CLASSIFICATION OF SPECIAL REDUCTIVE GROUPS

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ABSTRACT. We give a classification of special reductive groups over arbitrary fields that improves a theorem of M. Huruguen.

1. Introduction

An algebraic group G over a field F is called *special* if for every field extension K/F all G-torsors over K are trivial. Examples of special linear groups include:

1. The general linear group GL_n, and more generally the group $\mathrm{GL}_1(A)$ of invertible elements in a central simple F-algebra A;

2. The special linear group SL_n and the symplectic group Sp_{2n};

3. Quasi-trivial tori, and more generally invertible tori (direct factors of quasi-trivial tori).

4. If L/F is a finite separable field extension and G is a special group over L, then the Weil restriction $R_{L/F}(G)$ is a special group over F.

A. Grothendieck proved in [3] that a reductive group G over an algebraically closed field is special if and only if the derived subgroup of G is isomorphic to the product of special linear groups and symplectic groups.

In [4] M. Huruguen proved the following theorem.

Theorem. *Let G be a reductive group over a field F. Then G is special if and only if the following three condition hold:*

(1) *The derived subgroup G' of G is isomorphic to*

$$R_{L/F}(\mathrm{SL}_1(A)) \times R_{K/F}(\mathrm{Sp}(h))$$

where L and K are étale F-algebras, A an Azumaya algebra over L and h an alternating non-degenerate form over K.

(2) *The coradical G/G' of G is an invertible torus.*

(3) *For every field extension K of F, the abelian group $\mathfrak{S}(K;G)$ is trivial.*

The group $\mathfrak{S}(K;G)$ is a certain factor group of the group of isomorphism classes of Z'-torsors over $\operatorname{Spec} K$, where Z' is the center of G'. Unfortunately, as noticed in [4], condition (3) (which is in fact infinitely many conditions for all field extension K/F) is not easy to check in general.

In the present paper we replace condition (3) by solvability of a system of congruences over $\mathbb{Z}$ involving numerical (discrete) invariants of the reductive group G (Theorem 4.1), a condition that is relatively easy to check.

We use the following notation.

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F is the base field, F_{sep} a separable closure of F, $\Gamma = \Gamma_F := \mathrm{Gal}(F_{\mathrm{sep}}/F)$ the *absolute Galois group of* F;

$\mathbb{G}_m = \operatorname{Spec} F[t, t^{-1}]$ multiplicative group.

If G is an *algebraic group* (a group scheme of finite type) over F, and K/F is a field extension, we write $\mathrm{Tors}_G(K)$ for the pointed set of isomorphism classes of G-torsors over $\operatorname{Spec} K$. If G is commutative, $\mathrm{Tors}_G(K)$ is an abelian group.

If G is a reductive group over F, we let G' denote the *derived group* of G that is a semisimple group over F. The factor group G/G' is a torus that is called the *coradical* of G.

If X is a scheme over F and K/F is a field extension, we set $X_K := X \times_F \operatorname{Spec} K$. We also write X_{sep} for $X_{F_{\mathrm{sep}}}$.

2. Preliminary results

2.1. **Γ-lattices.** In this section, Γ is an arbitrary profinite group. A Γ-*lattice* is a free abelian group N of finite rank with a continuous Γ-action by group automorphisms. We write N^Γ for the subgroup of Γ-invariant elements in N.

The *dual lattice* $N^\vee$ is defined as $\mathrm{Hom}(N, \mathbb{Z})$ with the Γ-action given by $(\gamma f)(n) = f(\gamma^{-1} n)$ for $f \in N^\vee$. The pairing

$$N^\vee \otimes N \to \mathbb{Z}, \quad f \otimes n \mapsto \langle f, n\rangle := f(n)$$

is Γ-equivariant: $\langle \gamma f, \gamma n\rangle = \langle f, n\rangle$.

Let X be a finite Γ-set (with a continuous Γ-action). The free abelian group $\mathbb{Z}[X]$ with basis X is a Γ-lattice. A Γ-lattice N is *permutation* if N admits a Γ-invariant $\mathbb{Z}$-basis X, i.e., $N \simeq \mathbb{Z}[X]$.

The Γ-invariant bilinear form B on $\mathbb{Z}[X]$ defined by $B(x, x') = \delta_{x,x'}$ for $x, x' \in X$ yields a canonical isomorphism between the Γ-lattice $\mathbb{Z}[X]$ and its dual.

If N is a Γ-lattice and X a finite Γ-set, write $N[X]$ for the Γ-lattice $N \otimes_{\mathbb{Z}} \mathbb{Z}[X]$. An element $n = \sum_{x\in X} n_x \otimes x$ in $N[X]$ is Γ-invariant if and only if $\gamma n_x = n_{\gamma x}$ for all $\gamma \in \Gamma$ and $x \in X$. In particular, $n_x \in N^{\Gamma_x}$ where $\Gamma_x \subset \Gamma$ is the stabilizer of x.

Let X be the disjoint union of the Γ-orbits $X_1, X_2, \dots, X_s$. Choose representatives $x_i \in X_i$ and let $\Gamma_i \subset \Gamma$ be the stabilizer of x_i. Then the collection $(n_x)_{x\in X}$ such that $n \in N[X]^\Gamma$ is uniquely determined by n_{x_i} for $i = 1, 2, \dots, s$ which can be arbitrary elements in N^{Γ_i} respectively. This establishes a group isomorphism

$$N[X]^\Gamma \simeq N^{\Gamma_1} \oplus N^{\Gamma_2} \oplus \cdots \oplus N^{\Gamma_s}.$$

2.2. **Étale algebras.** Let L be an étale algebra over a field F. Write X for the finite Γ-set of all F-algebra homomorphisms $L \to F_{\mathrm{sep}}$. Note that L can be reconstructed from X as the F-algebra of all Γ-equivariant maps $X \to F_{\mathrm{sep}}$. The correspondence $L \leftrightarrow X$ extends to an anti-equivalence between the category of étale F-algebras and the category of finite Γ-sets.

Write a finite Γ-set X as the disjoint union of Γ-orbits $X_1, X_2, \dots, X_s$. The corresponding étale F-algebra L is the product $L_1 \times L_2 \times \cdots \times L_s$ of finite separable field extensions L_i/F such that $L_i \leftrightarrow X_i$.

If $L \leftrightarrow X$, then the F_{sep}-algebra $L \otimes_F F_{\text{sep}}$ is isomorphic to the F_{sep}-algebra $(F_{\text{sep}})^X$ of all maps $X \to F_{\text{sep}}$. It follows that the Γ-modules $K_0(L \otimes_F F_{\text{sep}})$ and the permutation Γ-module $\mathbb{Z}[X]$ are naturally isomorphic. Taking Γ-invariant elements we get an isomorphism $K_0(L) \simeq \mathbb{Z}[X]^\Gamma$.

We write $\Delta_L \in K_0(L \otimes_F L)$ for the class of L viewed as an $L \otimes_F L$-module via the product homomorphism $L \otimes_F L \to L$. Under the isomorphism $K_0(L \otimes_F L) \simeq \mathbb{Z}[X \times X]^\Gamma$ the element Δ_L corresponds to the diagonal element $\Delta_X = \sum_{x \in X}(x, x)$.

2.3. **Algebraic tori.** Let T be an algebraic torus over F. The *character* group

$$T^* := \operatorname{Hom}_{F_{\text{sep}}}(T_{\text{sep}}, (\mathbb{G}_m)_{\text{sep}})$$

is a Γ-lattice of rank $\dim(T)$. The dual group $T_* := (T^*)^\vee$ is the *co-character* Γ-lattice of all homomorphisms $(\mathbb{G}_m)_{\text{sep}} \to T_{\text{sep}}$ over F_{sep}.

The torus T can be reconstructed from the character Γ-lattice T^* as follows:

$$T = \operatorname{Spec}\bigl(F_{\text{sep}}[T^*]^\Gamma\bigr).$$

The correspondence $T \leftrightarrow T^*$ extends to an anti-equivalence between the category of algebraic tori over F and the category of Γ-lattices.

If R is a commutative F-algebra, then the group of R-points of a torus T is equal to

$$T(R) = \operatorname{Hom}_\Gamma\bigl(T^*, (R \otimes_F F_{\text{sep}})^\times\bigr) = \bigl(T_* \otimes_{\mathbb{Z}} (R \otimes_F F_{\text{sep}})^\times\bigr)^\Gamma.$$

Viewing every character in T^* as an invertible function on T_{sep} yields a Γ-equivariant embedding of T^* into $F_{\text{sep}}[T]^\times \subset F_{\text{sep}}(T)^\times = (F(T) \otimes_F F_{\text{sep}})^\times$ that represents the *generic point* of T in $T(F(T))$ over the function field $F(T)$.

A torus P is called *quasi-trivial* if P^* is a permutation Γ-lattice. A quasi-trivial torus is isomorphic to the Weil restriction $R_{L/F}(\mathbb{G}_{m,L})$, where L is the étale F-algebra corresponding to a Γ-invariant basis X of P^*. According to Section (2.1), the co-character Γ-lattice $P_* = (P^*)^\vee$ is also isomorphic to $\mathbb{Z}[X]$.

Let P be a quasi-trivial torus with $P^* = \mathbb{Z}[X]$ for a Γ-set X. The function field $F_{\text{sep}}(P)$ is the purely transcendental extension $F_{\text{sep}}(X)$ of F_{sep} in the independent variables from X. For every variable $x \in X$, the discrete x-adic valuation v_x on the field $F_{\text{sep}}(P) = F_{\text{sep}}(X)$ gives a group homomorphism $F_{\text{sep}}(P)^\times \to \mathbb{Z}$. All the valuations v_x for $x \in X$ yield a Γ-equivariant homomorphism $F_{\text{sep}}(P)^\times \to \mathbb{Z}[X] = P^*$ that splits the embedding of P^* into $F_{\text{sep}}(P)^\times$. Taking Γ-invariant elements, we get a factorization

$$K_0(L) = (P^*)^\Gamma \to F(P)^\times \to (P^*)^\Gamma = K_0(L)$$

of the identity of $K_0(L)$.

2.4. **Reduced norm.** Let A be a central simple algebra over F. The *degree* $\deg(A)$ of A is the square root of $\dim_F(A)$. By Wedderburn's theorem, $A \simeq M_k(D)$ for a central division algebra D over F. The *index* $\operatorname{ind}(A)$ of A is the degree of D. Thus, $\deg(A) = k \cdot \operatorname{ind}(A)$.

The exact forgetful functor *A-mod* $\to$ *F-mod* yields the *norm homomorphism*

$$N_i^{A/F} : K_i(A) \to K_i(F)$$

on K-groups. The group $K_0(A)$ is infinite cyclic generated by the class of the irreducible left A-module $D^k := D \oplus \cdots \oplus D$ (k times). Moreover,

$$N_0^{A/F}([D^k]) = \dim_F(D^k) = \deg(A) \cdot \operatorname{ind}(A) \in \mathbb{Z} = K_0(F).$$

Let $\operatorname{Nrd}_0^A : K_0(A) \to K_0(F) = \mathbb{Z}$ be the homomorphism taking $[D^k]$ to $\operatorname{ind}(A)$. Thus, $N_0^{A/F} = \deg(A) \cdot \operatorname{Nrd}_0^A$ and $\operatorname{Im}(\operatorname{Nrd}_0^A) = \operatorname{ind}(A) \cdot K_0(F)$.

More generally, let L be an étale F-algebra and A an Azumaya algebra over L. Write $L = L_1 \times L_2 \times \cdots \times L_s$, where L_i are fields and $A = A_1 \times A_2 \times \cdots \times A_s$, where each A_i is a central simple algebra over L_i. The component-wise maps Nrd_0 yield a homomorphism $\operatorname{Nrd}_0^A : K_0(A) \to K_0(L)$.

If A is a central simple algebra over F, denote by $\operatorname{Nrd}^A : K_1(A) \to K_1(F) = F^\times$ the *reduced norm homomorphism* satisfying $N_1^{A/F} = (\operatorname{Nrd}^A)^n$ (see [2]). More generally, if A is an Azumaya algebra over étale F-algebra L, we have a well defined homomorphism

$$\operatorname{Nrd}^A : K_1(A) \to K_1(L) = L^\times.$$

2.5. **Huruguen groups.** Let L be an étale F-algebra and A an Azumaya algebra over L. If $L = L_1 \times L_2 \times \cdots \times L_s$ is a product of fields, then $A = A_1 \times A_2 \times \cdots \times A_s$, where each A_i is a central simple algebra of some degree n_i over L_i and

$$R_{L/F}(\mathrm{SL}_1(A)) = R_{L_1/F}(\mathrm{SL}_1(A_1)) \times R_{L_2/F}(\mathrm{SL}_1(A_2)) \times \cdots \times R_{L_s/F}(\mathrm{SL}_1(A_s)).$$

Let K be an étale F-algebra and h an alternating non-degenerate form over K. If $K = K_1 \times K_2 \times \cdots \times K_t$ is a product of fields, then h is a product of alternating forms of some dimensions $2m_1, 2m_2, \ldots, 2m_t$ respectively, and

$$R_{K/F}(\mathrm{Sp}(h)) = R_{K_1/F}(\mathrm{Sp}_{2m_1}) \times R_{K_2/F}(\mathrm{Sp}_{2m_2}) \times \cdots \times R_{K_t/F}(\mathrm{Sp}_{2m_t}).$$

Let G be a reductive algebraic group over F. M. Huruguen proved in [4] that if G is special then

(1) The derived subgroup G' of G is isomorphic to

$$R_{L/F}(\mathrm{SL}_1(A)) \times R_{K/F}(\mathrm{Sp}(h)),$$

where L and K are étale F-algebras, A an Azumaya algebra over L and h an alternating non-degenerate form over K.

(2) The coradical G/G' of G is an invertible torus.

We call a reductive group G satisfying (1) and (2) a *Huruguen group*. Write $G' = G_1' \times G_2'$, where $G_1' = R_{L/F}(\mathrm{SL}_1(A))$ and $G_2' = R_{K/F}(\mathrm{Sp}(h))$.

Example 2.1. (Standard Huruguen groups) Following the notation as above set $G_1 := R_{L/F}(\mathrm{GL}_1(A))$ and $G_2 := R_{K/F}(\mathrm{GSp}(h))$, where $\mathrm{GSp}(h)$ is the group of *symplectic similitudes* (see [5, §23]). Then $G = G_1 \times G_2$ is a Huruguen group. Indeed,

$$G' = R_{L/F}(\mathrm{SL}_1(A)) \times R_{K/F}(\mathrm{Sp}(h))$$

and $G/G' = P_1 \times P_2$, where $P_1 = R_{L/F}(\mathbb{G}_{m,L})$ and $P_2 = R_{K/F}(\mathbb{G}_{m,K})$. The homomorphism $G_1 \to P_1$ is given by the reduced norm homomorphism Nrd^A.

The center Z of G is isomorphic to $P_1 \times P_2$, so the exact sequence $1 \to Z' \to Z \to S \to 1$, where Z' is the center of G', is isomorphic to

$$1 \to Z' \to P_1 \times P_2 \xrightarrow{\nu_1 \times \nu_2} P_1 \times P_2 \to 1. \tag{2.2}$$

The homomorphism $\nu_1 : P_1 \to P_1$ is the n_i-power map on the i-th component $R_{L_i/F}(\mathbb{G}_m)$ of P_1 and $\nu_2 : P_2 \to P_2$ is the square map. Note that the groups G_1, G_2 and G are special.

We show that every Huruguen group is the pull-back of the standard Huruguen group.

Proposition 2.3. *Let G be a Huruguen group and $\widetilde{G}$ the standard Huruguen group with $(\widetilde{G})' = G'$ as in Example 2.1. Then there is a homomorphism*

$$\bar{\lambda} := (\lambda_1, \lambda_2) : S = G/G' \to \widetilde{G}/\widetilde{G}' = P_1 \times P_2$$

of the coradical tori such that $G \simeq \widetilde{G} \times_{\widetilde{G}/\widetilde{G}'} S$. The homomorphism $\bar{\lambda}$ is uniquely determined modulo the image of $(\nu_1^*, \nu_2^*) : \mathrm{Hom}(S, P_1 \times P_2) \to \mathrm{Hom}(S, P_1 \times P_2)$.

Proof. The exact sequence (2.2) of groups of multiplicative type in Example 2.1 yields an exact sequence

$$\mathrm{Hom}(S, P_1 \times P_2) \xrightarrow{(\nu_1^*, \nu_2^*)} \mathrm{Hom}(S, P_1 \times P_2) \to \mathrm{Ext}(S, Z') \to \mathrm{Ext}(S, P_1 \times P_2).$$

As S is invertible and $P_1 \times P_2$ is a quasi-trivial torus, the group $\mathrm{Ext}(S, P_1 \times P_2)$ is trivial (see [1, Lemme 1]). It follows that the exact sequence $1 \to Z' \to Z \to S \to 1$ is the pull-back of (2.2) with respect to a group homomorphism $\bar{\lambda} : S \to P_1 \times P_2$. Therefore, the exact sequence $1 \to G' \to G \to S \to 1$ is the pull-back of $1 \to G' \to \widetilde{G} \to P_1 \times P_2 \to 1$ with respect to $\bar{\lambda}$. □

Let G be a Huruguen group with coradical S. Let $\widetilde{G}$ be the standard Huruguen group with $(\widetilde{G})' = G'$ as in Example 2.1. The coradical of $\widetilde{G}$ is $P_1 \times P_2$, where P_1 and P_2 are quasi-trivial tori. Write

$$\bar{\rho} = (\rho_1, \rho_2) : \widetilde{G} = \widetilde{G}_1 \times \widetilde{G}_2 \to P_1 \times P_2$$

for the canonical homomorphism. By Proposition 2.3, there is a group homomorphism

$$\bar{\lambda} = (\lambda_1, \lambda_2) : S \to P_1 \times P_2$$

such that G is the pull-back of $\widetilde{G}$ with respect to $\bar{\lambda}$.

Write for simplicity P for P_1 and $\lambda : S \to P$ for λ_1.

Lemma 2.4. *The following conditions are equivalent*

(1) *The group G is special.*

(2) *The homomorphism*

$$\widetilde{G}(K) \times S(K) \xrightarrow{(\bar{\rho}, \bar{\lambda})} P(K) \times P_2(K)$$

is surjective for all field extensions K/F.

(3) *The homomorphism*

$$\widetilde{G}_1(K) \times S(K) \xrightarrow{(\rho_1, \lambda)} P(K)$$

is surjective for all field extensions K/F.

Proof. In the diagram

$$\begin{array}{ccccccc} & & S(K) & \longrightarrow & \mathrm{Tors}_{G'}(K) & \longrightarrow & \mathrm{Tors}_{G}(K) & \longrightarrow & \mathrm{Tors}_{S}(K) \\ & & \downarrow{\scriptstyle\bar\lambda} & & \| & & \downarrow & & \\ \widetilde{G}(K) & \xrightarrow{\bar\rho} & P(K)\times P_2(K) & \longrightarrow & \mathrm{Tors}_{\widetilde{G}'}(K) & \longrightarrow & \mathrm{Tors}_{\widetilde{G}}(K) & & \end{array}$$

with exact rows the sets $\mathrm{Tors}_{\widetilde{G}}(K)$ and $\mathrm{Tors}_S(K)$ are singletons as $\widetilde{G}$ and S are special groups. The equivalence (1) $\Leftrightarrow$ (2) follows by the diagram chase. The equivalence (2) $\Leftrightarrow$ (3) follows from the fact that the map $\rho_2 : \widetilde{G}_2(K) \to P_2(K)$ is surjective as the group $\widetilde{G}'_2$ is special. $\square$

Proposition 2.5. *The Huruguen group G is special if and only if the generic point of P is in the image of the homomorphism*

$$(\mathrm{Nrd}, \lambda) : K_1(A \otimes_F F(P)) \times S(F(P)) \to (L \otimes_F F(P))^\times = P(F(P)).$$

Proof. Recall that $\widetilde{G}_1 = \mathrm{GL}_1(A)$ and for every field extension K/F the image of the map $\rho_1 : \widetilde{G}_1(K) \to P(K)$ coincides with the image of the reduced norm homomorphism

$$\mathrm{Nrd} : K_1(A \otimes_F K) \to (L \otimes_F K)^\times = P(K).$$

If F is a finite field, the algebra A is split and Nrd is surjective for every K and G is special by Lemma 2.4. Therefore, we may assume that F is infinite. The statement of the proposition is a consequence of Lemma 2.4 and the following lemma applied to the homomorphism $(\rho, \lambda) : G_1 \times S \to P$.

Lemma 2.6. *Let $H \to T$ be a homomorphism of algebraic groups with T a rational, smooth and connected group over an infinite field F. If the generic point of T in $T(F(T))$ is in the image of $H(F(T)) \to T(F(T))$, then the map $H(K) \to T(K)$ is surjective for every field extension K/F.*

Proof. As T is smooth and connected, T is geometrically integral. In particular, the function field $F(T)$ is defined. By assumption, the morphism $H \to T$ is split at the generic point of T, i.e., there is a nonempty open subset $U \subset T$ and a morphism $U \to H$ such that the composition $U \to H \to T$ is the inclusion. It follows that the subset $U(K) \subset T(K)$ is contained in the image of $H(K) \to T(K)$. Let $t \in T(K)$. Consider the nonempty open subset $W = t \cdot U_K^{-1} \cap U_K$ in T_K. As T is a rational variety and the field F is infinite, we have $W(K) \neq \emptyset$. Then $t \in U(K) \cdot U(K)$ is in the image of $H(K) \to T(K)$. $\square$

2.6. **Associated character.** We keep the notation of the previous section. Let X be the Γ-set corresponding to the F-algebra L, thus $P^* \simeq \mathbb{Z}[X]$. For every $x \in X$ write n_x for the degree of the central simple $A \otimes_L F_{\mathrm{sep}}$, where the tensor product is taken with respect to the homomorphism $x : L \to F_{\mathrm{sep}}$. Clearly, $n_x = n_i$ if $x \in X_i$.

The homomorphism $\lambda : S \to P$ determines an element $a \in S^*[X]^\Gamma$ via the isomorphisms

$$\mathrm{Hom}_F(S, P) = \mathrm{Hom}_\Gamma(S_*, \mathbb{Z}[X]) = (S^* \otimes \mathbb{Z}[X])^\Gamma = S^*[X]^\Gamma.$$

We call a the *character associated with G*.

We write $a = \sum_{x\in X} a_x \otimes x \in S^*[X]^\Gamma$ with $a_x \in (S^*)^{\Gamma_x}$ (see Section 2.1). By Proposition 2.3, a_x is uniquely determined modulo $n_x \cdot (S^*)^{\Gamma_x}$.

3. Two key propositions

Let L be an étale F-algebra and let X be the corresponding Γ-set. The Weil restriction

$$\mathbb{A}^X := R_{L/F}(\mathbb{A}^1_L) = \operatorname{Spec}(F_{\mathrm{sep}}[X])^\Gamma,$$

where $F_{\mathrm{sep}}[X]$ is a polynomial ring on the variables in X, is the affine space $\mathbb{A}(L)$ of the vector space L over F. The quasi-trivial torus $P = R_{L/F}(\mathbb{G}_{m,L})$ is the principal open subset of $\mathbb{A}^X$ given by the function $h = \prod_{x\in X} x \in F[\mathbb{A}^X]$, i.e.,

$$F[P] = F[\mathbb{A}^X][h^{-1}].$$

Note that h is the norm $N_{L(\mathbb{A}^X)/F(\mathbb{A}^X)}(x)$ for every $x \in X$.

For every variable $x \in X$ write v_x for the x-adic valuation on the rational function field $F_{\mathrm{sep}}(X)$ and also of its restriction to the subfield $F(\mathbb{A}^X) = F(P)$. Let $X_1, X_2, \dots, X_s$ be all Γ-orbits in X. For every i write h_i for the product of all $x \in X_i$. Then h is the product of all h_i and each h_i is an irreducible (prime) element of the polynomial ring $F[\mathbb{A}^X]$. Note that v_x is the discrete valuation on $F(\mathbb{A}^X)$ associated with h_i, where i is so that $x \in X_i$. In particular, $v_x = v_{x'}$ on $F(P)$ if x and x' lie in the same Γ-orbit. We will write v_i for v_x if $x \in X_i$.

For every i, let $Z_i \subset \mathbb{A}^X$ be the irreducible hypersurface given by the equation $f_i = 0$. The function field $F(Z_i)$ is the residue field of the valuation v_i. The scheme $(Z_i)_{\mathrm{sep}}$ is the union of $|X_i|$ irreducible hypersurfaces given by the equations $x = 0$ for $x \in X_i$. Fix a point $x \in X_i$ and identify L_i with the subfield $(F_{\mathrm{sep}})^{\Gamma_x}$, where Γ_x is the stabilizer of x in Γ. Let L'_i be the étale L_i-algebra corresponding to the Γ_x-set $X_i \setminus \{x\}$ and let P'_i be the quasi-trivial torus $R_{L'_i/L_i}(\mathbb{G}_{m,L'_i})$ over L_i. Viewing P'_i as a scheme over F, we see that P'_i is an open subscheme of Z_i (see [6, §2]). Therefore, $F(Z_i) = L_i(P'_i)$. In particular, the residue field $F(Z_i)$ of v_i is a purely transcendental field extension of L_i.

Proposition 3.1. *Let L and M be two étale F-algebras and A an Azumaya M-algebra. Then there is a commutative diagram*

$$\begin{array}{ccccc} K_0(A\otimes_F L) & \xrightarrow{f_A} & K_1(A\otimes_F F(P)) & \xrightarrow{g_A} & K_0(A\otimes_F L) \\ \downarrow{\scriptstyle \mathrm{Nrd}_0} & & \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}_0} \\ K_0(M\otimes_F L) & \xrightarrow{f_M} & K_1(M\otimes_F F(P)) & \xrightarrow{g_M} & K_0(M\otimes_F L) \end{array}$$

such that

(1) *The compositions $g_A \circ f_A$ and $g_M \circ f_M$ are the identity maps.*
(2) *If $M = L$, then $f_L(\Delta_L)$ is the generic point of P in $K_1(L \otimes_F F(P)) = P(F(P))^\times$.*

Proof. We may clearly assume that M is a field. Assume first that $M = F$. The homomorphisms f_F and g_F were defined in Section 2.3.

We define the map f_A as follows. Write $L = L_1 \times L_2 \times \cdots \times L_s$ as above. The group $K_0(A \otimes_F L)$ is the direct sum of $K_0(A \otimes_F L_i)$ over all i. Choose any $x \in X_i$, where $X_i \leftrightarrow L_i$. We can identify L_i with the field $(F_{\mathrm{sep}})^{\Gamma_x}$, where Γ_x is the stabilizer of x in

Γ. Thus, we can view x is a function in $L_i(P)^\times = K_1(L_i(P))$. Define the map f_A on $K_0(A \otimes_F L_i)$ as the composition

$$K_0(A \otimes_F L_i) \to K_1((A \otimes_F L_i) \otimes_{L_i} L_i(P)) = K_1(A \otimes_F L_i(P)) \to K_1(A \otimes_F F(P)),$$

where the first map is multiplication by $x \in K_1(L_i(P))$, i.e., it takes the class of a left projective $A \otimes_F L_i$-module Q to the class of the automorphism of multiplication by x in Q, and the last homomorphism in the norm map for the field extension $L_i(P)/F(P)$.

To prove commutativity of the left square of the diagram consider the following diagram

$$\begin{array}{ccccc} K_0(A \otimes_F L_i) & \longrightarrow & K_1(A \otimes_F L_i(P)) & \longrightarrow & K_1(A \otimes_F F(P)) \\ \downarrow{\scriptstyle \mathrm{Nrd}_0} & & \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}} \\ K_0(L_i) & \longrightarrow & K_1(L_i(P)) & \longrightarrow & K_1(F(P)), \end{array}$$

where the first map in each row is multiplication by x and the second map is the norm map for the field extension $L_i(P)/F(P)$. The right square is commutative as the reduced norm commutes with the usual norm. The composition in the top row is f_A and in the bottom row takes 1 to the product in $F(P)^\times = K_1(F(P))$ of all elements from the orbit X_i. Therefore, the composition in the bottom row coincides with f_F restricted to $K_0(L_i)$.

It suffices to prove that the left square in this diagram is commutative. Write $A \otimes_F L_i \simeq M_k(D)$, where D is a division algebra of degree $d = \mathrm{ind}(A \otimes_F L_i)$ over L_i. The group $K_0(A \otimes_F L_i) = K_0(M_k(D))$ is generated by the class $[Q]$ of the standard free left D-module $Q = D^k$. Identifying Q with the left ideal of $M_k(D)$ of all matrices with all terms but the first column zero, we see that the group $D^\times = \mathrm{Aut}_{M_k(D)}(Q)$ embeds into $\mathrm{Aut}_{M_k(D)}(M_k(D)) = \mathrm{GL}_k(D)$ via $d \mapsto \mathrm{diag}(d, 1, \dots, 1)$. It follows that the image of $[Q]$ in $K_1(A \otimes_F L_i(P)) = K_1(M_k(D) \otimes_{L_i} L_i(P))$ is given by the diagonal matrix $\mathrm{diag}(x, 1, \dots, 1) \in M_k(D \otimes_{L_i} L_i(P))$ whose reduced norm in $L_i(P)^\times = K_1(L_i(P))$ is equal to x^d. Finally, the image of $[Q]$ under Nrd_0 is equal to $d \cdot 1 \in K_0(L_i)$ and the bottom map in the left square of the diagram takes 1 to x. This proves commutativity of the left square of the above diagram and hence of the left square of the diagram in the statement of the proposition.

In order to define the map g_A we need the following statement.

Lemma 3.2. *Let A be a central simple F-algebra, K/F a field extension and v a discrete valuation on K over F with residue field $\overline{K}$. Then*

$$\mathrm{Im}\left[K_1(A \otimes_F K) \xrightarrow{\mathrm{Nrd}} K^\times \xrightarrow{v} \mathbb{Z}\right] \subset \mathrm{ind}(A \otimes_F \overline{K}) \cdot \mathbb{Z}.$$

Proof. Let $R \subset K$ be the valuation ring of v and let $\mathcal{C}_A$ be the Serre subcategory of R-torsion modules in the abelian category $\mathcal{M}(A \otimes_F R)$ of finitely generated left $A_F \otimes R$-modules. By dévissage (see [7, §5]), we have $K_i(\mathcal{C}_A) = K_i(A \otimes_F \overline{K})$ and the factor category $\mathcal{M}(A \otimes_F R)/\mathcal{C}_A$ is equivalent to $\mathcal{M}(A \otimes_F K)$.

Similarly, in the case $A = F$ we have the subcategory $\mathcal{C}_F \subset \mathcal{M}(R)$ such that $K_i(\mathcal{C}_F) = K_i(\overline{K})$ and $\mathcal{M}(R)/\mathcal{C}_F \simeq \mathcal{M}(K)$.

The exact functor $i : \mathcal{M}(A \otimes_F R) \to \mathcal{M}(R)$ induced by the natural homomorphism $R \to A \otimes_F R$ takes the subcategory $\mathcal{C}_A$ into $\mathcal{C}_F$. Therefore, we have a commutative diagram

$$\begin{array}{ccc} K_1(A\otimes_F K) & \xrightarrow{\partial_A} & K_0(A\otimes_F \overline{K}) \\ {\scriptstyle (i_1)_*}\downarrow & & \downarrow{\scriptstyle (i_0)_*} \\ K_1(K) & \xrightarrow{\partial_F} & K_0(\overline{K}), \end{array}$$

where the horizontal maps are the connecting homomorphisms in localization sequences and the vertical maps are induced by the functor i. By [7, Lemma 5.16], ∂_F coincides with $v : K^\times \to \mathbb{Z}$. We also have $(i_1)_* = N_{A/F} = \mathrm{Nrd}^n$ and the image of $(i_0)_*$ coincides with $n\cdot \mathrm{ind}(A\otimes_F \overline{K})\mathbb{Z}$, where $n = \deg(A)$ (see Section 2.4). The result follows. □

We apply this lemma to the field $K = F(P)$ and the discrete valuation v_i on K for some i. Recall that the residue field $\overline{K}_i$ of v_i is purely transcendental over L_i, hence $K_0(A\otimes_F \overline{K}_i) = K_0(A\otimes_F L_i)$. The image of Nrd_0 in $\mathbb{Z}^s = K_0(L)$ is equal to $\coprod_{i=1}^s \mathrm{ind}(A\otimes_F L_i)\cdot\mathbb{Z}$. It follows from Lemma 3.2 that the map g_A exists and unique.

The commutative diagram in the statement of the proposition is constructed in the case $M = F$. The case of a general M reduces to the one when W is a field. Finally, we apply the above case (when A is central over the base field) to the algebra A over M and the étale M-algebra $M\otimes_F L$ in place of L. Thus, the commutative diagram is constructed in the general case.

We know that $g_M\circ f_M$ is the identity. This implies that the composition $g_A\circ f_A$ is also the identity since the map Nrd_0 is injective.

Finally, to prove (2) we may assume that L is split, so $K_0(L\otimes_F L) = \mathbb{Z}[X\times X]$ and $\Delta_L = \Delta_X = \sum_{x\in X}(x,x)$. Then $f_L(\Delta_L)$ in $P(F(P)) = \mathrm{Hom}(P^*, F(P)^\times)$ is the homomorphism taking every $x\in X$ to $x\in F(P)^\times$, i.e., it is the canonical embedding of P^* into $F(P)^\times$ representing the generic point of P. □

Let L and M be étale F-algebras, and set $P = R_{L/F}(\mathbb{G}_{m,L})$, $Q = R_{M/F}(\mathbb{G}_{m,M})$.

Proposition 3.3. *Let S be an algebraic torus over F and $\mu : S\to Q$ a group homomorphism. Then there is a commutative diagram*

$$\begin{array}{ccccc} (S_*\otimes P^*)^\Gamma & \longrightarrow & S(F(P)) & \longrightarrow & (S_*\otimes P^*)^\Gamma \\ {\scriptstyle \mu_*\otimes\mathrm{id}}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle \mu_*\otimes\mathrm{id}}\downarrow \\ (Q_*\otimes P^*)^\Gamma & \longrightarrow & Q(F(P)) & \longrightarrow & (Q_*\otimes P^*)^\Gamma \\ \| & & \| & & \| \\ K_0(M\otimes_F L) & \xrightarrow{f_M} & K_1(M\otimes_F F(P)) & \xrightarrow{g_M} & K_0(M\otimes_F L). \end{array}$$

Proof. It suffices to consider the case when F is separably closed, so both L and M are split. Recall that P^* is the canonical direct summand of $F(P)^\times$. Since $S(F(P)) = S_*\otimes F(P)^\times$, we get the first row of the diagram. The second row is similar. The commutativity readily follows from the definitions. □

4. Classification theorem

Theorem 4.1. *Let G be a reductive group over a field F. Then G is special if and only if the following three conditions hold:*

1. *The derived subgroup G' of G is isomorphic to*
$$R_{L/F}(\mathrm{SL}_1(A)) \times R_{K/F}(\mathrm{Sp}(h))$$
where L and K are étale F-algebras, A an Azumaya algebra over L and h an alternating non-degenerate form over K;
2. *The coradical $S = G/G'$ of G is an invertible torus;*
3. *There exists $b = \sum_{x\in X} b_x \otimes x \in S_*[X]^\Gamma$ such that*
$$\langle a_x, b_y\rangle \equiv \begin{cases} 1 & \text{mod } d_{x,y}, \quad \text{if } x = y; \\ 0 & \text{mod } d_{x,y}, \quad \text{otherwise,} \end{cases}$$
where $X = \mathrm{Hom}_{F\text{-}alg}(L, F_{\mathrm{sep}})$, $a = \sum_{x\in X} a_x \otimes x \in S^*[X]^\Gamma$ is the character associated with G (see Section 2.6) and $d_{x,y} = \mathrm{ind}(A \otimes_L M_{x,y})$ with $M_{x,y} = \mathrm{Im}(L \otimes_F L \xrightarrow{(x,y)} F_{\mathrm{sep}})$ viewed as an L-algebra via $x : L \to M_{x,y}$.

Proof. By Propositions 2.5, 3.1 and 3.3, the group G is special if and only if Δ_X is contained in the image of the homomorphism
$$\theta = (\mathrm{Nrd}_0, a \otimes \mathrm{id}_{\mathbb{Z}[X]}) : K_0(A \otimes_F L) \oplus (S_* \otimes \mathbb{Z}[X])^\Gamma \to K_0(L \otimes_F L) = \mathbb{Z}[X \times X]^\Gamma.$$

If $(x, y) \in X \times X$ write $M_{x,y}$ for the image of $L \otimes_F L \to F_{\mathrm{sep}}$ taking $u \otimes v$ to $x(u) \cdot y(v)$. Note that $L \otimes_F L$ is isomorphic to the product of fields $M_{x,y}$ over a set of representatives (x, y) of all Γ-orbits in $X \times X$. We view $M_{x,y}$ as an L-algebra with respect to the homomorphism $L \to L \otimes_F L \to M_{x,y}$, where the first map takes l to $l \otimes 1$.

Since $M_{x,y}$ and $M_{x',y'}$ are isomorphic as L-algebras when (x, y) and (x', y') belong to the same Γ-orbit in $X \times X$, we have $d_{x,y} = d_{x',y'}$.

Denote by C the subgroup of $\mathbb{Z}[X \times X]$ generated by $d_{x,y} \cdot (x, y)$ over all $x, y \in X$. The image in $\mathbb{Z}[X \times X]^\Gamma$ of $K_0(A \otimes_F L)$ under θ coincides with C^Γ.

The map θ restricted to $(S_* \otimes \mathbb{Z}[X])^\Gamma$ takes an element $b = \sum_{x\in X} b_x \otimes x \in S_*[X]^\Gamma$ to
$$\sum_{(x,y)} \langle a_x, b_y\rangle(x, y) \in \mathbb{Z}[X \times X]^\Gamma.$$

Therefore, the diagonal $\Delta_X = \sum_{x\in X}(x, x)$ is contained in the image of θ if and only if there is $b \in S_*[X]^\Gamma$ such that $\langle a_x, b_y\rangle$ is congruent to 1 modulo $d_{x,y}$ if $x = y$ and is divisible by $d_{x,y}$ otherwise. □

Remark 4.2. According to Section 2.1 to give $b \in S_*[X]^\Gamma$ is the same as to give elements $b_i \in (S_*)^{\Gamma_i}$, where $\Gamma_i \subset \Gamma$ are the stabilizers of representatives of Γ-orbits in X. Also, the numbers $d_{x,y}$ and $\langle a_x, b_y\rangle$ stay the same if the pair (x, y) is replaced by a pair (x', y') in the same Γ-orbits in $X \times X$. Therefore, the conditions (3) in Theorem 4.1 for the pairs (x, y) and (x', y') are equivalent.

Example 4.3. Assume that G is a Huruguen group such that the algebra L is split, i.e., $L = F^s$ for some integer s. Then $X = \{1, 2, \dots, s\}$ and $\mathbb{Z}[X] = \mathbb{Z}^s$ with trivial Γ-action.

Write $A = A_1 \times A_2 \times \cdots \times A_s$, where A_i is a c.s.a. over F and let $d_i := \mathrm{ind}(A_i)$. We have $d_{i,j} = d_i$.

The character associated with G is a homomorphism $a : S \to (\mathbb{G}_m)^s$ given by a tuple $(a_1, a_2, \ldots, a_s)$ of characters in $(S^*)^\Gamma$. We write a_* for the associated map $S_* \to \mathbb{Z}^s$. We claim that G is special if and only if the composition

$$(S_*)^\Gamma \xrightarrow{q} \mathbb{Z}^s \xrightarrow{p} \coprod_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z},$$

where q is the restriction of a_* on $(S_*)^\Gamma$, is surjective. Indeed, by Theorem 4.1, G is special if and only if there is $b \in \mathrm{Hom}_\Gamma(\mathbb{Z}^s, S_*) = \mathrm{Hom}(\mathbb{Z}^s, (S_*)^\Gamma)$ such that $p \circ q \circ b = p$. The latter condition implies that $p \circ q$ is surjective. Conversely, if $p \circ q$ is surjective, then b exists as $\mathbb{Z}^s$ is a free abelian group.

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