# STEENROD OPERATIONS AND DEGREE FORMULAS

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Abstract. Some degree formulas associated with any morphism of projective algebraic varieties of the same dimension are proved. As an application certain conditions of non-compressibility of algebraic varieties are found. The basic tool is the action of the Steenrod algebra on the Chow groups of algebraic varieties modulo a prime integer.

## 1. Introduction

In the present paper we prove certain *degree formulas* associated with any morphism of projective algebraic varieties of the same dimension (Theorem 6.4) and derive some applications. The most general degree formula, which holds in the algebraic cobordism group of a variety, was proved by M. Levine and F. Morel in [9, Th. 13.7]. Our approach is elementary: we use Chow groups instead of cobordism groups and don't assume resolution of singularities.

In the first part of the paper we recall Steenrod operations on Chow groups as defined by P. Brosnan in [1]. We introduce certain characteristic classes $c_R$ and prove formulas involving Steenrod operations and values of the $c_R$ on the tangent vector bundle of a variety.

The degree formulas are proved in Section 6 and some applications are considered in Section 7. In particular, we find conditions of non-compressibility of algebraic varieties.

By a *variety* over a field $F$ we mean a quasi-projective integral scheme over $F$. For a variety $X$ over $F$ and a field extension $L/F$ we write $X(L)$ for the set $\operatorname{Mor}_F(\operatorname{Spec} L, X)$ of $L$-valued points of $X$.

Recall that the *degree* $\deg(f)$ of a rational morphism $f : Y \to X$ of varieties over $F$ of the same dimension is either zero, if $f$ is not dominant, or is equal to the degree of the field extension $F(Y)/F(X)$ otherwise.

## 2. Chow groups and correspondences

2.1. **Chow groups.** Let $X$ be a variety over a field $F$. We write $\mathrm{CH}_d(X)$ for the *Chow group* of rational equivalence classes of dimension $d$ algebraic cycles on the variety $X$ [2, Ch. 1]. We let $\mathrm{CH}(X)$ be the *graded* Chow group $\coprod_{d\geq 0} \mathrm{CH}_d(X)$.

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*Date*: September 16, 2002.

1991 *Mathematics Subject Classification.* Primary 14C15, 14E05 ; Secondary 14G05, 55S10.

Partially supported by the N.S.F.

For a fixed prime integer $p$, we write $A(X)$ and $A_d(X)$ for the modulo $p$ Chow groups $\mathrm{CH}(X) \otimes \mathbb{Z}/p\mathbb{Z}$ and for $\mathrm{CH}_d(X) \otimes \mathbb{Z}/p\mathbb{Z}$ respectively. If $Z \subset X$ is a closed subvariety, $[Z]$ denotes the class of $Z$ in either $\mathrm{CH}(X)$ or $A(X)$. A projective morphism of varieties $f : X \to Y$ induces a *push-forward* homomorphisms $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $f_* : A(X) \to A(Y)$ of graded groups [2, 1.4]. In particular, if $f$ is a projective morphism and $\dim(X) = \dim(Y)$, then $f_*([X]) = \deg(f) \cdot [Y]$.

2.2. **Correspondences.** Let $X$ and $Y$ be varieties over a field $F$, $d = \dim(X)$. A *correspondence from* $X$ *to* $Y$, denoted $\alpha\colon X \rightsquigarrow Y$, is a dimension $d$ algebraic cycle on $X \times Y$. A correspondence $\alpha$ is called *prime* if $\alpha$ is given by a prime (irreducible) cycle. Any correspondence is a linear combination with integer coefficients of prime correspondences.

Let $\alpha : X \rightsquigarrow Y$ be a prime correspondence. Suppose that $\alpha$ is given by a closed subvariety $Z \subset X \times Y$. We define *multiplicity* of $\alpha$ as the degree of the projection $Z \to X$. We extend the notion of multiplicity to arbitrary correspondences by linearity.

A rational morphism $X \to Y$ defines a multiplicity 1 prime correspondence $X \rightsquigarrow Y$ given by the closure of its graph. One can think of a correspondence of multiplicity $m$ as a "generically $m$-valued morphism".

## 3. The Steenrod algebra

We fix a prime integer $p$ in this section.

3.1. **Group scheme $G$.** Consider the polynomial ring

$$(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}] = (\mathbb{Z}/p\mathbb{Z})[b_1, b_2, \dots]$$

in infinitely many variables $b_1, b_2, \dots$ as a graded ring with $\deg b_i = p^i - 1$. The monomials

$$b^R = b_1^{r_1} b_2^{r_2} \dots,$$

where $R$ ranges over all sequences $(r_1, r_2, \dots)$ of non-negative integers such that almost all of the $r_i$'s are zero, form a basis of $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$ over $\mathbb{Z}/p\mathbb{Z}$. We set

$$|R| = \sum_{i \geq 1} r_i(p^i - 1).$$

Clearly, $\deg b^R = |R|$.

Denote the scheme $\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$ by $G$. For a commutative $\mathbb{Z}/p\mathbb{Z}$-algebra $A$ the set of $A$-points $G(A) = \mathrm{Hom}_{rings}\big((\mathbb{Z}/p\mathbb{Z})[\mathbf{b}], A\big)$ can be identified with the set of sequences $(a_1, a_2, \dots)$ of the elements of $A$ and, therefore, with the set of power series of the form

$$t + a_1 t^p + a_2 t^{p^2} + a_3 t^{p^3} + \dots \in A[[t]].$$

The composition law $(f_1 * f_2)(t) = f_2(f_1(t))$ makes $G$ a group scheme over $\mathbb{Z}/p\mathbb{Z}$ and $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$ a Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra.

We identify the tensor square of the ring $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$ with the polynomial ring $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}', \mathbf{b}'']$ where $\mathbf{b}'$ and $\mathbf{b}''$ are the two copies of the sets of variables $b'_i$ and $b''_i$ respectively. The coproduct ring homomorphism

$$\mu : (\mathbb{Z}/p\mathbb{Z})[\mathbf{b}] \to (\mathbb{Z}/p\mathbb{Z})[\mathbf{b}', \mathbf{b}'']$$

is given by the formula

$$\mu(b_k) = \sum_{i+j=k} (b'_i)^{p^j} \cdot b''_j. \tag{1}$$

Since $\mu$ is a homomorphism of graded rings, it extends to the homomorphism of power series rings

$$(\mathbb{Z}/p\mathbb{Z})[[\mathbf{b}]] \to (\mathbb{Z}/p\mathbb{Z})[[\mathbf{b}', \mathbf{b}'']],$$

which will be still denoted by $\mu$.

Let $\Lambda$ be an $\mathbb{Z}/p\mathbb{Z}$-algebra. We will keep the notation $\mu$ for the extended ring homomorphism

$$\Lambda[[\mathbf{b}]] \to \Lambda[[\mathbf{b}', \mathbf{b}'']].$$

Thus, $\mu$ acts identically on the coefficient ring $\Lambda$ and acts on the variables $b_k$ by formula (1).

For a power series $g \in \Lambda[[\mathbf{b}]]$, we write $g'$ and $g''$ for the corresponding power series in $\Lambda[[\mathbf{b}']]$ and $\Lambda[[\mathbf{b}'']]$ respectively.

Consider the *generic power series*

$$f(t) = t + b_1 t^p + b_2 t^{p^2} + b_3 t^{p^3} + \cdots \in (\mathbb{Z}/p\mathbb{Z})[t][[\mathbf{b}]].$$

By definition of the coproduct,

$$\mu\big(f(t)\big) = f''\big(f'(t)\big). \tag{2}$$

3.2. **Reduced Steenrod algebra.** We write $\mathcal{S}$ for the Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra dual to $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$. Thus, $\mathcal{S}$ is the graded Hopf algebra such that the $d$-component of $\mathcal{S}$ is the $\mathbb{Z}/p\mathbb{Z}$-space dual to the $d$-component of $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$. The algebra $\mathcal{S}$ is the *reduced Steenrod $\mathbb{Z}/p\mathbb{Z}$-algebra*, i.e., the Steenrod $\mathbb{Z}/p\mathbb{Z}$-algebra modulo the ideal generated by the Bockstein element [12, Th. 18.23].

Let $\{S^R\}$ be the basis of $\mathcal{S}$ dual to the basis $\{b^R\}$ of $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$. The power series

$$S = \sum_R S^R b^R \in \mathcal{S}[[\mathbf{b}]]$$

is called the *total Steenrod operation.* The product $\circ$ in $\mathcal{S}$ is dual to the coproduct in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{b}]$, therefore

$$\mu(S) = S' \circ S'' \in \mathcal{S}[[\mathbf{b}', \mathbf{b}'']]. \tag{3}$$

We write $S^i$ for the basis element $S^R$ with $R = (i, 0, \dots)$. The element $S^0$ is the identity of $\mathcal{S}$ and the algebra $\mathcal{S}$ is generated by the $S^i$, $i \geq 1$ [12, Ch. 18].

Let $M$ be a $\mathbb{Z}/p\mathbb{Z}$-space. Suppose for every sequence $R$ we are given an endomorphism $U^R \in \operatorname{End}(M)$. Set

$$U = \sum_R U^R b^R \in \operatorname{End}(M)[[\mathbf{b}]]. \tag{4}$$

The following statement readily follows from definitions.

**Proposition 3.1.** *The following two conditions are equivalent:*
1. $\mu(U) = U' \circ U''$ *in* $\mathrm{End}(M)[[\mathbf{b}', \mathbf{b}'']]$;
2. *The space* $M$ *has a structure of a left* $\mathcal{S}$*-module such that* $S^R m = U^R(m)$ *for every* $m \in M$.

**Example 3.2.** Let $f \in (\mathbb{Z}/p\mathbb{Z})[t][[\mathbf{b}]]$ be the generic power series. For an $\mathbb{Z}/p\mathbb{Z}$-algebra $\Lambda$ consider the operators $U^R$ on the polynomial ring $\Lambda[t]$ defined by the rule

$$U\big(g(t)\big) = g\big(f(t)\big)$$

for $g(t) \in \Lambda[t]$, where $U$ is the total operation as in (4). Since

$$\mu(U)\big(g(t)\big) = g\big(\mu(U)(t)\big) = g\big(f''(f'(t))\big) = U'\big(g(f''(t))\big) = (U' \circ U'')\big(g(t)\big),$$

by Proposition 3.1, we get an $\mathcal{S}$-module structure on $\Lambda[t]$ such that $S\big(g(t)\big) = g\big(f(t)\big)$ for every $g(t) \in \Lambda[t]$. Since $S(t) = f(t)$, it follows that

$$S^R(t) = \begin{cases} t^{p^n} & \text{if } R = (0, \dots, 0, \overset{n}{1}, 0, \dots), \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$S^i(t) = \begin{cases} t & \text{if } i = 0, \\ t^p & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Set

$$h(t) = f(t)/t = 1 + b_1 t^{p-1} + b_2 t^{p^2-1} + b_3 t^{p^3-1} + \cdots \in (\mathbb{Z}/p\mathbb{Z})[t][[\mathbf{b}]]. \tag{5}$$

It follows from (2) that

$$\mu\big(h(t)\big) = h'(t) \cdot h''\big(f'(t)\big) = h'(t) \cdot S'\big(h''(t)\big). \tag{6}$$

## 4. Chern classes

Consider the polynomial ring

$$\mathbb{Z}[\mathbf{c}] = \mathbb{Z}[c_1, c_2, \dots]$$

in infinite number of variables $c_1, c_2, \dots$ called the *Chern classes*. We call the elements of this ring the *characteristic classes*. For every $c \in \mathbb{Z}[\mathbf{c}]$ and a vector bundle $E$ over a smooth variety $X$, we have a well defined characteristic class $c(E) \in \mathrm{CH}(X)$ of $E$ [3, Appendix A].

Let $p$ be a prime integer. For every sequence $R = (r_1, r_2, \dots)$ consider the "smallest" symmetric polynomial $Q_R$ in the variables $X_1, X_2, \dots$ containing the monomial

$$(X_1 \dots X_{r_1})^{p-1} (X_{r_1+1} \dots X_{r_1+r_2})^{p^2-1} (X_{r_1+r_2+1} \dots X_{r_1+r_2+r_3})^{p^3-1} \dots$$

and write $Q_R$ as a polynomial on the standard symmetric functions:

$$Q_R = T_R(\sigma_1, \sigma_2, \dots).$$

We set

$$c_R = T_R(c_1, c_2, \dots) \in \mathbb{Z}[\mathbf{c}].$$

**Remark 4.1.** Another notation for the characteristic class $c_R$ is $c_\alpha$, where $\alpha$ is the partition

$$(p-1, \dots, p-1, p^2-1, \dots, p^2-1, \dots),$$

where $p^i - 1$ is repeated $r_i$ times [11, p. 388]. In particular, for $R = (0, \dots, 0, \overset{n}{1}, 0, \dots)$, the class $c_R = c_{(p^n-1)}$ is known as the *additive* Chern class.

**Remark 4.2.** If $p = 2$, the class $c_R$ for $R = (i, 0, 0 \dots)$ coincides with the Chern class $c_i$.

It follows from definition that for a line bundle $L$,

$$c_R(L) = \begin{cases} c_1(L)^{p^n-1} & \text{if } R = (0, \dots, 0, \overset{n}{1}, 0, \dots), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Consider the power series (the *total class*)

$$C = \sum_R c_R b^R \in \mathbb{Z}[\mathbf{c}][[\mathbf{b}]].$$

The constant term of the power series $C$ is the identity, hence $C$ is invertible.
It follows from (7) that for every line bundle $L$,

$$C(L) = \sum_R c_R(L) b^R = \sum_{i \geq 0} c_1(L)^{p^i-1} b_i. \tag{8}$$

By definition of the classes $c_R$, for a vector bundle $E$, having a filtration by subbundles with line factors $L_1, L_2, \dots, L_n$, we have

$$C(E) = C(L_1) \cdot C(L_2) \cdot \ldots \cdot C(L_n).$$

It follows from the splitting principle that the class $C$ is *multiplicative*, that is for an exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

we have

$$C(E) = C(E') \cdot C(E'').$$

Therefore, for every smooth variety $X$, the total class $C$ defines the homomorphism

$$K_0(X) \to \mathrm{CH}(X)[[\mathbf{b}]]^\times, \quad [E] \mapsto C(E).$$

In particular, we have a well defined class $c_R(a) \in \mathrm{CH}(X)$ for every $a \in K_0(X)$ and a sequence $R$.

## 5. Steenrod operations on Chow groups

We fix a field $F$ and a prime integer $p \neq \operatorname{char} F$. Let $X$ be a smooth variety over $F$. In [1, Def. 7.11], P. Brosnan has defined a structure of an $\mathcal{S}$-module on the modulo $p$ Chow group $A(X)$. The operation $S^R$ lowers the dimension of a cycle by $|R|$. The total operation $S$ satisfies the *Cartan formula* [1, Th. 8.2]

$$S(\alpha \cdot \beta) = S(\alpha) \cdot S(\beta)$$

for every $\alpha, \beta \in A(X)$ and takes the identity to itself, i.e., $S([X]) = [X]$.

For a characteristic class $c \in \mathbb{Z}[\mathbf{c}]$ we write $\bar{c}$ for its residue in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{c}]$. The class $\bar{c}$ takes values in $A(X)$.

Let $L$ be a line bundle over $X$. We have by (8),

$$\overline{C}(L) = h\big(\bar{c}_1(L)\big), \tag{9}$$

where $h$ is defined by equality (5). It was proved in [1, Prop. 8.4] that

$$S^i\big(\bar{c}_1(L)\big) = \begin{cases} \bar{c}_1(L) & \text{if } i = 0, \\ \bar{c}_1(L)^p & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases} \tag{10}$$

Recall the action of $\mathcal{S}$ on the ring $\mathbb{Z}/p\mathbb{Z}[t]$ considered in Example 3.2. It follows from (10) that every $S^i$ acts on $t$ the same way as the $S^i$ acts on $\bar{c}_1(L)$. Since the $S^i$ generate $\mathcal{S}$, we have by (6),

$$\mu\big(h(\bar{c}_1(L))\big) = h'(\bar{c}_1(L)) \cdot S'\big(h''(\bar{c}_1(L))\big). \tag{11}$$

**Lemma 5.1.** *For every $a \in K_0(X)$, we have $\mu\big(\overline{C}(a)\big) = \overline{C}'(a) \cdot S'\big(\overline{C}''(a)\big)$.*

*Proof.* By the splitting principle, we may assume that $a$ is the class of a line bundle $L$ over $X$. In this case the statement follows from (11) and (9). □

Fix an element $a \in K_0(X)$ and consider the new operations $U^R$ on $A(X)$ defined by the rule $U = \overline{C}(a) \cdot S$, where $U$ is the total operation. In particular, $U([X]) = \overline{C}(a)$, i.e., $U^R([X]) = \bar{c}_R(a)$ for every sequence $R$.

**Lemma 5.2.** *The operation $U$ satisfies $\mu(U) = U' \circ U''$.*

*Proof.*

$$\begin{aligned} U' \circ U'' &= \overline{C}'(a) \cdot S'\big(\overline{C}''(a) \cdot S''\big) \quad \text{(Cartan formula)} \\ &= \overline{C}'(a) \cdot S'(\overline{C}''(a)) \cdot (S' \circ S'') \quad \text{(equality (3))} \\ &= \overline{C}'(a) \cdot S'(\overline{C}''(a)) \cdot \mu(S) \quad \text{(multiplicativity of } \mu) \\ &= \overline{C}'(a) \cdot S'(\overline{C}''(a)) \cdot \mu\big(\overline{C}(a)\big)^{-1} \cdot \mu(U) \quad \text{(Lemma 5.1)} \\ &= \mu(U). \end{aligned}$$

□

It follows from Proposition 3.1 that the operation $U$ gives rise to a structure of a left $\mathcal{S}$-module on $A(X)$.

Take $a = -[T_X] \in K_0(X)$, where $T_X$ is the tangent vector bundle of $X$ and define the operation $U$ as above. P. Brosnan denoted the operations $U^i$ by $S_i$ in [1, Def. 7.13]. We set $S_R = U^R$ for every $R$. The operations $S_i$, and therefore, $S_R$ extend to $A(X)$ for every (non necessarily smooth) variety $X$ [1, §7]. These operations commute with projective push-forward homomorphisms [1, Prop. 8.11]. The operation $S_R$ lowers the dimension of a cycle by $|R|$.

We summarize some properties of the operations $S_R$ in the following

**Proposition 5.3.** (1) *For a projective morphism $f : X \to Y$ of varieties the diagram*

$$\begin{array}{ccc} A(X) & \xrightarrow{S_R} & A(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ A(Y) & \xrightarrow{S_R} & A(Y) \end{array}$$

*is commutative for every sequence $R$.*

(2) *Let $X$ be a smooth variety. Then $S_R([X]) = \bar{c}_R(-T_X)$ for every sequence $R$.*

## 6. Degree formulas

Let $X$ be a variety over $F$. For a closed point $x \in X$ we define its *degree* as the integer $\deg(x) = [F(x) : F]$ and set

$$n_X = \gcd \deg(x),$$

where the gcd is taken over all closed points of $X$. Clearly,

$$n_X = \gcd[L : F]$$

over all finite field extensions $L/F$ such that $X(L) \neq \emptyset$.

**Example 6.1.** Let $A$ be central simple algebra over $F$ and let $X = SB(A)$ be the associated Severi-Brauer variety [8, 1.16]. We have $X(L) \neq \emptyset$ for a field extension $L/F$ if and only if $L$ splits $A$ [8, Prop. 1.17]. The degree of every finite splitting field extension is divisible by the index $\operatorname{ind}(A)$ of the algebra $A$ and there are splitting field extensions of degree exactly $\operatorname{ind}(A)$ [4, Ch. 4]. Therefore, $n_X = \operatorname{ind}(A)$.

**Example 6.2.** Let $(V, q)$ be a nondegenerate quadratic form over $F$. Let $X$ be the projective quadric hypersurface in the projective space $\mathbb{P}(V)$ given by the equation $q(v) = 0$. We have $X(L) \neq \emptyset$ for a field extension $L/F$ if and only if the form $q_L = q \otimes_F L$ is isotropic. If $q$ is an anisotropic form, then it is isotropic over a quadratic extension of $F$ and for every odd degree extension $L/F$, the form $q_L$ is anisotropic by Springer's theorem, therefore, $n_X = 2$.

Fix a prime integer $p \neq \operatorname{char} F$. Let $X$ be a projective variety of dimension $d > 0$ and let $q : X \to \operatorname{Spec} F$ be the structure morphism. For every sequence $R$

with $|R| = d$, the group $A_d(\operatorname{Spec} F)$ in the commutative diagram (Proposition 5.3)

$$\begin{tikzcd} A_d(X) \arrow[r, "S_R"] \arrow[d, "q_*"'] & A_0(X) \arrow[d, "q_*=\deg"] & \\ A_d(\operatorname{Spec} F) \arrow[r, "S_R"] & A_0(\operatorname{Spec} F) \arrow[r, equal] & \mathbb{Z}/p \end{tikzcd}$$

is trivial. Hence the degree of a cycle $u_R^X \in \mathrm{CH}_0(X)$ representing the element $S_R([X]) \in A_0(X)$ is divisible by $p$. The class of the integer $\deg(u_R^X)/p$ modulo $n_X$ is independent on the choice of $u_R^X$; we denote it by

$$R^p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

Clearly, $p \cdot R^p(X) = 0$.

If $X$ is smooth, by Proposition 5.3(2), $S_R([X]) = \bar{c}_R(-T_X)$, hence we can take $u_R^X = c_R(-T_X)$. Thus, $\deg c_R(-T_X)$ is divisible by $p$ and

$$R^p(X) = \frac{\deg c_R(-T_X)}{p} + n_X\mathbb{Z}.$$

**Remark 6.3.** The degree of the 0-cycle $c_R(-T_X)$ does not change under field extensions. In particular, it can be computed over an algebraic closure of $F$.

**Theorem 6.4.** (Degree formula) *Let $f : X \to Y$ be a morphism of projective varieties over $F$ of dimension $d > 0$. Then $n_Y$ divides $n_X$ and for every sequence $R$ with $|R| = d$, and for any prime integer $p \neq \operatorname{char} F$, we have*

$$R^p(X) = \deg(f) \cdot R^p(Y) \in \mathbb{Z}/n_Y\mathbb{Z}.$$

*In particular, if $X$ and $Y$ are smooth, then*

$$\frac{\deg c_R(-T_X)}{p} \equiv \deg(f) \cdot \frac{\deg c_R(-T_Y)}{p} \quad \big(\mathit{mod}\ n_Y\big).$$

*Proof.* It follows from the commutativity of the diagram in Proposition 5.3(1) and the equality $f_*([X]) = \deg(f) \cdot [Y]$ that

$$f_* s_R([X]) = \deg(f) \cdot s_R([Y]) \in A_0(Y)$$

and therefore,

$$f_* u_R^X \equiv \deg(f) \cdot u_R^Y \quad \big(\mathit{mod}\ p\,\mathrm{CH}_0(Y)\big).$$

Applying the degree homomorphism, we get

$$\deg(u_R^X) = \deg(f_* u_R^X) \equiv \deg(f) \cdot \deg(u_R^Y) \quad (\mathit{mod}\ pn_Y),$$

whence the result. □

**Remark 6.5.** Using a different approach in [10], M. Rost defined the classes $R^p$ and proved the degree formula for the sequences of the form $R = (i, 0, 0, \dots)$.

**Remark 6.6.** For a field extension $L/F$ one can define an integer $n_L$ as the $\gcd[F(v):F]$ over all valuations $v$ on $L$ over $F$ with residue field $F(v)$ finite over $F$. Let $X$ be a variety over $F$, let $L = F(X)$ be the function field and let $v$ be a valuation on $L$ over $F$. If $X$ is projective, the generic point $\operatorname{Spec} L \to X$ factors through a morphism $f : \operatorname{Spec} \mathcal{O} \to X$, where $\mathcal{O}$ is the valuation ring of $v$. Let $m$ be the maximal ideal of $\mathcal{O}$ and let $x = f(m)$. Then the residue field $F(x)$ is isomorphic to a subfield of $\mathcal{O}/m = F(v)$, hence, $\deg(x)$ divides $[F(v):F]$ and therefore, $n_X$ divides $n_L$. If $X$ is smooth, for every closed point $x \in X$ there is a valuation $v$ on $L$ with residue field $F(x)$, so that $[F(v):F] = \deg(x)$ and therefore, $n_L$ divides $n_X$. Thus, if $X$ is smooth and projective, then $n_X = n_L$, i.e., the number $n_X$ is a birational invariant of a smooth projective variety over $F$.

## 7. Applications

We fix a prime integer $p$ and a field $F$ such that $\operatorname{char} F \neq p$. Let $R$ be a nonzero sequence and let $X$ be a projective variety over $F$ of dimension $|R|$. The variety $X$ is called *$R^p$-rigid* if $R^p(X) \neq 0 \in \mathbb{Z}/n_X\mathbb{Z}$.

We write $v_p$ for the $p$-adic valuation. For a cycle $u_R^X \in \mathrm{CH}_0(X)$ representing $S_R([X])$ we obviously have $v_p(n_X) \leq v_p(\deg u_R^X)$ since $n_X$ divides degree of any 0-cycle.

The following statement follows readily from the definition.

**Proposition 7.1.** *A projective variety $X$ of dimension $|R|$ is $R^p$-rigid if and only if $v_p(n_X) = v_p(\deg u_R^X)$. If $X$ is smooth, it is $R^p$-rigid if and only if $v_p(n_X) = v_p\big(\deg c_R(-T_X)\big)$.*

The following theorem is the main result of the section.

**Theorem 7.2.** *Let $X$ and $Y$ be projective varieties over $F$ and let $R$ be a sequence such that $\dim(X) = |R| > 0$. Suppose that*

(1) *There is a correspondence $\alpha : X \rightsquigarrow Y$ of multiplicity not divisible by $p$;*
(2) *$X$ is $R^p$-rigid;*
(3) *$v_p(n_X) \leq v_p(n_Y)$.*

*Then*

(1) $\dim(X) \leq \dim(Y)$*;*
(2) *If* $\dim(X) = \dim(Y)$*,*
  (2a) *There is a correspondence $\beta : Y \rightsquigarrow X$ of multiplicity not divisible by $p$;*
  (2b) *$Y$ is $R^p$-rigid;*
  (2c) *$v_p(n_X) = v_p(n_Y)$.*

*Proof.* Suppose that $m = \dim(X) - \dim(Y) \geq 0$ and set $Y' = Y \times \mathbb{P}_F^m$. Clearly, $n_{Y'} = n_Y$. We embed $Y$ into $Y'$ as $Y \times z$ where $z$ is a rational point of $\mathbb{P}_F^m$.

We may assume that $\alpha$ is a prime correspondence, replacing if necessary, $\alpha$ by one of its prime components. Let $Z \subset X \times Y$ be the closed subvariety representing $\alpha$. We have two natural morphisms $f : Z \to X$ and $g : Z \to Y \hookrightarrow Y'$. By assumption, $\deg(f)$ is not divisible by $p$.

We write the degree formulas of Theorem 6.4 for the morphisms $f$ and $g$:

$$R^p(Z) = \deg(f) \cdot R^p(X) \in \mathbb{Z}/n_X\mathbb{Z}, \tag{12}$$

$$R^p(Z) = \deg(g) \cdot R^p(Y') \in \mathbb{Z}/n_Y\mathbb{Z}. \tag{13}$$

The variety $X$ is $R^p$-rigid and the degree $\deg(f)$ is not divisible by $p$, hence it follows from (12) that $R^p(Z) \neq 0$ in $\mathbb{Z}/n_X\mathbb{Z}$. Since $v_p(n_X) \leq v_p(n_Y)$ and $p \cdot R^p(Z) = 0$, we have $R^p(Z) \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$ and it follows from the degree formula (13) that $\deg(g)$ is not divisible by $p$, so that $g$ is surjective, and $R^p(Y') \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$. The image of $g$ is contained in $Y$, therefore $Y = Y'$, i.e., $m = 0$ and $\dim(X) = \dim(Y)$.

The variety $Z$ defines a correspondence $\beta : Y \rightsquigarrow X$ of multiplicity $\deg(g)$ not divisible by $p$. Since $R^p(Y) = R^p(Y') \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$, it follows that $Y$ is $R^p$-rigid. Finally, $p \cdot R^p(Z) = 0$ and $R^p(Z)$ is nonzero in both $\mathbb{Z}/n_X\mathbb{Z}$ and $\mathbb{Z}/n_Y\mathbb{Z}$, therefore, we must have $v_p(n_X) = v_p(n_Y)$. □

We say that a variety $X$ over $F$ is *$p$-compressible* if there is a rational morphism $X \to Y$ to a variety $Y$ over $F$ such that $v_p(n_Y) \geq v_p(n_X)$ and $\dim(Y) < \dim(X)$. Since a rational morphism $X \to Y$ gives rise to a correspondence $X \rightsquigarrow Y$ of multiplicity 1, Theorem 7.2 yields:

**Corollary 7.3.** *An $R^p$-rigid variety is not $p$-compressible.*

**Remark 7.4.** The proof of Theorem 7.2 gives a slightly stronger statement than Corollary 7.3. Namely, let $f : X \to Y$ be a surjective morphism of varieties such that $X$ is $R^p$-rigid and $v_p(n_Y) \geq v_p(n_X)$. Then $\dim X = \dim Y$ and $\deg(f)$ is not divisible by $p$.

**Remark 7.5.** Let $X$ be a smooth variety with the function field $L = F(X)$. By Remark 6.6, $n_X = n_L$. If $Y$ is another smooth variety with $F(Y) \simeq L$, the the proof of Theorem 7.2 shows that $R^p(X) = R^p(Y)$ in $\mathbb{Z}/n_L\mathbb{Z}$, i.e., $R^p(X)$ is a birational invariant of a smooth projective variety $X$.

**Remark 7.6.** The number $v_p(n_X)$ is a birational invariant of an $R^p$-rigid projective variety $X$ (not necessarily smooth).

7.1. **Curves.** Let $X$ be a smooth curve over $F$ of characteristic not 2, let $p = 2$ and $R = (1, 0, 0. \dots)$. We have

$$R^p(X) = \frac{\deg c_1(-T_X)}{2} + n_X\mathbb{Z} = (g-1) + n_X\mathbb{Z} \in \mathbb{Z}/n_X\mathbb{Z}$$

where $g$ is the genus of $X$ [3, Ch. 4, Ex. 1.3.3]. The curve $X$ is $R^2$-rigid if and only if $n_X$ and $g$ are even. For example, a conic curve without rational points is $R^2$-rigid.

7.2. **Severi-Brauer varieties.** The class in $K_0(\mathbb{P})$ of the tangent bundle of the projective space $\mathbb{P} = \mathbb{P}^d$ over $F$ is equal to $(d+1)[L] - \mathbf{1}$, where $L$ is the canonical line bundle over $\mathbb{P}$ (with the sheaf of sections $\mathcal{O}(1)$) [3, Ch. 2, Ex. 8.20.1]. Hence, by additivity of the class $c_{(d)}$, we have

$$c_{(d)}(-T) = -(d+1)c_{(d)}(L) = -(d+1)c_1(L)^d = -(d+1)h^d,$$

where $h = c_1(L)$ is the class of a hyperplane in $\mathbb{P}$.

Assume that $d = p^n - 1$, where $p$ is prime such that $p \neq \operatorname{char} F$. Let $X = SB(A)$ be the Severi-Brauer variety of a central division algebra $A$ of index $p^n$. Then $n_X = \operatorname{ind}(A) = p^n$ by Example 6.1. Over an algebraic closure of $F$, the variety $X$ is isomorphic to $\mathbb{P}$ [8, Th. 1.18]. Hence for the sequence $R = (0, \dots, 0, \overset{n}{1}, 0, \dots)$ we have (see Remark 6.3)

$$R^p(X) = \frac{\deg c_{(d)}(-T)}{p} + n_X\mathbb{Z} = -p^{n-1} + n_X\mathbb{Z} \neq 0 \in \mathbb{Z}/n_X\mathbb{Z},$$

Therefore, $SB(A)$ is $R^p$-rigid. In particular, $SB(A)$ is not $p$-compressible.

**Remark 7.7.** If $A$ is not a division algebra or if $\operatorname{ind} A$ is not power of $p$, then one can show that $SB(A)$ is $p$-compressible and hence is not $R^p$-rigid for every $R$.

**Example 7.8.** Let $R = (r_1, r_2, \dots)$ be a sequence and let $p$ be a prime integer different from $\operatorname{char} F$. For every $i = 1, 2, \dots$ and $j = 1, 2, \dots, r_i$ choose a central division $F$-algebra $A_{ij}$ of index $p^i$. Denote by $X$ the product of (finitely many) Severi-Brauer varieties $SB(A_{ij})$. Over an algebraic closure of $F$ the variety $X$ is isomorphic to the product of projective spaces

$$(\mathbb{P}^{p-1})^{r_1} \times (\mathbb{P}^{p^2-1})^{r_2} \times (\mathbb{P}^{p^3-1})^{r_3} \times \cdots .$$

A straightforward computation (similar to one in 7.2) shows that

$$v_p\big(\deg c_R(-T_X)\big) = r_1 + 2r_2 + 3r_3 + \cdots .$$

Write $A$ for the tensor product of all the algebras $A_{ij}$ and set $Y = SB(A)$. Suppose $A$ is a division algebra. By example 6.1,

$$v_p(n_Y) = v_p\big(\operatorname{ind}(A)\big) = r_1 + 2r_2 + 3r_3 + \cdots .$$

The variety $X$ is identified canonically with a closed subvariety of $Y$, therefore, $n_Y$ divides $n_X$. It follows from Proposition 7.1 that $X$ is $R^p$-rigid. Thus, for every $R$ and $p$ there are examples of $R^p$-rigid varieties.

7.3. **Hypersurfaces.** Let $X$ be a smooth hypersurface of prime degree $p \neq \operatorname{char} F$ in the projective space $\mathbb{P} = \mathbb{P}^{d+1}$ over a field $F$. Let $i : X \hookrightarrow \mathbb{P}$ be the closed embedding. The normal bundle of $X$ in $\mathbb{P}$ is isomorphic to $i^*L^{\otimes p}$, where $L$ is the canonical line bundle over $\mathbb{P}$. Hence, there is an exact sequence of vector bundles over $X$:

$$0 \to T_X \to i^*T \to i^*L^{\otimes p} \to 0.$$

By additivity of the class $c_{(d)}$, we have

$$c_{(d)}(-T_X) = i^*\big(c_{(d)}(-T_P) + c_{(d)}(L^{\otimes p})\big) = -(d+2)h^d + p^d h^d = (p^d - d - 2)h^d,$$

where $h$ is the class of a hyperplane section of $X$.

Assume that $d = p^n - 1$ for some $n \geq 1$. Since $\deg(h^d) = p$, we have

$$R^p(X) = \frac{\deg c_{(d)}(-T_X)}{p} + n_X\mathbb{Z} = (p^d - d - 2) + n_X\mathbb{Z} \in \mathbb{Z}/n_X\mathbb{Z}$$

for $R = (0, \dots, 0, \overset{n}{1}, 0, \dots)$. If in addition, $n_X$ is divisible by $p$, we have $R^p(X) \neq 0$ and therefore, the hypersurface $X$ is $R^p$-rigid and not $p$-compressible.

In the case $p = 2$, $X$ is a smooth quadric. If $X$ is anisotropic, then $n_X = 2$ by Example 6.2. Therefore, $X$ is $R^2$-rigid and not 2-compressible.

**Remark 7.9.** Let $X$ be an anisotropic quadric of dimension $d$ corresponding to a Pfister neighbor. Choose an integer $n$ such that $2^n - 1 \leq d < 2^{n+1} - 1$. Let $Y$ be a subquadric in $X$ of dimension $2^n - 1$. It is known that $Y$ is isotropic over the function field $F(X)$. If $d \neq 2^n - 1$, the quadric $X$ can be compressed into $Y$ and therefore, $X$ is not $R^2$-rigid for every $R$.

7.4. **Algebras with involutions.** Let $A$ be a central simple algebra over $F$ and let $\sigma$ be an orthogonal involution on $A$ [8, §2]. We say that $\sigma$ is anisotropic if $\sigma(a)a \neq 0$ for every nonzero $a \in A$. It is conjectured that an anisotropic involution stays anisotropic over the function field of the Severi-Brauer variety $SB(A)$. We give another proof of the following partial result due to N. Karpenko [7, Th. 5.3].

**Proposition 7.10.** *Let $A$ be a division algebra with an orthogonal involution $\sigma$ and let $X$ the Severi-Brauer variety of $A$. Then the involution $\sigma$ is anisotropic over $F(X)$.*

*Proof.* The index of $A$ is a 2-power, hence by 7.2, $X$ is a $R^2$-rigid variety for a certain $R$. Let $Y = I(A, \sigma) \subset X$ be the involution variety (see [13]). The variety $Y$ has an $L$-valued point over a field extension $L/F$ is and only if $A$ is split over $L$ and $\sigma$ is isotropic over $L$.

Since $Y$ is a closed subvariety in $X$, the number $n_Y$ is divisible by $n_X = \mathrm{ind}(A)$. If $\sigma$ is isotropic over the field $F(X)$, the variety $Y$ has an $F(X)$-valued point and therefore, there is a rational morphism $X \to Y$. By Theorem 7.2, $\dim(X) \leq \dim(Y)$, a contradiction. $\square$

7.5. **Quadrics.** We give alternative proves of the following two propositions in algebraic theory of quadratic forms. The number $2^n - 1$ in these statements appears in connection with the Pfister forms (of dimension $2^n$) used in the original proofs. In our approach this number is the degree of a certain Steenrod operation.

**Proposition 7.11.** (Hoffmann [5, Th. 1]) *Let $X_1$ and $X_2$ be anisotropic quadrics. If* $\dim(X_1) \geq 2^n - 1$ *and $X_2$ is isotropic over $F(X_1)$, then* $\dim(X_2) \geq 2^n - 1$.

*Proof.* Let $X_1'$ be a smooth subquadric of $X_1$ of dimension $2^n - 1$. By assumption, $X_2$ is isotropic over $F(X_1)$ and therefore over $F(X_1 \times X_1')$. The quadric $X_1$ is isotropic over $F(X_1')$, hence the field extension $F(X_1 \times X_1')/F(X_1')$ is purely transcendental. It follows that $X_2$ is isotropic over $F(X_1')$, i.e., there is a rational morphism $X_1' \to X_2$. The variety $X_1'$ is $R^2$-rigid for a certain $R$ by 7.3 and $n_{X_1'} = n_{X_2} = 2$ (Example 6.2). Therefore, by Theorem 7.2, $\dim(X_2) \geq \dim(X_1') = 2^n - 1$. □

**Proposition 7.12.** (Izhboldin [6, Th. 0.2]) *Let $X_1$ and $X_2$ be anisotropic quadrics. If $\dim(X_1) \geq 2^n - 1 = \dim(X_2)$ and $X_2$ is isotropic over $F(X_1)$, then $X_1$ is isotropic over $F(X_2)$.*

*Proof.* As in the proof of Proposition 7.11, by Theorem 7.2 applied to the varieties $X_1'$ and $X_2$, there is a correspondence $X_2 \rightsquigarrow X_1'$ of odd multiplicity. In other words, $X_1'$ has a closed point of odd degree over $F(X_2)$. By Springer's theorem, $X_1'$, and therefore, $X_1$ is isotropic over $F(X_2)$. □

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