RATIONALITY PROBLEM FOR CLASSIFYING SPACES OF SPINOR GROUPS

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Abstract. We study stably rationality and retract rationality properties of the classifying spaces of split spinor groups $\mathbf{Spin}_n$ over a field F of characteristic not 2.

1. Introduction

Let G be an algebraic group over a field F, V a generically free representation of G (i.e., the stabilizer of the generic point in V is trivial) and $U \subset V$ a G-invariant open subset such that there is a G-torsor $f : U \longrightarrow U/G$. This is a *versal* G-torsor, i.e., every G-torsor over a field extension K/F with K infinite is isomorphic to the fiber of f over a K-point of U/G. Thus, the K-points of U/G parameterize all G-torsors over $\operatorname{Spec}(K)$.

We think of U/G as an approximation of the classifying space (stack) $\mathrm{B}G$ of all G-torsors. The stable birational and retract rational equivalence classes of U/G are independent of the choice of V and U. We simply say that $\mathrm{B}G$ is *stably rational* (respectively, *retract rational*) if so is U/G. In fact, $\mathrm{B}G$ is retract rational if and only if all the G-torsors over field extensions of F can be parameterized by algebraically independent variables.

We study the classifying spaces of split spinor groups $\mathbf{Spin}_n$ over a field F of characteristic not 2. The $\mathbf{Spin}_n$-torsors over a field extension K/F parameterize nondegenerate quadratic forms of dimension n over K of trivial discriminant and Clifford invariant. If $n \leq 6$, all such forms are isomorphic, hence $\mathrm{B}\,\mathbf{Spin}_n$ is stably rational. We also show that $\mathrm{B}\,\mathbf{Spin}_n$ is stably rational if $n \leq 10$ (at least over $F = \mathbb{C}$) and retract rational if $n \leq 16$.

We prove several reincarnations of the space $\mathrm{B}\,\mathbf{Spin}_n$. We show that $\mathrm{B}\,\mathbf{Spin}_n$ is stably birational to the Severi-Brauer variety over the classifying space $\mathrm{B}\mathbf{O}_n^+$ of the special orthogonal group corresponding to the Azumaya algebra whose class in the Brauer group if the Clifford invariant. As a consequence we show that $\mathrm{B}\,\mathbf{Spin}_n$ is stably birational to $\mathrm{B}\,\mathbf{Spin}_{n-1}$ if n is even. We also prove that $\mathrm{B}\,\mathbf{Spin}_n$ is stably birational to the classifying space of an extraspecial finite group of order 2^n if n is odd and 2^{n-1} if n is even.

We use the following notation.

A *variety* over a field F is an integral separated scheme of finite type over F.

An *algebraic group* over F is an affine group scheme of finite type over F.

$\mathbb{A}^n_F$ the affine space over F.

$\mathbf{G_m} = \mathbb{A}^1_F \setminus \{0\}$ the multiplicative group (torus).

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2. Rational and retract rational varieties

If X and Y are varieties over F, we write $X \approx Y$ if X and Y are *birationally isomorphic*, i.e., the rational function fields $F(X)$ and $F(Y)$ are isomorphic over F and $X \overset{\text{s.b.}}{\approx} Y$ if X and Y are *stably birational*, i.e., $X \times \mathbb{A}^m_F \approx Y \times \mathbb{A}^n_F$ for some m and n.

We say that X is a *rational* variety if $X \approx \mathbb{A}^n_F$ for some n and *stably rational* if $X \overset{\text{s.b.}}{\approx} \mathbb{A}^0_F = \operatorname{Spec} F$.

We will use the following elementary lemma.

Lemma 2.1. *Let $f : Y \to X$ be a morphism of varieties over F. Suppose that for every field extension K/F and every point $x \in X(K)$, the fiber of f over x is a rational variety over K. Then $Y \overset{\text{s.b.}}{\approx} X$.*

Proof. By assumption, the generic fiber Z of the morphism f is a rational variety over the function field $F(X)$. The result follows since $F(Y) \simeq F(X)(Z)$. □

A morphism of varieties $f : Y \to X$ over a field F is called *weakly split* if there is a rational morphism $g : X \dashrightarrow Y$ such that $f \circ g$ is the identity of X. We say that f is *split* if for every nonempty open subset $U \subset Y$ there is a rational morphism $g : X \dashrightarrow Y$ such that $\operatorname{Im}(g) \cap U \neq \emptyset$ and $f \circ g = \mathrm{id}_X$.

A variety X over F is *weakly retract rational* (respectively, *retract rational*) if there is a nonempty open subvariety $Y \subset \mathbb{A}^n_F$ for some n and a weakly split (respectively, split) morphism $f : Y \to X$ over F.

Every stably rational variety is retract rational and hence weakly retract rational (see [11, §2]).

3. Versal torsors and classifying spaces

Let G be an algebraic group over F. A G-torsor $Y \to X$ over a variety X is called *versal* if for every G-torsor $E \to \operatorname{Spec}(K)$ for a field extension K/F with K an infinite field and every nonempty open subset $U \subset X$, there is a point $x \in U(K)$ such that the G-torsor $E \to \operatorname{Spec}(K)$ is isomorphic to the pull-back of $Y \to X$ with respect to x (see [6]). Thus a versal G-torsor $Y \to X$ parameterizes all G-torsors over field extensions K/F by the points of X over K.

Let G be an algebraic group over F, V a generically free representation of G over F. A nonempty G-invariant open subset U of the affine space $\mathbb{A}(V)$ of V such that there exists a G-torsor $U \to U/G$ for a variety U/G over F is called a *friendly open subset* of V or a *friendly G-variety*. Friendly open subset always exist (see [14, Proposition 4.7]) and the torsor $U \to U/G$ is versal (see [6]). It is called a *standard versal G-torsor*.

Example 3.1. Let $G = (\boldsymbol{\mu}_n)^r$ for some n and r, where $\boldsymbol{\mu}_n$ is the group of roots of unity of degree n. Then the natural representation F^r of G is generically free and $(\mathbf{G_m})^r$ is a friendly open subset of $\mathbb{A}^r_F = \mathbb{A}(F^r)$ with the G-torsor $(\mathbf{G_m})^r \to (\mathbf{G_m})^r/G = (\mathbf{G_m})^r$, so $(\mathbf{G_m})^r$ is an approximation of $\mathrm{B}G$. Note that a G-torsor over a field extension K/F is isomorphic to $\operatorname{Spec} K(a_1^{1/n}, a_2^{1/n}, \dots, a_r^{1/n}) \to \operatorname{Spec} K$ for a point $(a_1, a_2, \dots, a_r) \in (\mathbf{G_m})^r(K)$ with $a_i \in K^\times$.

We think of U/G as an "approximation" of the stack $\mathrm{B}G$ of all G-torsors, which we call the *classifying space* of G. The stable birational type of U/G does not depend on the choice of V and U. The space $\mathrm{B}G$ is *retract rational* or *stably rational* if so is U/G.

We say that the G-torsors over field extensions of F are *rationally parameterized* if there is a versal G-torsor $Y \to X$ with X a rational variety. The following statement was proved in [11, Corollary 5.9].

Proposition 3.2. *The G-torsors over field extensions of F are rationally parameterized if and only if the classifying space $\mathrm{B}G$ is retract rational over F.*

Let U be a friendly G-variety and let $H \subset G$ be a subgroup. Then U is a friendly H-variety. We think of the natural morphism $U/H \to U/G$ as an approximation of the morphism $\mathrm{B}H \to \mathrm{B}G$.

Let x be a K-point of U/G for a field extension K/F and J the corresponding G-torsor over K, i.e., J is the inverse image of x under $U \to U/G$. It follows that the fiber of the morphism $U/H \to U/G$ over x is equal to J/H. Lemma 2.1 then yields the following proposition.

Proposition 3.3. *Let G be an algebraic group over F and $H \subset G$ a subgroup. Suppose that for every field extension K/F, and every G-torsor J over K, the variety J/H is rational over K. Then $\mathrm{B}H \overset{\text{s.b.}}{\approx} \mathrm{B}G$.*

Example 3.4. Let G be a reductive algebraic group over F and $T \subset G$ a maximal torus over F. Let N be the normalizer of T in G. For a G-torsor J the group $G^J := \mathrm{Aut}_G(J)$ is the twist of G by J. The morphism from J to the variety $\mathrm{MaxTori}(G^J)$ of maximal tori in G^J taking j in J to the maximal torus of all φ in G^J such that $\varphi(j) \in jT$ yields an isomorphism $J/N \xrightarrow{\sim} \mathrm{MaxTori}(G^J)$ (This is the twist of the isomorphism $G/N \xrightarrow{\sim} \mathrm{MaxTori}(G)$ taking gN to gTg^{-1}.) The variety $\mathrm{MaxTori}(G^J)$ is known to be rational [3, Theorem 7.9]. Hence $\mathrm{B}N \overset{\text{s.b.}}{\approx} \mathrm{B}G$. This was proved in [1, Lemma 2.4] when F is algebraically closed.

4. Quadratic forms

The references for the algebraic theory of quadratic forms are [10], [8] and [5].

Let F be a field of characteristic different from 2 and let $q : V \to F$ be a nondegenerate quadratic form of dimension n over F. In an orthogonal basis of V the form q is diagonal: $q(x) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$ for $a_1, a_2, \ldots, a_n \in F^\times$. We write

$$q = \langle a_1, a_2, \ldots, a_n\rangle. \tag{4.1}$$

The *discriminant* of q is $\mathrm{disc}(q) = (-1)^{n(n-1)/2}a_1a_2\cdots a_n \in F^\times/F^{\times 2}$.

Write $C(q)$ for the *Clifford algebra* of q (of dimension 2^n) and $C_0(q)$ for the *even Clifford algebra*. If n is even (respectively, odd), $C(q)$ (respectively, $C_0(q)$) is a central simple algebra over F.

If n is even and $\mathrm{disc}(q)$ is trivial, $C_0(q)$ is the product of two copies of a central simple algebra $C^+(q)$. If n is odd, we set $C^+(q) := C_0(q)$. Thus, $C^+(q)$ is a central simple algebra over F of degree 2^m, where m is so that

$$n = \begin{cases} 2m+1, & \text{if } n \text{ is odd;} \\ 2m+2, & \text{if } n \text{ is even.} \end{cases}$$

The class of $C^+(q)$ in the Brauer group $\mathrm{Br}(F)$ is the *Clifford invariant* of q.

If R is a commutative ring with $2 \in R^\times$ and $a, b \in R^\times$, we write (a, b) for the *(generalized) quaternion R-algebra* generated by two elements x and y that are subject to the relations $x^2 = a$, $y^2 = b$ and $yx = -xy$. It is an *Azumaya algebra* of rank 4 over R.

If q is a form as in (4.1), the algebra $C^+(q)$ is the tensor product of m quaternion F-algebras:

$$C^+(q) = \begin{cases} (a_1, a_2) \otimes (-a_1a_2a_3, -a_1a_2a_4) \otimes \cdots, & \text{if } n \text{ is odd;} \\ (-a_1a_2, -a_1a_3) \otimes (a_1a_2a_3a_4, a_1a_2a_3a_5) \otimes \cdots, & \text{if } n \text{ is even.} \end{cases} \tag{4.2}$$

We write $\mathbf{O}(q)$ and $\mathbf{O}^+(q)$ for the *orthogonal* and *special orthogonal* groups, respectively. The *even Clifford group* $\mathbf{\Gamma}^+(q)$ is a subgroup of the multiplicative group of the even Clifford algebra $C_0(q)$. For a field extension K/F the group $\mathbf{\Gamma}^+(q)(K)$ of K-points consists of all products of even number of anisotropic vectors in the space $V_K = V \otimes_F K$. The *spinor group* $\mathbf{Spin}(q)$ is the kernel of the *spinor norm* homomorphism $\mathrm{Sn} : \mathbf{\Gamma}^+(q) \to \mathbf{G_m}$ taking $v_1v_2 \cdots v_{2s}$ to the product $q(v_1)q(v_2) \cdots q(v_{2s})$.

Let $q_h = \langle 1, -1, 1, \dots, (-1)^{n-1} \rangle$. This is a *split form*, i.e., a quadratic form of dimension n over F, trivial discriminant and maximal Witt index. The form q_h is hyperbolic if n is even.

We write $\mathbf{O}_n$, $\mathbf{O}_n^+$, $\mathbf{\Gamma}_n^+$ and $\mathbf{Spin}_n$ for $\mathbf{O}(q_h)$, $\mathbf{O}^+(q_h)$, $\mathbf{\Gamma}^+(q_h)$ and $\mathbf{Spin}(q_h)$, respectively. The groups $\mathbf{O}_n^+$ and $\mathbf{Spin}_n$ are split semisimple groups.

By [8, Chaper VII], there are the following bijections:

$$\mathbf{O}_n\text{-torsors over } K \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{Quadratic forms}\\ \text{of dimension } n \text{ over } K\end{array}}$$

$$\mathbf{O}_n^+\text{-torsors over } K \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{Quadratic forms of dimension } n\\ \text{over } K \text{ of trivial discriminant}\end{array}}$$

The connecting map $H^1(K, \mathbf{O}_n^+) \to H^2(K, \mathbf{G_m}) = \mathrm{Br}(K)$ for the exact sequence

$$1 \to \mathbf{G_m} \to \mathbf{\Gamma}_n^+ \xrightarrow{\theta} \mathbf{O}_n^+ \to 1,$$

where θ sends the product $v_1v_2 \cdots v_{2s}$ to the product of reflections with respect to the v_i's, takes a quadratic form q to the Clifford invariant of q. It follows that there is a bijection

$$\mathbf{\Gamma}_n^+\text{-torsors over } K \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{Quadratic forms of dimension } n \text{ over } K\\ \text{of trivial discriminant and Clifford invariant}\end{array}}$$

Quadratic forms of dimension n of trivial discriminant and Clifford invariant are parameterized by independent parameters if $n \leq 14$ (see [12, Theorem 4.4]). In other words, by Proposition 3.2, the space $\mathrm{B}\mathbf{\Gamma}_n^+$ is retract rational if $n \leq 14$.

Lemma 4.3. *Let $1 \to H \to G \to \mathbf{G_m} \to 1$ be an exact sequence. Then* $\mathrm{B}H \overset{\text{s.b.}}{\approx} \mathrm{B}G$.

Proof. The morphism $\mathrm{B}H \to \mathrm{B}G$ is approximated by $f : U/H \to U/G$ for a friendly G-variety U. The morphism f is a $\mathbf{G_m}$-torsor, hence it is generically split. Thus $\mathrm{B}H \approx \mathrm{B}G \times \mathbf{G_m} \overset{\text{s.b.}}{\approx} \mathrm{B}G$. □

Corollary 4.4. *The spaces* $\mathrm{B}\,\mathbf{Spin}_n$ *and* $\mathrm{B}\boldsymbol{\Gamma}_n^+$ *are stably birational. In particular,* $\mathrm{B}\,\mathbf{Spin}_n$ *is retract rational if* $n \leq 14$.

Proof. Apply the Lemma 4.3 to the exact sequence

$$1 \to \mathbf{Spin}_n \to \boldsymbol{\Gamma}_n^+ \xrightarrow{\mathrm{Sn}} \mathbf{G_m} \to 1.$$ □

Remark 4.5. It is proved in [12, Theorem 4.4] (see also [4, Theorem 4.15]) that $\mathrm{B}\,\mathbf{Spin}_n$ is weakly retract rational if $n \leq 14$.

Every quadratic form of trivial discriminant and Clifford invariant of dimension at most 6 is split, i.e., the group $\mathrm{B}\boldsymbol{\Gamma}_n^+$ is special. This implies that the spaces $\mathrm{B}\boldsymbol{\Gamma}_n^+$ and $\mathrm{B}\,\mathbf{Spin}_n$ are stably rational if $n \leq 6$. We will see (Remark 5.8) that this is actually true for $n \leq 10$ if $F = \mathbb{C}$.

Let $q : V \to F$ be a nondegenerate quadratic form of dimension n over F of characteristic not 2. An *orthogonal decomposition* of V is a tuple $L = (L_1, L_2, \dots, L_n)$ of 1-dimensional subspaces of V such that $V = L_1 \perp L_2 \perp \cdots \perp L_n$. Orthogonal decompositions of V form a variety $\mathrm{Orth}(q)$ over F. Every orthogonal decomposition L yields a full flag of subspaces $V_i = L_1 \perp \cdots \perp L_i$ of V. Conversely, every full flag (V_i) of subspaces of V such that the restriction of q to every V_i is nondegenerate, yields an orthogonal decomposition L with L_i the orthogonal complement of V_{i-1} in V_i. It follows that the variety $\mathrm{Orth}(q)$ is birational to the full flag variety of V and therefore, $\mathrm{Orth}(q)$ is a rational variety.

Choose an orthogonal decomposition L and consider the subgroup $H(q)$ of all elements in $\mathbf{O}^+(q)$ fixing L. Thus, $H(q)$ is a finite subgroup of $\mathbf{O}^+(q)$ of order 2^{n-1} acting by ± 1 on each L_i. As $\mathbf{O}^+(q)$ act transitively on $\mathrm{Orth}(q)$, we have $\mathrm{Orth}(q) \simeq \mathbf{O}^+(q)/H(q)$.

The group $H(q)$ is canonically isomorphic to the kernel of the product homomorphism $(\boldsymbol{\mu}_2)^n \to \boldsymbol{\mu}_2$. An $H(q)$-torsor over F is given by a tuple $a = (a_1, a_2, \dots, a_n)$ of elements in $F^\times$ with trivial product. The embedding of $H(q)$ into $\mathbf{O}^+(q)$ induces a map taking an n-tuple a to the quadratic form $\perp_{i=1}^n a_i(q|_{L_i})$.

An $\mathbf{O}^+(q)$-torsor J over a field extension K/F is the variety of isomorphisms between q_K and a quadratic form q' over K of the same dimension and discriminant as q_K. Moreover, the variety $J/H(q)$ is isomorphic to $\mathbf{O}^+(q')/H(q') = \mathrm{Orth}(q')$. Hence the fiber of the natural morphism

$$\mathrm{B}H(q) \to \mathrm{B}\mathbf{O}^+(q)$$

over q' is isomorphic to $\mathrm{Orth}(q')$, and therefore, is a rational variety. We have proved the following lemma.

Lemma 4.6. *For every* $\mathbf{O}^+(q)$*-torsor* J *over a field extension* K/F*, the variety* $J/H(q)$ *is rational over* K*. Each fiber of the natural morphism* $\mathrm{B}H(q) \to \mathrm{B}\mathbf{O}^+(q)$ *is a rational variety.*

5. Severi-Brauer varieties

Let $\mathcal{A}$ be an Azumaya algebra of degree n over a variety X over F and let $\mathrm{SB}(X, \mathcal{A})$ be the Severi-Brauer variety over X (see [7]). By definition $\mathrm{SB}(X, \mathcal{A})$ is an X-scheme locally isomorphic to the projective space $\mathbb{P}^{n-1}$ for the étale topology on X. The fiber over a point $x \in X(K)$ is the variety of right ideals of dimension n in the central simple K-algebra $\mathcal{A}(x)$.

Suppose we have an exact sequence

$$1 \to \boldsymbol{\mu} \xrightarrow{i} G \to N \to 1,$$

where $\boldsymbol{\mu}$ is subgroup of $\mathbf{G_m}$ (thus, $\boldsymbol{\mu} = \mathbf{G_m}$ or $\boldsymbol{\mu}_n$ for some n) and a representation $\rho : G \to \mathbf{GL}(V)$ such that the composition $\rho \circ i$ coincides with the natural embedding $\boldsymbol{\mu} \hookrightarrow \mathbf{G_m} \hookrightarrow \mathbf{GL}(V)$. We then have an induced homomorphism $N \to \mathbf{PGL}(V)$.

An N-torsor J over a variety X yields then an Azumaya algebra

$$\mathcal{A} := (\mathrm{End}(V) \times J)/N$$

over X that is the twist of $\mathcal{A}$ by J. The twist

$$\mathbb{P}(V)^J := (\mathbb{P}(V) \times J)/N$$

of the projective space $\mathbb{P}(V)$ is the Severi-Brauer variety $\mathrm{SB}(X, \mathcal{A})$ over X.

Let W be a generically free representation of N and $U \subset W$ a friendly open subset. Then the twist by the standard versal N-torsor $U \to U/N$ yields an Azumaya algebra $\mathcal{A}$ over the approximation U/N of $\mathrm{B}N$ and a Severi-Brauer variety

$$(\mathbb{P}(V) \times U)/N$$

over U/N which we denote by $\mathrm{SB}(\mathrm{B}N, \mathcal{A})$. The stable birational type of $\mathrm{SB}(\mathrm{B}N, \mathcal{A})$ does not depend on the choice of W and U.

Proposition 5.1. *The classifying space* $\mathrm{B}G$ *is stably birational to* $\mathrm{SB}(\mathrm{B}N, \mathcal{A})$.

Proof. Let $\widetilde{G} := (\mathbf{G_m} \times G)/\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is embedded into $\mathbf{G_m} \times G$ via $s \mapsto (s, \rho(s^{-1}))$. The representation ρ extends to a homomorphism $\tilde{\rho} : \widetilde{G} \to \mathbf{GL}(V)$. Moreover, $\mathbf{G_m}$ is a subgroup of $\widetilde{G}$ and $\widetilde{G}/\mathbf{G_m} \simeq N$. Then

$$\mathrm{SB}(\mathrm{B}N, \mathcal{A}) = (\mathbb{P}(V) \times U)/N = ((V \setminus 0) \times U)/\widetilde{G}$$

and $(V \setminus 0) \times U$ is a friendly open subset in $V \oplus W$ for the group $\widetilde{G}$, i.e., $((V \setminus 0) \times U)/\widetilde{G}$ is an approximation of $\mathrm{B}\widetilde{G}$, hence $\mathrm{SB}(\mathrm{B}N, \mathcal{A}) \overset{\text{s.b.}}{\approx} \mathrm{B}\widetilde{G}$.

On the other hand, G is a subgroup of $\widetilde{G}$ and the group $\widetilde{G}/G \simeq \mathbf{G_m}/\boldsymbol{\mu}$ is either trivial or isomorphic to $\mathbf{G_m}$. Therefore, $\mathrm{B}G \overset{\text{s.b.}}{\approx} \mathrm{B}\widetilde{G}$ by Lemma 4.3. □

Consider the exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}_n \to \mathbf{O}_n^+ \to 1$$

and a (half-)spin representation $\mathbf{Spin}_n \to \mathbf{GL}_1(C_n^+) = \mathbf{GL}_{2^m}$. We have then a projective representation $\mathbf{O}_n^+ \to \mathbf{PGL}_{2^m}$ and the associated Azumaya algebra $\mathcal{C}_n^+$ over $\mathrm{B}\mathbf{O}_n^+$. The fiber of $\mathcal{C}_n^+$ over a quadratic form q of trivial discriminant (that is an $\mathbf{O}_n^+$-torsor) is the algebra $C^+(q)$.

By Proposition 5.1,

$$\mathrm{B}\,\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}\mathbf{O}_n^+, \mathcal{C}_n^+\big). \tag{5.2}$$

Let $q_h = \langle 1, -1, 1, \ldots, (-1)^{n-1} \rangle$ be a split quadratic form of dimension n and $H_n := H(q_h)$ the finite subgroup of $\mathbf{O}_n^+$ defined in Section 4 for the standard orthogonal basis. Write $\mathcal{B}_n$ for the pull-back of $\mathcal{C}_n^+$ under the morphism $\mathrm{B}H_n \to \mathrm{B}\mathbf{O}_n^+$. If $b = (b_1, b_2, \ldots, b_n)$ is a tuple representing an H_n-torsor, then the fiber of $\mathcal{B}_n$ over b is the algebra $C_n^+(q(b))$, where

$$q(b) = \langle b_1, -b_2, b_3, \ldots, (-1)^{n-1} b_n \rangle \tag{5.3}$$

We have a pull-back diagram

$$\begin{tikzcd}
\mathrm{SB}(\mathrm{B}H_n, \mathcal{B}_n) \arrow[r] \arrow[d] & \mathrm{SB}(\mathrm{B}\mathbf{O}_n^+, \mathcal{C}_n^+) \arrow[d] \\
\mathrm{B}H_n \arrow[r] & \mathrm{B}\mathbf{O}_n^+ .
\end{tikzcd}$$

The fiber of the top morphism over a K-point x of $\mathrm{SB}(\mathrm{B}\mathbf{O}_n^+, \mathcal{C}_n^+)$ is naturally isomorphic to the fiber of the bottom map over the image of x in $\mathrm{B}\mathbf{O}_n^+(K)$. By Lemma 4.6, all such fibers are rational varieties. In view of Lemma 2.1 and equation (5.2),

$$\mathrm{B}\,\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}\mathbf{O}_n^+, \mathcal{C}_n^+\big) \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}H_n, \mathcal{B}_n\big). \tag{5.4}$$

Write as above, $n = 2m + 1$ if n is odd and $n = 2m + 2$ if n is even. Consider the torus $(\mathbf{G_m})^{2m}$ with coordinates $x_1, \ldots, x_m, y_1, \ldots, y_m$ as an approximation of $\mathrm{B}(\boldsymbol{\mu}_2)^{2m}$ (see Example 3.1). Write $\mathcal{A}_m$ for the tensor product

$$(x_1, y_1) \otimes (x_2, y_2) \ldots \otimes (x_m, y_m)$$

of m quaternion algebras over the Laurent polynomial algebra

$$F[(\mathbf{G_m})^{2m}] = F[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}].$$

Thus $\mathcal{A}_m$ is an Azumaya algebra over $\mathrm{B}\,(\boldsymbol{\mu}_2)^{2m}$.

The kernel T of the product homomorphism $(\mathbf{G_m})^n \to \mathbf{G_m}$ is an approximation of the classifying space $\mathrm{B}H_n$. If $b = (b_1, b_2, \ldots, b_n)$ is a point of T, it follows from (4.2) and (5.3) that in the case n is odd, we have

$$\mathcal{B}(b) = C^+(q_b) = (b_1, -b_2) \otimes (b_1 b_2 b_3, -b_1 b_2 b_4) \otimes \cdots$$

is a tensor product of m quaternion algebras. The isomorphism between T and $(\mathbf{G_m})^{2m}$ defined by $x_1 = b_1, y_1 = -b_2, x_2 = b_1 b_2 b_3, y_2 = -b_1 b_2 b_4, \ldots$ takes the algebra $\mathcal{B}_n$ to $\mathcal{A}_m$.

If n is even, we have

$$C^+(q) = (b_1 b_2, -b_1 b_3) \otimes (b_1 b_2 b_3 b_4, -b_1 b_2 b_3 b_5) \otimes \cdots$$

is a tensor product of m quaternion algebras. The isomorphism between T and $\mathbf{G_m} \times (\mathbf{G_m})^{2m}$ (with coordinates t, x_i and y_i) defined by $t = b_1, x_1 = b_1 b_2, y_1 = -b_1 b_3, x_2 = b_1 b_2 b_3 b_4, y_2 = -b_1 b_2 b_3 b_5, \ldots$ takes the algebra $\mathcal{B}_n$ to the pull-back of $\mathcal{A}_m$ with respect to the projection $\mathbf{G_m} \times (\mathbf{G_m})^{2m} \to (\mathbf{G_m})^{2m}$.

We have shown that in eather case,

$$\mathrm{SB}\big(\mathrm{B}H_n, \mathcal{B}_n\big) \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}(\boldsymbol{\mu}_2)^{2m}, \mathcal{A}_m\big).$$

It follows from (5.4) that

$$\mathrm{B}\,\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}(\boldsymbol{\mu}_2)^{2m}, \mathcal{A}_m\big) \overset{\text{s.b.}}{\approx} \mathrm{SB}\big((\mathbf{G_m})^{2m}, \mathcal{A}_m\big). \tag{5.5}$$

We have proved the following theorem.

Theorem 5.6. *Let $n = 2m+1$ or $n = 2m+2$ for some m. Let $\mathcal{A}_m$ be the tensor product $(x_1, y_1) \otimes (x_2, y_2) \ldots \otimes (x_m, y_m)$ of m quaternion Azumaya algebras over $(\mathbf{G_m})^{2m} = \operatorname{Spec} F[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$. Then*

$$\mathrm{B}\,\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{SB}\big((\mathbf{G_m})^{2m}, \mathcal{A}_m\big).$$

Corollary 5.7. *The classifying spaces $\mathrm{B}\,\mathbf{Spin}_{2m+1}$ and $\mathrm{B}\,\mathbf{Spin}_{2m+2}$ are stably birational.*

Corollary 5.8. *If $F = \mathbb{C}$, the classifying space $\mathrm{B}\mathbf{Spin}_n$ is stably rational for $n \leq 10$.*

Proof. We have noticed that $\mathrm{B}\,\mathbf{Spin}_n$ is stably rational for $n \leq 6$ (over any field). It is proved in [9] that $\mathrm{B}\,\mathbf{Spin}_n$ is stably rational for $n = 7$ and $n = 10$ if $F = \mathbb{C}$. □

Let A_m be the pull-back of $\mathcal{A}_m$ to the generic point of $(\mathbf{G_m})^{2m}$, i.e, A_m is the tensor product of quaternion algebras (x_i, y_i) over the field of rational functions $K = F(x, y)$. The reduced norm map $\mathrm{Nrd}_m : A_m \to K$ for the algebra A_m is given by a polynomial in $2^{2m} + 2m$ variables: 2^{2m} coordinate functions on A_m (in some basis for A_m) and $x_1, \ldots, x_m, y_1, \ldots, y_m$. This polynomial is homogeneous of degree 2^m in the first set of 2^{2m} variables. By [13, Theorem 4.2], the Severi-Brauer variety is stably birational to the hypersurface given by the reduced norm polynomial.

Corollary 5.9. *The classifying space $\mathrm{B}\,\mathbf{Spin}_n$ is stably birational to the hypersurface in the affine space $\mathbb{A}^{2^{2m}+2m}$ given by the equation $\mathrm{Nrd}_m = 0$.*

6. Comparison with the classifying space of finite groups

Suppose a quadratic form $q : V \to F$ over a field F with $\mathrm{char}(F) \neq 2$ admits an orthogonal basis $v_1, v_2, \ldots, v_n$ such that $q(v_i) = 1$ for all i, i.e., q is the sum of squares in that basis. Consider the subgroup $H(q)$ corresponding to the orthogonal decomposition of V into orthogonal sum of the subspaces Fv_i (see Section 4). Write D_n for the pre-image of $H(q)$ under the natural homomorphism $\mathbf{Spin}(q) \to \mathbf{O}_n^+$ with kernel $\boldsymbol{\mu}_2$.

Since the group $H(q)$ consists of all products of even number of reflections with respect to the vectors v_i, the group D_n consists of all products $\pm v_{i_1} v_{i_2} \cdots v_{i_k}$ in the Clifford algebra of q with $i_1 < i_2 < \cdots < i_k \leq n$ and k even. In particular, D_n is a finite constant group of order 2^n.

The group D_n is generated by the following elements:

$$c := -1, \; x_i := v_0 v_i \in D_n \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

We have the following relations:

$$c^2 = [c, x_i] = 1 \quad \text{and} \quad x_i^2 = [x_i, x_j] = c \quad \text{for all} \quad i \neq j.$$

Thus, D_n is a central extension of an elementary abelian 2-group of order 2^{n-1} generated by the cosets of the x_i's by the cyclic subgroup of order 2 generated by c.

Theorem 6.1. *Let q be the sum of n squares over a field F of characteristic different from 2. Then $\mathbf{Spin}(q) \overset{\text{s.b.}}{\approx} \mathrm{B}D_n$. If -1 is a square in F, then $\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{B}D_n$.*

Proof. Let I be a $\mathbf{Spin}(q)$-torsor over a field extension K/F. Write J for the push-forward of I with respect to the natural homomorphism $\mathbf{Spin}(q) \to \mathbf{O}^+(q)$, i.e., $J = I/\boldsymbol{\mu}_2$. Thus, J is an $\mathbf{O}^+(q)$-torsor over K. By Lemma 4.6, the variety

$$I/D_n \simeq J/H_n$$

is rational. It follows from Proposition 3.3 applied to the subgroup D_n of $\mathbf{Spin}(q)$ that $\mathbf{Spin}(q) \overset{\text{s.b.}}{\approx} \mathrm{B}D_n$. If -1 is a square in F, the form q is split and $\mathbf{Spin}(q) = \mathbf{Spin}_n$. □

If $n = 2m + 1$, the center $C = \{1, c\}$ of D_n is cyclic of order 2 and the factor group G/C is elementary abelian of order 2^{2m}, hence D_n is an extraspecial 2-group. Corollary 5.7 yields the following statement.

Corollary 6.2. *Let -1 be a square in F and let $n = 2m + 1$ or $n = 2m + 2$ for some m. Then $\mathrm{B}\,\mathbf{Spin}_n$ is stably birational to the classifying space $\mathrm{B}D_{2m+1}$ of the extraspecial 2-group D_{2m+1}. In particular, $\mathrm{B}D_{2m+1}$ is retract rational for $m \leq 6$.*

Remark 6.3. There are exactly two extraspecial 2-groups of order 2^{2m+1} up to isomorphism. It is proved in [2] that their classifying spaces are stably birational if F is algebraically closed field of characteristic zero. In fact, it is sufficient to assume that -1 is a square in F.

7. More on quaternion algebras

In this section we prove that the classifying space of $\mathbf{Spin}_n$ is stably birational to that of a certain semisimple group of type A.

The center of $(\mathbf{SL}_2)^m$ is the group $(\boldsymbol{\mu}_2)^m$. Let C_m be the kernel of the product homomorphism $(\boldsymbol{\mu}_2)^m \to \boldsymbol{\mu}_2$. Write S_m for the factor group $(\mathbf{SL}_2)^m/C_m$, thus, we have an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to S_m \to (\mathbf{PGL}_2)^m \to 1.$$

The mth tensor power $(\mathbf{SL}_2)^m \to \mathbf{GL}_{2^m}$ of the tautological representation of $\mathbf{SL}_2$ yields a representation $S_m \to \mathbf{GL}_{2^m}$ and a homomorphism $(\mathbf{PGL}_2)^m \to \mathbf{PGL}_{2^m}$. The associated Azumaya algebra $\mathcal{D}_m$ on $\mathrm{B}(\mathbf{PGL}_2)^m$ is the tensor product $Q_1 \otimes \cdots \otimes Q_m$, where Q_i is the tautological quaternion Azumaya algebra over the ith factor $\mathrm{B}\mathbf{PGL}_2$ of $\mathrm{B}(\mathbf{PGL}_2)^m$.

By Proposition 5.1,

$$\mathrm{B}S_m \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}(\mathbf{PGL}_2)^m, \mathcal{D}_m\big) \tag{7.1}$$

Consider the composition

$$(\boldsymbol{\mu}_2)^2 \simeq H_3 \hookrightarrow \mathbf{O}_3^+ \simeq \mathbf{PGL}_2,$$

where the first isomorphism takes (x, y) to (x, xy, y).

By Lemma 4.6, every fiber of $\mathrm{B}(\boldsymbol{\mu}_2)^2 \to \mathrm{B}\mathbf{PGL}_2$ is a rational variety. In fact, this map takes a pair $\{a, b\}$ of elements in $K^\times$ to the quaternion algebra (a, b) over K. The restriction of the algebra $\mathcal{D}_m$ under the map $\mathrm{B}(\boldsymbol{\mu}_2)^{2m} \to \mathrm{B}(\mathbf{PGL}_2)^m$ is the algebra $\mathcal{A}_m$ defined in the previous section. Therefore, the fiber of the natural morphism

$$\mathrm{SB}\big(\mathrm{B}(\boldsymbol{\mu}_2)^{2m}, \mathcal{A}_m\big) \to \mathrm{SB}\big(\mathrm{B}(\mathbf{PGL}_2)^m, \mathcal{D}_m\big)$$

is a rational variety. By Lemma 2.1,

$$\mathrm{SB}\big(\mathrm{B}(\boldsymbol{\mu}_2)^{2m}, \mathcal{A}_m\big) \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}(\mathbf{PGL}_2)^m, \mathcal{D}_m\big).$$

It follows from (7.1) that

$$\mathrm{B}S_m \overset{\text{s.b.}}{\approx} \mathrm{SB}\big(\mathrm{B}(\boldsymbol{\mu}_2)^{2m}, \mathcal{A}_m\big). \tag{7.2}$$

Then (5.5) and (7.2) yield the following:

Theorem 7.3. *Let $n = 2m+1$ or $n = 2m+2$ for some m. Let S_m be the factor group $(\mathbf{SL}_2)^m/C_m$, where C_m is the kernel of the product homomorphism $(\boldsymbol{\mu}_2)^m \to \boldsymbol{\mu}_2$. Then*

$$\mathrm{B}\,\mathbf{Spin}_n \overset{\text{s.b.}}{\approx} \mathrm{B}S_m.$$

Note that the group S_m is a semisimple group of type $\mathsf{A}_1 + \cdots + \mathsf{A}_1$ (m times) and $\mathbf{Spin}_n$ is a simply connected semisimple group of type B_m if n is odd and of type D_{m+1} if n is even.

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