ESSENTIAL DIMENSION OF CERTAIN SUBFUNCTORS

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Let F be a field and let $\mathcal{F} : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. For every prime integer p or $p = 0$, we write $\operatorname{ed}_p(\mathcal{F})$ for the essential p-dimension of $\mathcal{F}$. For example, if X is an algebraic variety over F, viewed as a functor of points, then $\operatorname{ed}_p(X) = \dim(X)$. If G is an algebraic group over F, then $\operatorname{ed}_p(G)$ is the essential p-dimension of the functor G- *torsors* taking a field K to $H^1(K, G)$. If Y is a classifying variety for G, then there is a surjective map of functors $Y \to G$- *torsors*, i.e., the functor G- *torsors* is a factor functor of Y.

In the present note we consider subfunctors of Y for an algebraic variety Y over F. Moore specifically, let $f : X \to Y$ be a dominant morphism of (integral) varieties over F. We consider the functor $I_f : \mathit{Fields}/F \to \mathit{Sets}$ defined by

$$I_f(K) = \operatorname{Im}\big(X(K) \xrightarrow{f_K} Y(K)\big) \subset Y(K)$$

for a field extension K/F. Thus, I_f is a subfunctor of the functor Y.

Theorem 0.1. *Let $f : X \to Y$ be a dominant morphism of varieties over a field F and let X' be the generic fiber of f. Then*

$$\dim(Y) + \operatorname{cdim}_p(X') \le \operatorname{ed}_p(I_f) \le \operatorname{ed}(I_f) \le \dim(X)$$

for every $p \ge 0$.

Proof. As there is a surjection $X \to I_f$, we have $\operatorname{ed}_p(I_f) \le \operatorname{ed}(I_f) \le \dim(X)$.

Let $K = F(Y)$, let E/K be a field extension and $x' \in X'(E)$. Write x for the image of x' in $X(E)$ and set $y := f_E(x)$ in $Y(E)$. We view y as a point in $I_f(E)$. By the definition of the essential dimension of $I_f(E)$, there is a field extension L/E of degree prime to p, a subfield $L_0 \subset L$ over F and an element $y_0 \in I_f(L_0)$ such that $(y_0)_L = y_L$ and $\operatorname{tr.deg}(L_0/F) \le \operatorname{ed}_p(I_f)$. It follows that the images of y_0 and y in Y coincide with the generic point of Y, hence K can be viewed as a subfield of L_0.

As $y_0 \in I_f(L_0)$, there is a point $x_0 \in X(L_0)$ such that $f_{L_0}(x_0) = y_0$. We can view x_0 as a point in $X'(L_0)$. By the definition of the canonical dimension of X', we have

$$\operatorname{cdim}_p(x') \le \operatorname{tr.deg}(L_0/K) = \operatorname{tr.deg}(L_0/F) - \operatorname{tr.deg}(K/F) \le \operatorname{ed}_p(I_f) - \dim(Y).$$

It follows that $\operatorname{cdim}_p(X') \le \operatorname{ed}_p(I_f) - \dim(Y)$. □

Corollary 0.2. *If the generic fiber X' is p-incompressible, then* $\operatorname{ed}_p(I_f) = \operatorname{ed}(I_f) = \dim(X)$.

Proof. As X' is p-incompressible, we have $\operatorname{cdim}_p(X') = \dim(X')$. Note that $\dim(X) = \dim(Y) + \dim(X')$. □

Example 0.3. Let F be a field of characteristic zero and $\alpha \in H^n\big(F, \mu_p^{\otimes(n-1)}\big)$ be a non-zero symbol. Consider the functor

$$\mathcal{F}_\alpha(K) = \Big\{a \in K^\times \quad \text{such that} \quad (a) \cup \alpha_K = 0 \quad \text{in} \quad H^{n+1}\big(K, \mu_p^{\otimes n}\big)\Big\} \subset K^\times.$$

We claim that

$$\mathrm{ed}_p(\mathcal{F}_\alpha) = p^n.$$

Let Z_α be a p-generic splitting norm variety of α of dimension $p^{n-1} - 1$. Write $\widetilde{S}^p(Z_\alpha)$ for the symmetric pth power of Z_α with all the diagonals removed. A geometric point of $\widetilde{S}^p(Z_\alpha)$ is a zero-cycle $z = z_1 + \cdots + z_p$ of degree p with all z_i distinct. There is a vector bundle $E \to \widetilde{S}^p(Z_\alpha)$ with the fiber over a point z as above the degree p algebra $F(z) := F(z_1) \times \cdots \times F(z_p)$. Leaving only invertible elements in each fiber we get an open subvariety X in E. Note that $\dim(X) = p \dim(Z_\alpha) + p = p^n$. A K-point of X is a pair (z, u), where z is an effective zero-cycle on Z_α over K of degree p and $u \in K(z)^\times$.

Consider the morphism $f : X \to \mathbf{G}_\mathrm{m}$ taking a pair (z, u) to $N_{K(z)/K}(u)$ and the functor I_f.

Lemma 0.4. *For any field extension K/F we have:*
1. $I_f(K) \subset \mathcal{F}_\alpha(K)$.
2. If K is a p-special field, then $\mathcal{F}_\alpha(K) = I_f(K)$.

Proof. 1. Suppose $a \in I_f(K)$, i.e., $a = N_{K(z)/K}(u)$ for a point $(z, u) \in X(K)$. We have

$$(a) \cup \alpha_K = N_{E/K}\big((u) \cup \alpha_{K(z)}\big) = 0$$

as $\alpha_{K(z)} = 0$ since Z_α is a splitting field of α. Thus, $a \in \mathcal{F}_\alpha(K)$.

2. Let $a \in \mathcal{F}_\alpha(K)$, i.e., $(a) \cup \alpha_K = 0$ for an element $a \in K^\times$. By [3], there is a degree p field extension E/K and an element $u \in E^\times$ such that $a = N_{E/K}(u)$ and $\alpha_E = 0$. It follows that $Z(E) \neq \emptyset$ and therefore, Z has a closed point z of degree p with $F(z) = E$. We have $(z, u) \in X(K)$ and $f(z, u) = a$, hence $a \in I_f(K)$. □

It follows from the lemma that the inclusion of functors $I_f \hookrightarrow \mathcal{F}_\alpha$ is a p-bijection, hence $\mathrm{ed}_p(I_f) = \mathrm{ed}_p(\mathcal{F}_\alpha)$.

The generic fiber X' of f is a standard norm variety for the $(n+1)$-symbol $(t) \cup \alpha$ over the rational function field $F(t)$. As the symbol $(t) \cup \alpha$ is not trivial, the variety X' is p-incompressible. By Corollary 0.2, $\mathrm{ed}_p(I_f) = \dim(X) = p^n$.

Example 0.5. Let (V, q) be a non-degenerate quadratic form over F of characteristic different from 2 and let $D(q)$ be the functor of values of q, i.e.,

$$D(q)(K) = \big\{q(v), \quad v \in V_K \quad \text{an anisotropic vector}\big\} \subset K^\times.$$

If the form q is isotropic, then $D(q)(K) = K^\times$ for all K and hence $\mathrm{ed}_2\big(D(q)\big) = \mathrm{ed}\big(D(q)\big) = 1$.

We claim that if q is anisotropic, then

$$\mathrm{ed}_2\big(D(q)\big) = \mathrm{ed}\big(D(q)\big) = \dim(q).$$

Let $X \subset \mathbb{A}(V)$ be the open subscheme of anisotropic vectors in the affine space of V. The restriction of q on X yields a morphism $f : X \to \mathbf{G}_{\mathrm{m}}$. The generic fiber X' of f is the affine quadric given by the quadratic form $h := q \perp \langle -t \rangle$ over the rational function field $F(t)$.

Lemma 0.6. *The first Witt index of h is equal to 1.*

Proof. Over the function field $F(t)(h)$ of h, we have:

$$q_{F(t)(h)} \perp \langle -t \rangle = h_{F(t)(h)} = h' \perp \langle t, -t \rangle$$

for a quadratic form h' over $F(t)(h)$. Then the form h' is a subform of $q_{F(t)(h)}$. As the field extension $F(t)(h)/F$ is purely transcendental, the form $q_{F(t)(h)}$ is anisotropic, hence so is h'. □

It follows from the lemma and [1, Th.75.4] that the generic fiber X' is 2-incompressible. The claim follows from Corollary 0.2.

Corollary 0.7. *Let $q(x) = q(x_1, \dots, x_n)$ be an anisotropic quadratic form over a field F with $\mathrm{char}(F) \neq 2$. Let L be a subfield of the rational function field $F(x)$ containing $F\big(q(x)\big)$. If the generic value $q(x)$ of q is a value of q over L, then $[F(x) : L] < \infty$. Moreover, the degree $[F(x) : L]$ is odd.*

Proof. The last statement follows from [2, Th. 6.4]. □

Example 0.8. Let L/F be a finite separable field extension. Let $f : R_{L/F}(\mathbf{G}_{\mathrm{m},L}) \to \mathbf{G}_{\mathrm{m}}$ be the norm map. Consider the functor I_f. The set $I_f(K)$ is the group of all non-zero norms for the extension $K \otimes_F L/K$. The generic fiber X' of f is the generic torsor for the norm one torus $T = R^{(1)}_{L/F}(\mathbf{G}_{\mathrm{m},L})$.

Question 0.9. *When is the generic T-torsor p-incompressible?*

Suppose that L/F is a cyclic extension. Then the generic fiber is an open subscheme of the Severi-Brauer variety of the cyclic (division) algebra $\big(L(t)/F(t), t\big)$ over $F(t)$. This variety is known to be p-incompressible if $[L : F]$ is a power of the prime p by [2]. In this case, $\mathrm{ed}_p(I_f) = \mathrm{ed}(I_f) = [L : F]$.

References

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