilde{t}$ under the maps $T(O) \to T\big(F(t)\big)$ and $T(O) \to T\big(F(t')\big)$ coincide with y and y' respectively. The statement follows now from Example 3.8. □

Let $\theta : A \to B$ be a morphism of functors from *Fields*$/F$ to *Sets* with specializations (for example, the functors $L \mapsto T(L)/R$ or $L \mapsto \mathrm{CH}_0(X_L)$). We say that θ *commutes with specializations* if for every DVR as above, the diagram

$$\begin{tikzcd}
A(L) \arrow[r, "\theta_L"] \arrow[d, "s_A"'] & B(L) \arrow[d, "s_B"] \\
A(K) \arrow[r, "\theta_K"] & B(K)
\end{tikzcd}$$

is commutative.

Proposition 3.10. *Let T be an algebraic torus over F and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the morphism φ in* (16) *commutes with specializations.*

Proof. Let O be a DVR over F with quotient field L and residue field K. For an O-point p of T let $[p]$ denote the class of its graph in $\mathrm{CH}_1(X_O)$. Consider the diagram

$$\begin{array}{ccccc} T(K) & \longleftarrow & T(O) & \longrightarrow & T(L) \\ {\scriptstyle\varphi_K}\big\downarrow & & {\scriptstyle\varphi_O}\big\downarrow & & {\scriptstyle\varphi_L}\big\downarrow \\ \mathrm{CH}_0(X_K) & \longleftarrow & \mathrm{CH}_1(X_O) & \longrightarrow & \mathrm{CH}_0(X_L) \end{array}$$

where $\varphi_O(p) = [p] - [1]$ and the bottom maps are the pull-back homomorphisms. The statement follows from the commutativity property of the diagram. To prove commutativity let E be either K or L and $f : \operatorname{Spec} E \to \operatorname{Spec} O$, $g : X_E \to X_O$ the natural morphisms. Let $p \in T(O)$ be a point and $q \in T(E)$ its image. We view p and q as morphisms $p : \operatorname{Spec} O \to X_O$ and $q : \operatorname{Spec} E \to X_E$. By [7, Th. 6.2(a)], the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(\operatorname{Spec} O) & \xrightarrow{f^*} & \mathrm{CH}_0(\operatorname{Spec} E) \\ {\scriptstyle p_*}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ \mathrm{CH}_1(X_O) & \xrightarrow{g^*} & \mathrm{CH}_0(X_E) \end{array}$$

is commutative. It follows that $[q] = q_*(1_E) = q_* f^*(1_O) = g^* p_*(1_O) = g^*([p])$ and the result follows. □

Proposition 3.11. *Let T be an algebraic torus over F and $\theta, \theta' : T(?)/R \to B$ two morphisms of functors commuting with specializations. Suppose that $\theta_{F(T)}$ and $\theta'_{F(T)}$ coincide at the generic point of T. Then $\theta = \theta'$.*

Proof. Let $p : \operatorname{Spec} L \to T$ be a point of T over a field extension L over F. We need to prove that $\theta_L(p) = \theta'_L(p)$. Let $t \in T$ be the point in the image of p. We view t as a point of T over the residue field $F(t)$. As $F(t) \subset L$ and p is the image of t under the map $T\big(F(t)\big) \to T(L)$, it suffices to show that $\theta_{F(t)}(t) = \theta'_{F(t)}(t)$.

We prove this by induction on $\operatorname{codim}(t)$. By assumption, the statement holds if t is the generic point. Otherwise let $t' \in T$ be a point such that t is a direct specialization of t'. Then the local ring $O_{t',t}$ is a DVR with quotient field $F(t')$ and residue field $F(t)$. As θ and θ' commute with specializations, it follows from Lemma 3.9 that

$$\theta_{F(t)}(t) = \theta_{F(t)}\big(s(t')\big) = s_B\big(\theta_{F(t')}(t')\big) = s_B\big(\theta'_{F(t')}(t')\big) = \theta'_{F(t)}\big(s(t')\big) = \theta'_{F(t)}(t).$$

□

Proposition 3.12. *Let T be an algebraic torus of dimension 3 over F and X a smooth proper toric model of T. Then the morphism of functors ρ in (15) commutes with specializations.*

Proof. Let O be a DVR over F of geometric type with quotient field L and residue field K. The diagram

$$\begin{array}{ccccc} H^1(X_K, K_2) & \longleftarrow & H^1(X_O, K_2) & \longrightarrow & H^1(X_L, K_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{CH}^3(X_K) & \longleftarrow & \mathrm{CH}^3(X_O) & \longrightarrow & \mathrm{CH}^3(X_L) \end{array}$$

where the middle vertical map is the differential in the E_2-term of the BGQ spectral sequence (2) for X_O. The right square is commutative since the morphism $X_L \to X_O$ is flat (cf. [18, §7, Th. 5.4]).

The pull-back homomorphism f^* for the morphism $f : X_K \to X_O$ in K-theory is defined as follows (cf. [18, §7.2.5]). Let $\pi \in O$ be a prime element and $M(X_O, f)$ the full subcategory of the category $M(X_O)$ of coherent sheaves on X_O consisting of sheaves G with π a non-zero-divisor in G. Then f^* is the composition of the inverse of the isomorphism induced by the inclusion functor $\alpha : M(X_O, f) \to M(X_O)$ on K-groups and the map induced by the restriction $\beta : M(X_O, f) \to M(X_K)$ of the unverse image functor $M(X_O) \to M(X_K)$. Note that functors α and β take sheaves supported in codimension p into $M^p(X_O)$ and $M^p(X_K)$ respectively. Hence f induces a pull-back map of the BGQ spectral sequences for X_O and X_K. It follows that the left square of the diagram is commutative too.

As the map $H^1(X, K_2) \to H^1(X_{\mathrm{sep}}, K_2)$ is injective by Proposition 2.6, we may consider the split situation. In the diagram

$$\begin{array}{ccccc} S^\circ(K) & \longleftarrow & S^\circ(O) & \longrightarrow & S^\circ(L) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X_K, K_2) & \longleftarrow & H^1(X_O, K_2) & \longrightarrow & H^1(X_L, K_2) \end{array}$$

the vertical maps are the product maps. The commutativity follows from the projection formula in K-cohomology (cf. [19, §14.5]).

Finally, it follows from the definition that the isomorphism $T(L)/R \xrightarrow{\sim} S^\circ(L)/R$ of functors in (15) commutes with specializations. ☐

4. Main theorem

Let T be a torus over F and $1 \to S \to P \to T \to 1$ a flasque resolution.

4.1. **The group $T\big(F(T)\big)/R$.** Tensoring the exact sequence

$$0 \to F_{\mathrm{sep}}^\times \oplus T^* \to F_{\mathrm{sep}}(T)^\times \to \mathrm{Div}(T_{\mathrm{sep}}) \to 0$$

with S_* and applying Galois cohomology yields a surjective homomorphism

$$H^1(F, S) \oplus H^1(F, S_* \otimes T^*) \to H^1\big(F(T), S\big)$$

since $H^1\big(F, S_* \otimes \mathrm{Div}(T_{\mathrm{sep}})\big) = 0$ as S is flasque.

Tensoring (4) with S_* yields a surjective homomorphism

$$\mathrm{End}_F(S) = H^0(F, S_* \otimes S^*) \to H^1(F, S_* \otimes T^*)$$

as $H^1(F, S_* \otimes P^*) = 0$. Combining these two surjections we get another surjective homomorphism

$$\big(T(F)/R\big) \oplus \mathrm{End}_F(S) \to T\big(F(T)\big)/R.$$

Note that the group $T(L)/R = H^1(L, S)$ is a left module over the ring $\mathrm{End}_F(S)$ for any field extension L/F. The image of an element $\alpha \in \mathrm{End}_F(S)$ in $T\big(F(T)\big)/R$ is equal to $\alpha(\xi)$ (up to sign), where ξ is the generic point of T.

We have proven

Proposition 4.1. *Every element of the group $T\big(F(T)\big)/R$ is of the form $t{\cdot}\alpha(\xi)$ where $t \in T(F)/R$ and $\alpha \in \mathrm{End}_F(S)$.*

Now assume that $\dim T = 3$ and X is a smooth proper toric model of T.

Corollary 4.2. *There is an $\alpha \in \mathrm{End}_F(S)$ such that the composition $\rho^{-1} \circ \varphi$ takes every $t \in T(L)/R$ over a field extension L/F to $\alpha(t)$.*

Proof. By Propositions 3.10, 3.11 and 3.12, it is sufficient to prove the statement in the case when t is the generic point ξ of T. By Proposition 4.1, $(\rho^{-1} \circ \varphi)(\xi) = t{\cdot}\alpha(\xi)$ for some $\alpha \in \mathrm{End}_F(S)$ and $t \in T(F)/R$. As $(\rho^{-1} \circ \varphi)(1) = 1$, specializing at 1, we get $t = 1$. □

Example 3.3 then yields:

Corollary 4.3. *The composition $\rho^{-1} \circ \varphi$ commutes with norms.*

4.2. **Main theorem.**

Theorem 4.4. *Let T be an algebraic torus of dimension 3 and X a smooth proper geometrically irreducible variety over F containing T as an open subset. Then the map $\varphi : T(F)/R \to A_0(X)$ is an isomorphism.*

Proof. In view of Proposition 2.9, we may assume that X is a smooth proper toric model of T. By Proposition 3.6 and Corollary 4.3, φ commutes with norms. It follows from Proposition 3.5 that φ is an isomorphism. □

Remark 4.5. The following is an alternative proof of Theorem 4.4. This proof avoids the machinery of Section 3, but it is based on deep, albeit classical, arithmetic-geometric result. We may assume that the field F is finitely generated over the prime subfield. By [5, Th.1], the group $T(F)/R$ is finite. It follows from (15) that $A_0(X)$ is also finite of the same order. As φ is injective, it is a bijection. Therefore, φ is an isomorphism of groups as we have a homomorphism of groups ψ with $\psi \circ \varphi = \mathrm{id}$.

The statement of the following theorem (but not the proof) does not involve a toric model.

Theorem 4.6. *Let T be an algebraic torus of dimension 3. Then there is a natural isomorphism $T(F)/R \simeq H^1(F, T^\circ)/R$.*

Proof. The sequence dual to (5)

$$1 \to T^\circ \to P^\circ \to S^\circ \to 1$$

and [5, Th. 2] (cf. Example 1.3) yield an isomorphism

$$S^\circ(F)/R \simeq H^1(F, T^\circ)/R.$$

On the other hand, by (8), $S^\circ(F)/R \simeq H^1(F, S) \simeq T(F)/R$. □

In the following examples we give two applications of Theorem 4.6.

Example 4.7. Let L/F be a degree 4 separable field extension and T the norm 1 torus for L/F, i.e.,

$$T = \operatorname{Ker}\big(R_{L/F}(\mathbb{G}_{m,L}) \xrightarrow{N_{L/F}} \mathbb{G}_m\big).$$

Then $T^\circ = R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$ and

$$H^1(F, T^\circ) = \operatorname{Br}(L/F),$$

the relative Brauer group of the extension L/F. Thus by Theorem 4.6, we have a canonical isomorphism

$$\operatorname{Br}(L/F)/R \simeq T(F)/R.$$

The case of a biquadratic extension L/F was considered in [21, p. 427].

Example 4.8. Let L and K be finite separable field extensions of a field F and set $M := K \otimes_F L$. Let T be the kernel of the norm homomorphism

$$N_{M/L} : R_{M/F}(\mathbb{G}_{m,M})/R_{K/F}(\mathbb{G}_{m,K}) \to R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m.$$

We have

$$T(F) = \{x \in M^\times \quad \text{such that} \quad N_{M/L}(x) \in F^\times\}/K^\times.$$

The dual torus T° is the kernel of the norm homomorphism

$$N_{M/K} : R_{M/F}(\mathbb{G}_{m,M})/R_{L/F}(\mathbb{G}_{m,L}) \to R_{K/F}(\mathbb{G}_{m,K})/\mathbb{G}_m.$$

We have an exact sequence

$$K^\times \to H^1(F, T^\circ) \to \operatorname{Br}(M/L) \to \operatorname{Br}(K/F).$$

Now suppose that $[K : F] = 2$ and $[L : F] = 4$. Then T is a 3-dimensional torus and the last homomorphism in the exact sequence is isomorphic to the norm map

$$N_{L/F} : L^\times/N_{M/L}(M^\times) \to F^\times/N_{K/F}(K^\times).$$

Let U be the subtorus of $R_{L/F}(\mathbb{G}_{m,L}) \times R_{K/F}(\mathbb{G}_{m,K})$ consisting of all pairs (l, k) with $N_{L/F}(l) = N_{K/F}(k)$. It follows that

$$T(F)/R \simeq H^1(F, T^\circ)/R \simeq U(F)/R.$$

This isomorphism was known when L/F is a biquadratic extension (cf. [20, Cor. 1.13] and [9, Prop. 3]).

5. Chow group of a 3-dimensional torus

Let T be an algebraic torus over a field F and X a smooth proper geometrically irreducible variety containing T as an open subset. Set $Z = X \setminus T$.

Lemma 5.1. (cf. [5, Lemme 12], [22, Prop. 17.3] and [10, Prop. 1.1]) *The torus T is isotropic if and only if $Z(F) \neq \emptyset$.*

Proof. Suppose T is isotropic. Then T contains a subgroup isomorphic to $\mathbb{G}_m$. The embedding of $\mathbb{G}_m$ into T extends to a regular morphism $f : \mathbb{P}^1 \to X$. Then $f(0)$ or $f(\infty)$ is a rational point of Z.

Conversely, suppose Z has a rational point z. Since z is regular on X, there is a geometric valuation v of $F(X)$ dominating z with residue field $F = F(z)$. Suppose that T is anisotropic. Then there is a proper geometrically irreducible variety X' containing T as an open subset such that $X' \setminus T$ has no rational points (cf. [5, Lemme 12], [22, Prop. 17.3]). But v dominates a rational point on $X' \setminus T$, a contradiction. □

Write i_T (respectively n_Z) for the greatest common divisor of the integers $[L : F]$ for all finite field extensions L/F such that T is isotropic over L (respectively $Z(L) \neq \emptyset$).

Corollary 5.2. *The number i_T coincides with n_Z. In particular, the integer n_Z does not depend on the smooth proper geometrically irreducible variety X containing T as an open subset.*

Proposition 5.3. *The order of the class* $[1]$ *in* $\mathrm{CH}_0(T)$ *is equal to* i_T*.*

Proof. If T is isotropic, there is a subgroup H of T isomorphic to $\mathbb{G}_m$. As $\mathrm{CH}_0(\mathbb{G}_m) = 0$, we have $[1] = 0$ in $\mathrm{CH}_0(H)$ and therefore in $\mathrm{CH}_0(T)$. In the general case, let L be a finite field extension such that T_L is isotropic. By the first part of the proof, $[1]$ is trivial in $\mathrm{CH}_0(T_L)$, hence applying the norm map for the extension L/F yields $[L : F] \cdot [1] = 0$ in $\mathrm{CH}_0(T)$. Therefore, $i_T \cdot [1] = 0$.

Now let $m \cdot [1] = 0$ in $\mathrm{CH}_0(T)$ for some integer m. Hence the cycle $m \cdot [1]$ in $\mathrm{CH}_0(X)$ belongs to the image of the push-forward map $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$ (cf. [7, Prop. 1.8]). In particular, there is a zero-cycle on Z of degree m, hence $i_F = n_Z$ divides m. □

Consider the map

$$\alpha_T : T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$$

taking a pair (t, k) to the cycle $[t] + (k - 1) \cdot [1]$.

Theorem 5.4. *Let T be a torus of dimension at most* 3*. Then the map $\alpha_T : T(F)/R \oplus \mathbb{Z}/i_T\mathbb{Z} \to \mathrm{CH}_0(T)$ is an isomorphism.*

Proof. The Chow group $\mathrm{CH}_0(T)$ is the factor group of $\mathrm{CH}_0(X) = A_0(X) \oplus \mathbb{Z} \cdot [1]$ by the image of $\mathrm{CH}_0(Z)$. Let $z \in Z$ be a closed point. By Lemma 5.1, the torus $T_{F(z)}$ is isotropic and hence is stably birational to a 2-dimensional torus. Therefore, $T_{F(z)}$ is rational, $A_0(X_{F(z)}) = 0$ and the image of the class of z in

$A_0(X) \oplus \mathbb{Z} \cdot [1]$ is equal to $0 \oplus \deg(z) \cdot [1]$. Hence $\mathrm{CH}_0(T)$ is isomorphic to $A_0(X) \oplus \mathbb{Z}/i_T\mathbb{Z}$. The result follows from Theorem 4.4. $\square$

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