THE MOD 2 NEGLIGIBLE COHOMOLOGY OF GROUPS OF ORDER AT MOST 8

SANGHOON BAEK AND ALEXANDER S. MERKURJEV

Abstract. Let G be a finite group and F a field of characteristic not equal to two. We study the ideal of negligible cohomology classes in the mod two cohomology ring of G, consisting of elements whose associated cohomological invariants vanish on all G-extensions over field extensions of F. We give a complete description of the negligible ideal for all finite groups of order at most eight that depends explicitly on the arithmetic properties of the field F, precisely the presence of certain 2-primary roots of unity and the level of F.

Contents

1. Introduction

One of the central problems in Galois theory is to characterize absolute Galois groups of fields among profinite groups. In this article, we investigate restrictions to the profinite groups given by negligible cohomology.

Let F be a field and let $\Gamma_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ be the *absolute* Galois group of F. Let G be a finite group and let E/F be a G-*extension* that is a Galois extension of rings with $G = \mathrm{Gal}(E/F)$ in the sense of Galois theory of commutative rings (or equivalently, a Galois G-algebra). There is the associated continuous group homomorphism j :

$\Gamma_F \to G$ that is uniquely determined up to conjugation by an element in G (see [8, Section 18.B]). The map j is surjective if and only if E is a field.

For every (discrete) Γ_F-module M we write $H^n(F, M)$ for the nth cohomology group $H^n(\Gamma_F, M)$.

Let M be a G-module and let Γ be a profinite group. A continuous group homomorphism $j : \Gamma \to G$ yields the pullback homomorphism

$$j^* : H^d(G, M) \to H^d(\Gamma, M),$$

where we view M as a Γ-module via j.

Let $\mathcal{C}$ be a class of profinite groups. We define the subgroup of *negligible elements relative to* $\mathcal{C}$

$$H^d(G, M)_{\mathrm{neg},\mathcal{C}} = \bigcap \operatorname{Ker}(j^*) \subset H^d(G, M),$$

where the intersection is taken over all continuous group continuous homomorphisms $\Gamma \to G$ for Γ in $\mathcal{C}$. The larger the class $\mathcal{C}$, the smaller the subgroup of negligible elements relative to $\mathcal{C}$. For example, if $\mathcal{C}$ is the class of all profinite groups, then the group $H^d(G, M)_{\mathrm{neg},\mathcal{C}}$ is trivial. Thus, the subgroup of negligible elements relative to $\mathcal{C}$ provides some information about the "size" of $\mathcal{C}$.

Let F be a field and let $\mathcal{C}$ be the class of the absolute Galois groups Γ_K for all field extensions K/F. We write $H^d(G, M)_{\mathrm{neg},F}$ for $H^d(G, M)_{\mathrm{neg},\mathcal{C}}$. This group consists of all elements $u \in H^d(G, M)$ such that $j^*(u) = 0$ for all continuous group homomorphisms $j : \Gamma_K \to G$ for all field extensions K/F. Such elements are called *negligible over* F. The group $H^d(G, M)_{\mathrm{neg},F}$ was introduced by Serre in [12]. When $M = \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, the negligible over F elements in the cohomology ring $H^*(G, \mathbb{F}_p)$ form an ideal, called the *negligible ideal*.

The degree 2 negligible cohomology over F admits an interpretation via the classical Galois embedding problem. Let

$$1 \to M \to H \xrightarrow{i} G \to 1 \tag{1}$$

be an exact sequence of groups with M an abelian group, so M is a G-module via the conjugation. Let E/K be a G-extension. A solution to the embedding problem associated with the sequence (1) and the extension E/K is an H-extension L/E such that $L^M = E$. Equivalently, if $j : \Gamma_K \to G$ is a group homomorphism corresponding to the G-extension E/K, then the embedding problem is solvable if j extends to a homomorphism $k : \Gamma_K \to H$, that is $i \circ k = j$. The latter condition is equivalent to the triviality in $H^2(K, M)$ of the pullback of the class $u \in H^2(G, M)$ of the exact sequence (1) with respect to j. Thus the class u is negligible over F if and only if the embedding problem associated with (1) is *universally solvable* over F, i.e., it is solvable for every G-extension E/K for all field extensions K/F.

There is a relation of negligible cohomology with the cohomological invariants over F as follows. Let M be a Γ_F-module. A degree d *cohomological invariant* of G over F with values in M is a collection of natural in K assignments for every G-extension E/K with K/F a field extension of an element in $H^d(K, M)$. All such invariants form a group $\operatorname{Inv}^d(G, M)$ (see [4]).

Suppose that M is a trivial Γ_F-module. We also view M as a G-module with trivial action. There is a homomorphism

$$\mathrm{inv} : H^d(G, M) \to \mathrm{Inv}^d(G, M),$$

taking an element $u \in H^d(G, M)$ to the invariant of G that assigns to a G-extension E/K the element $j^*(u)$ in $H^d(K, M)$, where $j : \Gamma_K \to G$ is a homomorphism corresponding to E/K. In this notation, the group of negligible over F elements coincides with the kernel of inv:

$$H^d(G, M)_{\mathrm{neg},\, F} = \mathrm{Ker}(\mathrm{inv}).$$

Let M be an arbitrary G-module. The group $H^d(G, M)_{\mathrm{neg},F}$ is trivial if $d = 0$ or 1 (see [5, Corollary 2.2]). The degree 2 negligible over F cohomology (under a mild hypothesis on F) was determined in [11]. If M is a trivial G-module, the degree 2 negligible cohomology was determined in [5]. Serre provided explicit descriptions of negligible classes over $\mathbb{Q}$ mod 2 negligible elements of elementary abelian 2-groups in [4, Part I, Lemma 26.4]. These results were later extended in [6, Theorems 2.1 and 2.3], where the mod p negligible over F cohomology of elementary abelian p-groups was computed over an arbitrary field F. In fact, the negligible ideal in $H^*(G, \mathbb{F}_p)$ is quite large (see [6]): the quotient ring by the ideal is finite if F is not formally real or G has odd order and otherwise it has Krull dimension 1. Recall that the Krull dimension of the ring $H^*(G, \mathbb{F}_p)$ is equal to the p-rank of G (see [2, Chapter IV, Section 5]).

In the present paper, we work with coefficients $M = \mathbb{F}_2$ over a field F of characteristic different from 2. We determine the negligible ideal for all finite groups of order at most 8. For any group G of odd order, one has $H^*(G, \mathbb{F}_2) = \mathbb{F}_2$, and hence $H^*(G, \mathbb{F}_2)_{\mathrm{neg},\, F} = 0$. Specifically, the negligible ideals are computed in Theorems 3.1, 5.1, and 4.1 for cyclic groups C_4 and C_8, and for the direct product $C_4 \times C_2$. For nonabelian groups, the negligible ideals of the symmetric group S_3, the dihedral group D_8, and the quaternion group Q_8 are determined in Lemma 2.5, Theorem 6.1, and Theorem 7.1, respectively.

Our descriptions of the negligible ideals depend on the presence of certain roots of unity in the base field, as well as on the level of the field. The proofs are based on the analysis of the cohomological invariants of the form $\mathrm{inv}(y)$ for certain degree 2 elements y. We use different techniques depending on the group. If $G = C_8$, the G-extensions cannot be rationally parameterized over an arbitrary field, so a reasonable formula would not be expected for $\mathrm{inv}(y)$, where y is a generator of $H^2(C_4, \mathbb{F}_2)$. Instead, we employ a reduction argument to the C_4-case. In the case $G = Q_8$ there is a generator z in $H^4(Q_8, \mathbb{F}_2)$. In order to compute $\mathrm{inv}(z)$, we embed Q_8 into the symmetric group S_8 and prove that $\mathrm{inv}(z)$ extends to an invariant of S_8 given by the 4th Stiefel-Whitney class of the trace form of the Q_8-extension.

2. Preliminaries

In this section, we introduce the notation and recall basic notions that will be used throughout the paper. The base field F is assumed to be of characteristic different from 2.

2.1. **Iterated quadratic étale algebras.** Let R be a commutative ring with $2 \in R^\times$ and $a \in R^\times$. We write R_a for the quadratic étale R-algebra $R[t]/(t^2 - a)$. We can iterate this construction: given $A \in (R_a)^\times$, consider $(R_a)_A = R_a[s]/(s^2 - A)$, which is an étale R-algebra of degree 4, and so on. For $a, b \in R^\times$, we write $R_{a,b}$ for the biquadratic étale R-algebra $(R_a)_b = R[t,s]/(t^2 - a, s^2 - b)$ and $N_a : (R_a)^\times \to R^\times$ for the norm map.

2.2. **Mod 2 cohomology ring of a field.** For a field F, we have commutative graded *cohomology ring* of F

$$H^*(F, \mathbb{F}_2).$$

The product $\cdot$ in this ring is the cup-product. The degree 1 component $H^1(F, \mathbb{F}_2)$ is canonically isomorphic to $F^\times/(F^\times)^2$. For every $a \in F^\times$ we write (a) for the corresponding class in $H^1(F, \mathbb{F}_2)$. By Voevodsky's theorem, the ring $H^*(F, \mathbb{F}_2)$ is generated by the classes (a) for all $a \in F^\times$.

The product $(a_1) \cdot (a_2) \cdot \ldots \cdot (a_n)$ in $H^n(F, \mathbb{F}_2)$ is denoted by $(a_1, a_2, \ldots, a_n)$. We frequently use the following relations:

$$(a, a) = (a, -1), \quad (a, b) = (b, a)$$

for $a, b \in F^\times$. Note that $(a, b) = 0$ if and only if $b = x^2 - ay^2$ for some $x, y \in F$. For $u \in H^d(F, \mathbb{F}_2)$ and any $i \geq 1$, we have

$$u^i = u \cdot (-1)^{d(i-1)}.$$

A field extension K/F gives rise to the *restriction* and (in the case K/F is finite) to the *corestriction* maps

$$\mathrm{res} : H^d(F, \mathbb{F}_2) \to H^d(K, \mathbb{F}_2), \quad \mathrm{cor} : H^d(K, \mathbb{F}_2) \to H^d(F, \mathbb{F}_2)$$

such that the composition $\mathrm{cor} \circ \mathrm{res}$ is the multiplication by $[K : F]$.

2.3. **Quadratic forms and Stiefel-Whitney classes.** We write $q = \langle a_1, \ldots, a_n \rangle$ for the diagonal quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$ of rank n over F, where $a_i \in F^\times$. A quadratic form

$$\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$

is an *n-fold Pfister form* of dimension 2^n.

For each $0 \leq d \leq n$, the d-th Stiefel-Whitney class of q is given by

$$w_d(q) = \sum (a_{j_1}, \ldots, a_{j_d}) \in H^d(F, \mathbb{F}_2), \quad \text{where } 1 \leq j_1 < \cdots < j_d \leq n.$$

For an étale F-algebra E of dimension n and a quadratic form $p : V \to E$ over E, we write $\mathrm{Tr}_{E/F}(p)$ for the quadratic form q on V over F defined by

$$q(v) := \mathrm{Tr}_{E/F}(p(v)) \in F.$$

The *trace form of* E/F is the quadratic form

$$q_E := \mathrm{Tr}_{E/F}\langle 1\rangle$$

over F.

Define the *Galois Stiefel-Whitney class* of E with values in $H^d(F,\mathbb{F}_2)$ by the equality

$$w_d^{\mathrm{gal}}(E) = \begin{cases} w_d(q_E), & \text{if } d \text{ is odd;} \\ w_d(q_E) + (2)\cdot w_{d-1}(q_E), & \text{if } d \text{ is even.} \end{cases} \tag{2}$$

The *total* Galois Stiefel-Whitney class is defined as $w^{\mathrm{gal}} = 1 + w_1^{\mathrm{gal}} + w_2^{\mathrm{gal}} + \cdots$.

Since there is a canonical bijection between isomorphism classes of étale F-algebras of dimension n and S_n-extensions, we can view w_d^{gal} as the degree d invariant of S_n.

Proposition 2.1. [4, Part I, §25.2] *Let* $E = F_{a_1}\times F_{a_2}\times\cdots\times F_{a_m}$ *for some* $a_1, a_2, \ldots a_m \in F^\times$*. Then*

$$w^{\mathrm{gal}}(E) = (1+(a_1))\cdot(1+(a_2))\cdot\ldots\cdot(1+(a_m)).$$

2.4. **Level of a field.** The *level* of a field F is defined as

$$\min\{\, n \mid -1 \text{ is a sum of } n \text{ squares in } F\,\}.$$

The level is equal to ∞ if and only if -1 is not a sum of squares in F, that is, F is a *formally real field*. The level is equal to 1 if and only if $\mu_4 \subset F^\times$. If the level is finite, it is always a power of two.

The following result is a consequence of the standard properties of Pfister forms (see [9, Chapter X]) and the Milnor Conjecture (now Voevodsky's theorem).

Lemma 2.2. *The following statements for a field* F *are equivalent:*

(1) *The level of* F *is at most* 2^n*.*
(2) -1 *is a sum of* 2^n *squares in* F*.*
(3) *The* $(n+1)$*-fold Pfister form* $\langle\langle -1,\ldots,-1\rangle\rangle$ *is hyperbolic over* F*.*
(4) $(-1)^{n+1} = 0$ *in* $H^{n+1}(F,\mathbb{F}_2)$*.*

2.5. **Negligible ideal of elementary abelian 2-groups.** Let G be an elementary abelian 2-group of order 2^n. Choose a basis $\{x_1, x_2, \ldots, x_n\}$ for the character group $\mathrm{Hom}(G,\mathbb{F}_2) = H^1(G,\mathbb{F}_2)$. Then

$$H^*(G,\mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \ldots, x_n].$$

Theorem 2.3. [6, Theorem 4.1] *Let* G *be an elementary abelian* 2*-group of order* 2^n *and let* F *be a field of level* 2^l*. Then*

$$H^*(G,\mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (x_i^2 : 1\le i\le n) & \text{if } l=0,\\ (x_ix_j^2 + x_jx_i^2 : 1\le i<j\le n,\ x_i^{l+2} : 1\le i\le n) & \text{if } 1\le l<\infty,\\ (x_ix_j^2 + x_jx_i^2 : 1\le i<j\le n) & \text{if } l=\infty. \end{cases}$$

In particular, $H^*(C_2,\mathbb{F}_2) = \mathbb{F}_2[x]$.

Corollary 2.4. *Let F be a field of level 2^l. Then*

$$H^*(C_2, \mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (x^{l+2}) & \text{if } l < \infty, \\ 0 & \text{if } l = \infty. \end{cases}$$

The negligible ideal in $H(S_3, \mathbb{F}_2)$ coincides with the negligible ideal in $H(C_2, \mathbb{F}_2)$ as follows:

Lemma 2.5. *The restriction map* $\mathrm{res} : H^*(S_3, \mathbb{F}_2) \to H^*(C_2, \mathbb{F}_2) = \mathbb{F}_2[x]$ *is an isomorphism. It induces an isomorphism of negligible ideals*

$$H^*(S_3, \mathbb{F}_2)_{\mathrm{neg},\,F} \simeq H^*(C_2, \mathbb{F}_2)_{\mathrm{neg},\,F}.$$

Proof. Consider the projection $p : S_3 \twoheadrightarrow C_2$ with a section $i : C_2 \hookrightarrow S_3$. Since $\mathrm{cor} \circ \mathrm{res} = 1$ and $i^* \circ p^* = 1$, the map $\mathrm{res} = i^*$ is an isomorphism. By [5, Prop. 2.3(1)], both i^* and p^* preserve negligible ideals, thus i^* induces the desired isomorphism. □

By [3, §3.2], the mod 2 cohomology ring of the cyclic group C_{2^n} of order 2^n (for $n \geq 2$) is given by

$$H^*(C_{2^n}, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^2) \tag{3}$$

with $\deg(x) = 1$ and $\deg(y) = 2$.

Lemma 2.6. *Let F be a field of level 2^l. Assume that the class $y^d \in H^{2d}(C_{2^n}, \mathbb{F}_2)$ is negligible over F for some $d \geq 2$. Then $l \leq 2d - 2$, or equivalently, $(-1)^{2d-1} = 0$ in $H^{2d-1}(F, \mathbb{F}_2)$.*

Proof. The restriction map $\mathrm{res} : H^{2d}(C_{2^n}, \mathbb{F}_2) \to H^{2d}(C_2, \mathbb{F}_2)$ is an isomorphism of cyclic groups of order 2. The claim then follows from the fact that the restriction preserves negligible over F elements by [5, Proposition 2.3] together with Corollary 2.4. □

2.6. **Degree 2 negligible cohomology.** Let G be a finite group. Following [5], the natural identification $G^* = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Z})$ induces an inclusion

$$G^*/2G^* \hookrightarrow H^2(G, \mathbb{F}_2).$$

Proposition 2.7. [5, Theorem 4.2] *Let $t \geq 1$ be the maximal integer (or ∞) such that $\mu_{2^t} \subset F$. Then*

$$H^2(G, \mathbb{F}_2)_{\mathrm{neg},\,F} = \big(G^*[2^{t-1}] + 2G^*\big)/2G^*.$$

Corollary 2.8. *For any $n \geq 2$, the group $H^2(C_{2^n}, \mathbb{F}_2)_{\mathrm{neg},\,F}$ is trivial if and only if $\mu_{2^{n+1}} \not\subset F$.*

3. Cyclic group C_4

Recall from (3) that the mod 2 cohomology ring of C_4 is given by

$$H^*(C_4, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^2),$$

where $\deg(x) = 1$ and $\deg(y) = 2$.

Theorem 3.1. *Let F be a field of level 2^l. Then,*

$$H^*(C_4, \mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (y) & \text{if } \mu_8 \subset F, \\ (xy,\ y^2) & \text{if } \mu_8 \not\subset F \text{ and } l = 0, \\ (xy,\ y^{[\frac{l+3}{2}]}) & \text{if } 1 \le l < \infty, \\ (xy) & \text{if } l = \infty. \end{cases}$$

Lemma 3.2. [7, Theorem 3], [10, Example 3.1] *A quadratic field extension K_a/K can be embedded in a C_4-extension E/K if and only if $(-1, a) = 0$ in $H^2(K, \mathbb{F}_2)$, i.e., a is the sum of two squares in K. Every C_4-extension is of the form*

$$E = (K_a)_{rA}$$

for some $r \in K^\times$ and $A \in K_a^\times$ with $N_a(A) = a$. Conversely, for every $a \in K$ and $A \in K_a$ as above, $(K_a)_{rA}/K$ is a C_4-extension for every $r \in K^\times$. Moreover, we have

$$\mathrm{inv}(x)(E) = (a) \quad \text{and} \quad \mathrm{inv}(y)(E) = (2, a) + (-1, r). \quad \square$$

Proof of Theorem 3.1. By Corollary 2.8, the class y is negligible over F if and only if $\mu_8 \subset F$. We have

$$\mathrm{inv}(xy)(E) = (a) \cdot [(2, a) + (-1, r)] = (2, a, a) + (a, -1, r) = 0,$$

so xy is always negligible over F. Since $(-1, 2) = 0$, Lemma 3.2 gives

$$\mathrm{inv}(y^d)(E) = \mathrm{inv}(y)(E) \cdot (-1)^{2d-2} = [(2, a) + (-1, r)] \cdot (-1)^{2d-2} = (-1)^{2d-1} \cdot (r)$$

for any $d \ge 2$. Therefore, by Lemma 2.6, we conclude that y^d is negligible over F if and only if $l \le 2d - 2$. The result follows. $\square$

4. Direct product $C_4 \times C_2$

Recall from (3) that $H^*(C_4, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^2)$ and $H^*(C_2, \mathbb{F}_2) = \mathbb{F}_2[z]$. Then, by the Künneth formula [3, §2.5], the cohomology ring of $C_4 \times C_2$ is

$$H^*(C_4 \times C_2, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(x^2), \tag{4}$$

where $\deg(x) = \deg(z) = 1$ and $\deg(y) = 2$.

Theorem 4.1. *Let F be a field of level 2^l. Then,*

$$H^*(C_4 \times C_2, \mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (y,\ z^2) & \text{if } \mu_8 \subset F, \\ (xy,\ y^2,\ z^2) & \text{if } \mu_8 \not\subset F,\ l = 0, \\ (xy,\ xz^2,\ y^2,\ z^{l+2},\ yz^{l+1}) & \text{if } l = 1, 2, \\ (xy,\ xz^2,\ y^2z + yz^3,\ z^{l+2},\ yz^{l+1},\ y^{[\frac{d+3}{2}]}) & \text{if } 3 \le l < \infty, \\ (xy,\ xz^2,\ y^2z + yz^3) & \text{if } l = \infty. \end{cases}$$

Lemma 3.2 yields the following statement.

Lemma 4.2. *Every $C_4 \times C_2$-extension E/K is of the form*

$$E = (K_{a,b})_{rA},$$

for some $r, a, b \in K^\times$ and $A \in K_a^\times$ with $N_a(A) = a$. Conversely, for every $a, b, \in K$ and $A \in K_a$ as above, $(K_{a,b})_{rA}/K$ is a $C_4 \times C_2$-extension for every $r \in K^\times$. Moreover, we have

$$\text{(5)} \qquad \operatorname{inv}(x)(E) = (a), \quad \operatorname{inv}(y)(E) = (2, a) + (-1, r), \quad \textit{and} \quad \operatorname{inv}(z)(E) = (b).$$

Lemma 4.3. *The class yz is negligible over F if and only if $\mu_8 \subset F$.*

Proof. If $\mu_8 \subset F$, then by Corollary 2.8, the class y is negligible over F, and hence so is xy. Conversely, suppose $\mu_8 \not\subset F$. Then again by Corollary 2.8, there exists a C_4-extension L/K such that $\operatorname{inv}(y)(L) \neq 0$. Let b be a variable over K, and set $K' = K(b)$, $L' = L(b)$. Then, $(L')_b/K'$ is a $C_4 \times C_2$-extension. Consider the b-adic residue map

$$\partial_b \colon H^3(K', \mathbb{F}_2) \to H^2(K, \mathbb{F}_2).$$

Since $\operatorname{inv}(yz)((L')_b)$ has residue $\operatorname{inv}(y)(L) \neq 0$ under ∂_b, it follows that yz is not negligible over F. □

Proof of Theorem 4.1. Since $(a, a) = (a, -1) = 0$, it follows from Lemma 4.2 that

$$\operatorname{inv}(xy)(E) = 0 \text{ and } \operatorname{inv}(xz^2)(E) = (a, b, b) = (a, b, -1) = 0,$$

thus xy and xz^2 are always negligible over F. Assume that a and b are independent variables over F. Then, $(a, b) \neq 0$, thus xz is not negligible over F. Hence, it suffices to consider sums of terms of the form $y^f z^e$ with $f, e \geq 0$

Assume that $f, e \leq 1$. By Corollary 2.8, the class y is negligible over F if and only if $\mu_8 \subset F$. Hence, by Lemma 4.3, the class yz^e is negligible over F if and only if $\mu_8 \subset F$ for $e = 0, 1$. On the other hand, since $\deg(z) = 1$, the class z is not negligible over F.

Now assume that $f \geq 2$ or $e \geq 2$. Then, by Lemma 4.2 we obtain

$$\text{(6)} \qquad \operatorname{inv}(y^f z^e)(E) = \begin{cases} (b) \cdot (-1)^{e-1} & \text{if } f = 0,\ e \geq 2, \\ (r, b) \cdot (-1)^{e} & \text{if } f = 1,\ e \geq 2, \\ (r) \cdot (-1)^{2f-1} & \text{if } f \geq 2,\ e = 0, \\ (r, b) \cdot (-1)^{2f+e-2} & \text{if } f \geq 2,\ e \geq 1. \end{cases}$$

In particular, we have $\operatorname{inv}(yz^3)(E) = \operatorname{inv}(y^2 z)(E) = (r, b) \cdot (-1)^3$, thus the class $yz^3 + y^2 z$ is always negligible over F.

We obtain the following equivalences:

$$\begin{aligned} z^e \in H^*(C_4 \times C_2, \mathbb{F}_2)_{\mathrm{neg}, F} &\text{ if and only if } l(F) \leq e - 2, \\ yz^e \in H^*(C_4 \times C_2, \mathbb{F}_2)_{\mathrm{neg}, F} &\text{ if and only if } l(F) \leq e - 1 \text{ and } e \geq 2, \\ y^f \in H^*(C_4 \times C_2, \mathbb{F}_2)_{\mathrm{neg}, F} &\text{ if and only if } l(F) \leq 2f - 2 \text{ and } f \geq 2, \\ y^f z^e \in H^*(C_4 \times C_2, \mathbb{F}_2)_{\mathrm{neg}, F} &\text{ if and only if } l(F) \leq 2f + e - 3,\ f \geq 2,\ e \geq 1. \end{aligned}$$

The backward implications follow directly from equation (6) and Lemma 2.2. The forward implications follow by assuming that r and b are algebraically independent variables and applying the residue map at the r-adic and b-adic valuations.

Note that for any $d \geq 3$, the ideal $(yz^{d+1},\ y^{\lceil \frac{d+3}{2} \rceil},\ y^2z+yz^3)$ contains every class $y^f z^e$ with $f \geq 2$, $e \geq 1$, and $2f+e \geq d+3$. The claim now follows by a straightforward calculation. $\square$

5. Cyclic group C_8

Recall from (3) that the mod 2 cohomology ring of C_8 is given by

$$H^*(C_4, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^2),$$

where $\deg(x) = 1$ and $\deg(y) = 2$.

Theorem 5.1. *Let F be a field of level 2^l. Then,*

$$H^*(C_8, \mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (y) & \text{if } \ \mu_{16} \subset F, \\ (xy,\ y^2) & \text{if } \ \mu_{16} \not\subset F \text{ and } l = 0, \\ (xy,\ y^{\lceil \frac{l+3}{2} \rceil}) & \text{if } \ 1 \leq l < \infty, \\ (xy) & \text{if } \ l = \infty. \end{cases}$$

Proposition 5.2. *Let E/K be a C_8-extension. Then* $\mathrm{inv}(y^2)(E)$ *is divisible by* $(-1)^3 = (-1,-1,-1)$.

Proof. Let K_a/K be a quadratic subextension of E/K and let $j\colon \Gamma_F \to \mathrm{Gal}(E/K)$ be the corresponding homomorphism. Then, the extension E/K is of the form $E = ((K_a)_A)_B$ for some $A \in (K_a)^\times$ and $B \in ((K_a)_A)^\times$.

Let $i : \mathrm{Gal}(E/K_a) \to \mathrm{Gal}(E/K)$ be the embedding of C_4 into C_8 and let $k : \Gamma_{K_a} \to \mathrm{Gal}(E/K_a)$ be the restriction of j. Then, we have a commutative diagram

$$\begin{tikzcd} H^2(C_8, \mathbb{F}_2) \arrow[r, "j^*"] \arrow[d, "i^*=\mathrm{res}"] & H^2(K, \mathbb{F}_2) \arrow[d, "\mathrm{res}"] \\ H^2(C_4, \mathbb{F}_2) \arrow[r, "k^*"] & H^2(K_a, \mathbb{F}_2). \end{tikzcd}$$

Since the restriction map $i^* : H^2(C_8, \mathbb{F}_2) \to H^2(C_4, \mathbb{F}_2)$ is an isomorphism, it follows from Lemma 3.2 aplied to the C_4-extension E/K_a that

$$y_a := \mathrm{res}\big(j^*(y)\big) = k^*\big(i^*(y)\big) = (A, 2) + (R, -1) \in H^2(K_a, \mathbb{F}_2) \tag{7}$$

for some $R \in (K_a)^\times$. Then, we have

$$0 = 2j^*(y) = \mathrm{cor}(y_a) = (N_a(A), 2) + (N_a(R), -1). \tag{8}$$

Choose $z \in K_a$ such that $N_a(z) = -1$. Then, by the projection formula and (7), we have

$$j^*(y^2) = j^*(y) \cdot (-1,-1) = \mathrm{cor}\,\big(y_a \cdot (z, -1)\big) = \mathrm{cor}(A, 2, z, -1) + \mathrm{cor}(R, -1, z, -1).$$

Since $(A,-1)=0$ in $H^2(K_a,\mathbb{F}_2)$, we get $j^*(y^2)=\operatorname{cor}(R,z,-1,-1)$. Since $(2,-1)=0$, it follows from Lemma 5.3 below and (8) that

$$\begin{aligned}\operatorname{inv}(y^2)(E) &= j^*(y^2)\\ &= \operatorname{cor}(R,z,-1,-1)\\ &= (b,N_a(R),-1,-1)+(c,N_a(z),-1,-1)\\ &= (b,N_a(A),2,-1)+(c,-1,-1,-1)\\ &= (c,-1,-1,-1)\end{aligned}$$

for some $b,c\in K^\times$. □

Lemma 5.3. *For every $X,Y\in(K_a)^\times$, there exist $b,c\in K^\times$ such that in $H^2(K,\mathbb{F}_2)$*

$$\operatorname{cor}(X,Y)=(b,N_a(X))+(c,N_a(Y))$$

for some $b,c\in F^\times$.

Proof. Since $\dim_K K_a=2$, the elements $1,X,Y$ are K-linearly dependent. Hence, there exist $b,c\in K$, not both zero, such that $cX+bY=1$. Assume that both b,c are nonzero. Then, $(cX,bY)=0\in H^2(K_a,\mathbb{F}_2)$. Applying $\operatorname{cor}:H^2(K_a,\mathbb{F}_2)\to H^2(K,\mathbb{F}_2)$ to this relation yields the result. If $b=0$, then $X=c^{-1}$ and $\operatorname{cor}(X,Y)=(1,N_a(X))+(c,N_a(Y))$. □

Corollary 5.4. *For any $d\geq 2$, $\operatorname{inv}(y^d)(E)$ is divisible by $(-1)^{2d-1}$.*

Proof. By Proposition 5.2, the element

$$\operatorname{inv}(y^d)(E)=\operatorname{inv}(y)(E)\cdot(-1)^{2d-2}=\operatorname{inv}(y)(E)\cdot(-1,-1)\cdot(-1)^{2d-4}$$

is divisible by $(-1,-1,-1)\cdot(-1)^{2d-4}=(-1)^{2d-1}$. □

Corollary 5.5. *For any $d\geq 2$, the class y^d is negligible over F if and only if $(-1)^{2d-1}=0$.*

Proof. If $(-1)^{2d-1}=0$, then it follows from Corollary 5.4 that y^d is negligible over F. Conversely, if y^d is negligible over F, then Lemma 2.6 implies $(-1)^{2d-1}=0$. □

Proposition 5.6. *The class xy is negligible over F.*

Proof. Since $\operatorname{inv}(x)(E)=(a)$ and $(A,A)=(A,-1)=0$, by (7) we have

$$\operatorname{inv}(xy)(E)=(a)\cdot\operatorname{inv}(y)(E)=N_a((A)\cdot y_a)=N_a(A,A,2)+N_a(A,R,-1)=0.$$ □

Proof of Theorem 5.1. It follows from Corollary 2.8 that $y\in H^2(C_8,\mathbb{F}_2)_{\mathrm{neg},F}$ if and only if $\mu_{16}\subset F$. By Corollary 5.5 and Lemma 2.2, we have

$$y^d\in H^{2d}(C_8,\mathbb{F}_2)_{\mathrm{neg},F}\quad\text{if and only if}\quad l\leq 2d-2$$

for any $d\geq 2$. Hence, the statement follows from Proposition 5.6. □

6. Dihedral group D_8

Write $D_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^4 = 1\rangle$ for the dihedral group of order 8. By [2, Theorem 2.7] the cohomology ring is

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy),$$

where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. The degree 1 generators are the characters

$$x(\sigma) = 1,\ x(\tau) = 0, \qquad y(\sigma) = 0,\ y(\tau) = 1.$$

In the cohomology group $H^2(D_8, \mathbb{F}_2)$ there is a unique nonzero class z whose restrictions to the cyclic subgroups $\langle\sigma\rangle$ and $\langle\tau\rangle$ both vanish. This class represents the central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to D_{16} \to D_8 \to 1,$$

where D_{16} is the dihedral group of order 16 (see [2, Chapter IV, Theorem 2.7]).

Theorem 6.1. *Let F be a field of level 2^l. Then,*

$$H^*(D_8, \mathbb{F}_2)_{\mathrm{neg},\,F} = \begin{cases} (xz,\ yz,\ x^{l+2},\ y^{l+2},\ \ z^{[\frac{l+4}{2}]}) & \text{if } 0 \le l < \infty, \\ (xz,\ yz) & \text{if } l = \infty. \end{cases}$$

Lemma 6.2. [10, Example 4.3] *Let $F_{a,b}$ be a biquadratic field extension such that*

$$\sigma(\sqrt{a}) = -\sqrt{a}, \quad \sigma(\sqrt{b}) = \sqrt{b}, \quad \tau(\sqrt{a}) = \sqrt{a}, \quad \tau(\sqrt{b}) = -\sqrt{b}$$

The field $K_{a,b}$ can be embedded in a D_8-extension E/K if and only if $(a, b) = 0$ in $H^2(K, \mathbb{F}_2)$, i.e., $u^2 - av^2 = b$ for some $u, v \in K$. Every D_8-extension E over K is of the form

$$E = (K_{a,b})_A$$

for some $A \in (K_a)^\times$ such that $N_a(A) \in b \cdot (F^\times)^2$ and $\mathrm{Tr}_a(A) \neq 0$. Conversely, for every $a, b, \in K$ and $A \in K_a$ as above, $(K_{a,b})_A/K$ is a D_8-extension. Moreover, we have

$$\mathrm{inv}(x)(E) = (a), \quad \mathrm{inv}(y)(E) = (b), \quad \text{and} \quad \mathrm{inv}(z)(E) = (b, 2) + (-ab, 2\,\mathrm{Tr}_a(A)). \quad \square$$

Proof of Theorem 6.1. Since $(a, b) = 0$ and $(a, -a) = 0$, it follows from Lemma 6.2 below that

$$\mathrm{inv}(xz)(E) = (a) \cdot \big[(b, 2) + (-ab, 2\,\mathrm{Tr}_a(A))\big] = (a, b, 2) + (a, -ab, 2\,\mathrm{Tr}_a(A)) = 0,$$

thus xz is negligible over F. Symmetrically yz is negligible over F. Let a, r, u, v be algebraically independent variables over F and let $K = F(a, r, u, v)$. Set

$$b = u^2 - av^2 \in K^\times, \qquad A = r\big(u + v\sqrt{a}\big) \in K_a^\times$$

and consider the D_8-extension

$$E = \big(K_{a,b}\big)_A/K.$$

For every integer $d \ge 1$,

$$\mathrm{inv}(x^d)(E) = (a)^d = (a) \cdot (-1)^{d-1} \in H^d(K, \mathbb{F}_2).$$

Let v_a be the a-adic valuation on K with residue field $F(r,u,v)$. Applying the residue map

$$\partial_a\colon H^d(K,\mathbb{F}_2) \longrightarrow H^{d-1}\big(F(r,u,v),\mathbb{F}_2\big)$$

to $\operatorname{inv}(x^d)(E)$ shows that, if x^d is negligible over F, then $(-1)^{d-1}=0$ in $H^{d-1}\big(F(r,u,v),\mathbb{F}_2\big)$ and hence in $H^{d-1}(F,\mathbb{F}_2)$. Thus,

$$x^d \text{ is negligible over } F \iff (-1)^{d-1}=0 \text{ in } H^{d-1}(F,\mathbb{F}_2) \iff l\le d-2.$$

By a similar argument (interchanging a and b, x and y) we obtain

$$y^d \text{ is negligible over } F \iff (-1)^{d-1}=0 \text{ in } H^{d-1}(F,\mathbb{F}_2) \iff l\le d-2.$$

For every $d\ge 1$,

$$\operatorname{inv}(z^d)(E) \;=\; j(z)\cdot(-1)^{2d-2} \;=\; \big[(b,2)+(-ab,ru)\big]\cdot(-1)^{2d-2} \;\in\; H^{2d}(K,\mathbb{F}_2).$$

Assume z^d is negligible over F. Then,

$$\partial_a\big(\operatorname{inv}(z^d)(E)\big) \;=\; (ru)\cdot(-1)^{2d-2}=0 \quad \text{in } H^{2d-1}\big(F(r,u,v),\mathbb{F}_2\big),$$

and

$$\partial_r\big(\operatorname{inv}(z^d)(E)\big) \;=\; (-1)^{2d-2}=0 \quad \text{in } H^{2d-2}\big(F(u,v),\mathbb{F}_2\big),$$

hence $(-1)^{2d-2}=0$ in $H^{2d-2}(F,\mathbb{F}_2)$. Therefore,

$$z^d \text{ is negligible over } F \iff (-1)^{2d-2}=0 \text{ in } H^{2d-2}(F,\mathbb{F}_2). \quad \square$$

7. Quaternion group Q_8

By [2, Lemma 2.10], the mod 2 cohomology ring of the quaternion group Q_8 of order 8 is given by

$$H^*(Q_8,\mathbb{F}_2)=\mathbb{F}_2[x,y,z]/(x^2+y^2+xy,\,x^2y+xy^2), \tag{9}$$

where $\deg(x)=\deg(y)=1$ and $\deg(z)=4$.

Theorem 7.1. *Let F be a field of level 2^l. Then,*

$$H^*(Q_8,\mathbb{F}_2)_{\mathrm{neg},\,F}=\begin{cases}(x^2,\,y^2,\,x^2y,\,z) & \text{if } l=0,\\ (x^2y,\,z) & \text{if } l=1,\,2,\\ (x^2y,\,xz,\,yz,\,z^{[\frac{l+5}{4}]}) & \text{if } 3\le l<\infty,\\ (x^2y,\,xz,\,yz) & \text{if } l=\infty.\end{cases}$$

Lemma 7.2. [7, Theorem 4] *Let L/K be a $C_2\times C_2$-extension, i.e. $L=K_{a,b}$ for some $a,b\in K^\times$. Then, the following statements are equivalent :*

(1) *The extension L/F can be embedded in a Q_8-extension E/K.*

(2) *$\langle a,b,ab\rangle\simeq\langle 1,1,1\rangle$ over K.*

(3) *$(-a,-b)=(-1,-1)$ in $H^2(K,\mathbb{F}_2)$.*

(4) *$a=u^2+v^2+w^2$, $b=p^2+q^2+s^2$, and $up+vq+ws=0$ for some $u,v,w,p,q,s\in K$.*

In this case, every Q_8-extension E/F is of the form

$$E = (K_{a,b})_{rB}, \qquad B = 1 + \frac{u}{\sqrt{a}} + \frac{s}{\sqrt{b}} + \frac{us - wp}{\sqrt{ab}} \tag{10}$$

for some $r \in K^\times$. Conversely, for every $a, b, u, v, w, p, q, s \in K$ as above, $(K_{a,b})_{rB}/K$ is a Q_8-extension for every $r \in K^\times$. □

Lemma 7.3. *Consider the hypersurface*

$$X \;=\; \left\{(u, v, w, p, q, s) \in \mathbb{A}^6_F \mid up + vq + ws = 0\right\} \subset \mathbb{A}^6_F.$$

Let $A = (u^2 + v^2 + w^2, -1) \in H^2(F(X), \mathbb{F}_2)$. Then,

$$A = 0 \iff -1 \in (F^\times)^2.$$

Proof. Let $\pi : X \to \mathbb{A}^3_F$ be the projection $(u, v, w, p, q, s) \mapsto (p, q, s)$ and set $L = F(p, q, s)$. Let $Y \subset X$ be the hypersurface defined by $u^2 + v^2 + w^2 = 0$. The generic fiber of $\pi|_Y$ over $\operatorname{Spec} L$ is a smooth, geometrically integral conic curve, so L is algebraically closed in the function field $F(Y)$. Since L/F is purely transcendental, F is also algebraically closed in $F(Y)$; hence

$$-1 \in (F(Y)^\times)^2 \iff -1 \in (F^\times)^2. \tag{11}$$

The residue map

$$\partial_Y : H^2(F(X), \mathbb{F}_2) \longrightarrow H^1(F(Y), \mathbb{F}_2) = F(Y)^\times/(F(Y)^\times)^2$$

gives $\partial_Y(A) = (-1)$. By (11) this residue vanishes exactly when -1 is a square in F, proving the equivalence. □

Lemma 7.4. *Let z be the generator of $H^4(Q_8, \mathbb{F}_2)$. Then for every Q_8-extension E/K we have*

$$\operatorname{inv}(z)(E) = w_4(q_E) + (2) \cdot w_3(q_E),$$

where q_E is the trace form of E/K.

Proof. Consider the embedding $\eta : Q_8 \hookrightarrow S_8$ such that the generator of the center of Q_8 maps to $\varepsilon := (12)(34)(56)(78)$. We have the following commutative diagram:

$$\begin{tikzcd}
Q_8 \arrow[d, hook, "\eta"] & S_2 = \langle \varepsilon \rangle \arrow[l, hook', "\kappa"'] \arrow[r, "\Delta_2"] \arrow[d, "(\iota,\iota)"] & S_2 \times S_2 \arrow[d, hook, "\theta"] \\
S_8 & S_4 \times S_4 \arrow[l, hook', "\psi"'] & S_4, \arrow[l, "\Delta_4"']
\end{tikzcd}$$

where $\iota : S_2 \to S_4$ satisfies $\iota(\varepsilon) = (12)(34)$, Δ_2, Δ_4 denote the diagonal maps and κ, ψ, θ are the natural embeddings.

By [1, p.402], there are elements $\sigma_4 \in H^4(S_8, \mathbb{F}_2)$ and $\sigma_2 \in H^2(S_4, \mathbb{F}_2)$ such that

$$\psi^*(\sigma_4) = \sigma_2 \otimes \sigma_2.$$

Clearly, $\Delta_4^*(\sigma_2\otimes\sigma_2)=\sigma_2^2$. By [2, Example 1.13, p. 185], we have $\theta^*(\sigma_2)=x\otimes x$ for $x\in H^1(S_2,\mathbb{F}_2)=\mathbb{F}_2\cdot x$, thus

$$(\theta\circ\Delta_2)^*(\sigma_2)=\Delta_2^*(x\otimes x)=x^2\in H^2(S_2,\mathbb{F}_2)=\mathbb{F}_2\cdot x^2.$$

Hence, $(\eta\circ\kappa)^*(\sigma_4)=x^4$.

Recall that $H^4(Q_8)=\mathbb{F}_2\cdot z$ and z is not nilpotent. Since $\langle\varepsilon\rangle\simeq S_2$ is the onl elementary abelian 2-subgroup of Q_8, it follows from [13] that $\kappa^*(z)\neq 0$, hence $\kappa^*(z)=x^4$. It follows that

$$\eta^*(\sigma_4)=z, \tag{12}$$

i.e., the invariant $\mathrm{inv}(z)$ of Q_8 extends to the invariant $\mathrm{inv}(\sigma_4)$ of S_8.

The pullback map for the composition

$$S_2\times S_2\times S_2\times S_2\xrightarrow{\theta\times\theta}S_4\times S_4\xrightarrow{\psi}S_8,$$

takes σ_4 to $x\otimes x\otimes x\otimes x$. It follows that for every field extension K/F and any elements $a,b,c,d\in K^\times$,

$$\mathrm{inv}(\sigma_4)(E)=(a,b,c,d)\in H^4(K,\mathbb{F}_2),$$

where $E=K_a\times K_b\times K_c\times K_d$. By Proposition 2.1, $w_4^{\mathrm{gal}}(E)$ is also equal to (a,b,c,d). Hence, the two invariants $\mathrm{inv}(\sigma_4)$ and w_4^{gal} of S_8 coincide on multi-quadratic étale algebras of degree 8, thus by [4, Part I, Theorem 24.9], they are equal: $\mathrm{inv}(\sigma_4)=w_4^{\mathrm{gal}}$. Restricting to Q_8 with respect to η and taking into account (2) and (12) we get

$$\mathrm{inv}(z)(E)=\mathrm{inv}(\sigma_4)(E)=w_4^{\mathrm{gal}}(E)=w_4(q_E)+(2)\cdot w_3(q_E)$$

for all Q_8-extensions E/K. □

Lemma 7.5. *Let $L=K_a$ be a quadratic extension over K. If L_A is a cyclic quartic extension over K with $A\in L^\times$ such that $Tr_{L/K}(A)\neq 0$, then*

$$\mathrm{Tr}_{L_A/K}\langle 1\rangle\simeq\langle 1,\ a,\ 2\,\mathrm{Tr}_{L/K}(A),\ 2\,\mathrm{Tr}_{L/K}(A)\rangle.$$

Proof. Let σ be a generator of $\mathrm{Gal}(L_A/K)$ and let $x=\sqrt{A}\cdot\sigma(\sqrt{A})$. Since x is fixed by σ^2, we have $x\in L$. Since

$$-\sqrt{A}=\sigma^2(\sqrt{A})=\sigma(x)/\sigma(\sqrt{A})=\sigma(x)\sqrt{A}/x$$

we get $\sigma(x)=-x$, thus $x=b\sqrt{a}$ for some $b\in K^\times$. Hence,

$$N_{L/K}(A)=A\cdot\sigma(A)=x^2=b^2a. \tag{13}$$

By [9, Theorem 6.18] and (13), we get

$$\begin{aligned}\mathrm{Tr}_{L_A/K}\langle 1\rangle&\simeq\langle 1,\ a,\ 2\,\mathrm{Tr}_{L/K}(A),\ N_{L/K}(A)\cdot a\cdot 2\,\mathrm{Tr}_{L/K}(A)\rangle\\&\simeq\langle 1,\ a,\ 2\,\mathrm{Tr}_{L/K}(A),\ 2\,\mathrm{Tr}_{L/K}(A)\rangle.\end{aligned}$$

□

Proposition 7.6. *Let E/K be a Q_8-extension as in* (10)*. Then,*

$$\mathrm{inv}(z)(E) = (r,\ -1,\ -1,\ -1) \in H^4(K, \mathbb{F}_2).$$

Proof. Let $L = K_a$. Then, $E = (L_b)_{rB}/L$ is a cyclic quartic extension. Since $\mathrm{Tr}_{L/K}(rB) = 2r\big(1 + \frac{u}{\sqrt{a}}\big)$, it follows from Lemma 7.5 that

$$\mathrm{Tr}_{E/L}\langle 1\rangle = \langle 1,\ b,\ r\big(1 + \frac{u}{\sqrt{a}}\big),\ r\big(1 + \frac{u}{\sqrt{a}}\big)\rangle.$$

Since $N_{L/K}(1 + \frac{u}{\sqrt{a}}) = \frac{a-u^2}{a} = \frac{v^2+w^2}{a}$, by [9, Lemma 6.17] we get

$$\mathrm{Tr}_{L/K}\langle 1, b\rangle = \langle 2, 2a, 2b, 2ab\rangle \text{ and } \mathrm{Tr}_{L/K}\langle r\big(1 + \frac{u}{\sqrt{a}}\big)\rangle = \langle 2r, 2r(u^2 + w^2)\rangle.$$

Therefore, by Lemma 7.2 (2), we have

$$\begin{aligned} q_E = \mathrm{Tr}_{E/K}\langle 1\rangle &= \langle 2, 2a, 2b, 2ab\rangle \perp 2r\langle 1, v^2 + w^2, 1, v^2 + w^2\rangle \\ &= 2\langle\langle -a, -b\rangle\rangle \perp 2r\langle\langle -1, -v^2 - w^2\rangle\rangle \\ &= 2\langle\langle -1, -1\rangle\rangle \perp 2r\langle\langle -1, -1\rangle\rangle \\ &= 2\langle\langle -r, -1, -1\rangle\rangle = \langle\langle -r, -1, -1\rangle\rangle \end{aligned}$$

as $2\langle\langle -1, -1\rangle\rangle = \langle\langle -1, -1\rangle\rangle$.

Since $w_4(E) = w_4(q_E) = (r, r, r, r) = (r, -1, -1, -1)$ and $w_3(E) = 0$, it follows from Lemma 7.4 that

$$\mathrm{inv}(z)(E) = w_4(E) + (2) \cdot w_3(E) = (r, -1, -1, -1). \qquad \square$$

Proof of Theorem 7.1. Let E/K be a Q_8-extension as in (10). By Proposition 7.6 we have

$$\mathrm{inv}(x)(E) = (a), \quad \mathrm{inv}(y)(E) = (b), \quad \text{and} \quad \mathrm{inv}(z)(E) = (r, -1, -1, -1).$$

Since

$$\mathrm{inv}(x^2)(E) = (a, a) = (a, -1) \text{ and } \mathrm{inv}(y^2)(E) = (b, b) = (b, -1)$$

in $H^2(F, \mathbb{F}_2)$, it follows from Lemmas 7.2 and 7.3 that

$$x^2 \in H^*(Q_8, \mathbb{F}_2)_{\mathrm{neg},\,F} \ \Leftrightarrow\ y^2 \in H^*(Q_8, \mathbb{F}_2)_{\mathrm{neg},\,F} \ \Leftrightarrow\ l(F) = 0.$$

Since a and b are the sums of three squares by Lemma 7.2 (4), we have

$$(a, -1, -1) = (b, -1, -1) = 0, \tag{14}$$

thus it follows from Lemma 7.2 (2) that

$$\mathrm{inv}(xy^2)(E) = (a, b, b) = (a, b, -1) = (ab, -1, -1) = (a, -1, -1) + (b, -1, -1) = 0,$$

i.e., $xy^2 \in H^*(Q_8, \mathbb{F}_2)_{\mathrm{neg},\,F}$. By (14), we have

$$\mathrm{inv}(xz)(E) = (r, a, -1, -1, -1) = 0 \text{ and } \mathrm{inv}(yz)(E) = (r, b, -1, -1, -1) = 0,$$

i.e., $xz,\ yz \in H^*(Q_8, \mathbb{F}_2)_{\mathrm{neg},\,F}$. Therefore, the elements

$$xz^d,\ yz^d,\ x^2z^d,\ y^2z^d$$

are always negligible over F for any $d \geq 1$.

For any $d \geq 1$, we have $\mathrm{inv}(z^d)(E) = (r) \cdot (-1)^{4d-1}$, thus

$$z^d \in H^*(Q_8, \mathbb{F}_2)_{\mathrm{neg},\,F} \text{ if and only if } l(F) \leq 4d-2.$$

Hence, the statement follows. □

References

[1] A. Adem, J. Maginnis, and R. J. Milgram, *Symmetric invariants and cohomology of groups*, Math. Ann. **287** (1990), no. 3, 391–411.
[2] A. Adem and R. J. Milgram, *Cohomology of Finite Groups*, Springer–Verlag Grundlehren **309** (2004).
[3] L. Evens, *The cohomology of groups*, Oxford University Press (1991).
[4] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, American Mathematical Society, Providence, RI, (2003).
[5] M. Gherman and A. Merkurjev, *negligible degree two cohomology of finite groups*, J. Algebra **611** (2022), 82–93.
[6] M. Gherman and A. Merkurjev, *Krull dimension of the negligible quotient in* $\bmod p$ *cohomology of a finite group*, J. Pure Appl. Algebra **228** (2024), no. 3, Paper No. 107489, 8.
[7] I. Kiming, *Explicit classifications of some* 2*-extensions of a field of characteristic different from* 2, Canad. J. Math. **42** (1990), no. 5, 825–855.
[8] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tit.
[9] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.
[10] A. Ledet, *On* 2*-groups as Galois groups*, Canad. J. Math. **47** (1995), no. 6, 1253–1273.
[11] A. Merkurjev and F. Scavia, *Galois representations modulo* p *that do not lift modulo* p^2, To appear in JAMS.
[12] J.-P. Serre, *Oeuvres/Collected papers. III. 1972–1984*, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013.
[13] D. Quillen and B. B. Venkov. *Cohomology of finite groups and elementary abelian subgroups*, Topology **11** (1972), 317–318.

(Sanghoon Baek) Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-701, Republic of Korea
Email address: sanghoonbaek@kaist.ac.kr
URL: https://mathsci.kaist.ac.kr/~sbaek/

(Alexander S. Merkurjev) Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
Email address: merkurev@math.ucla.edu
URL: https://www.math.ucla.edu/~merkurev/