SUSLIN’S CONJECTURE ON THE REDUCED WHITEHEAD GROUP OF A SIMPLE ALGEBRA

ALEXANDER MERKURJEV

Abstract. In 1991, A. Suslin conjectured that if the index of a central simple algebra $A$ is not square-free, then the reduced Whitehead group of $A$ is nontrivial generically. We prove this conjecture in the present paper.

1. Introduction

Let $A$ be a central simple algebra over a field $F$. The reduced norm homomorphism $A^\times \to F^\times$ yields a homomorphism

$$\text{Nrd} : K_1(A) \to F^\times = K_1(F).$$

The kernel $SK_1(A)$ of Nrd is the reduced Whitehead group of $A$. Wang proved in [25] that if $\text{ind}(A)$ is a square-free integer, then $SK_1(A) = 0$. He also proved that the reduced Whitehead group is always trivial if $F$ is a number field.

Platonov found examples of $A$ with nontrivial $SK_1(A)$ (see [17]).

In 1991, Suslin conjectured in [23] that if $\text{ind}(A)$ is not square-free, then the reduced Whitehead group $SK_1(A)$ of $A$ is generically nontrivial, i.e., there is a field extension $L/F$ such that $SK_1(A \otimes_F L) \neq 0$.

Suslin’s Conjecture was proved in the case when $\text{ind}(A)$ is divisible by 4 (see [13] and [15]).

In this paper we prove Suslin’s Conjecture (Theorem 8.1):

**Theorem.** Let $A$ be a central simple $F$-algebra. If $\text{ind}(A)$ is not square-free, then there is a field extension $L/F$ such that $SK_1(A \otimes_F L) \neq 0$.

Note that the group $SK_1(A)$ coincides with the group of $R$-equivalence classes in the special linear group $\text{SL}_1(A)$. In particular, generic non-triviality of the reduced Whitehead group of $A$ implies that $\text{SL}_1(A)$ is not a retract rational variety (Corollary 8.2).

2. Cycle modules and spectral sequences

Let $Z$ be a variety over a field $F$ and let $M_n$ be a cycle module over $Z$ (see [20, §2]). This is a collection of group $M_n(z)$ for $n \in \mathbb{Z}$ and a point $z : L \to Z$.
over \( F \) having certain compatibility properties. We write \( K_* \) for the cycle module over \( \text{Spec} \ F \) given by (Quillen’s) \( K \)-groups (see [20, Remark 2.5]).

For every integer \( r \geq 0 \), denote by \( Z^{(r)} \) the set of points of \( Z \) of codimension \( r \). We write \( A^r(Z, M_n) \) for the homology group of the complex [20, §5]

\[
\prod_{z \in Z^{(r-1)}} M_{n-r+1}F(z) \xrightarrow{\partial} \prod_{z \in Z^{(r)}} M_{n-r}F(z) \xrightarrow{\partial} \prod_{z \in Z^{(r+1)}} M_{n-r-1}F(z).
\]

For example, \( A^r(Z, K_r) \) is the Chow group \( \text{CH}^r(Z) \) of classes of codimension \( r \) algebraic cycles on \( Z \). If \( Z \) is smooth, \( A^r(Z, K_*) \) is a bi-graded commutative ring.

If \( f : Y \to Z \) is a flat morphism of equidimensional varieties and \( M \) a cycle module over \( Y \), for every \( n \in \mathbb{Z} \), there is a spectral sequence [20, Corollary 8.2]

\[
E_1^{r,s}(f, n) = \prod_{z \in Z^{(r)}} A^s(f^{-1}(z), M_{n-r}) \Rightarrow A^{r+s}(Y, M),
\]

where \( f^{-1}(z) \) is the fiber of \( f \) over \( z \in Z \).

Very often we will be considering the projections \( f : W \times Z \to Z \) with \( Z \) and \( W \) smooth varieties. In this case \( f^{-1}(z) = W_{f(z)} \). The associated spectral sequences have the following functorial properties. A morphism \( h : W \to W' \) of smooth varieties yields a pull-back morphism of spectral sequences

\[
h^* : E_\ast^{r,s}(f' , n) \to E_\ast^{r,s}(f, n)
\]

for every \( n \) (here \( f' : W' \times Z \to Z \) is the projection). If \( h \) is a closed embedding of codimension \( c \), we have a push-forward morphism of spectral sequences

\[
h_* : E_\ast^{r,s}(f, n) \to E_\ast^{r,s+c}(f', n + c).
\]

More generally, every correspondence \( \lambda \) between \( W \) and \( W' \) of degree \( d \) (see [3, §63]) yields a morphism

\[
\lambda^* : h^* : E_\ast^{r,s}(f' , n) \to E_\ast^{r,s-d}(f, n - d).
\]

This is because the four basic maps of complexes of \( W \times Z \) and \( W' \times Z \) respect the filtration when projected to \( Z \) (see [20, §3]).

3. Chern classes

Let \( X \) be a smooth variety. There are Chern classes (see [7]):

\[
c_{i,n} : K_n(X) \to A^{i-n}(X, K_i)
\]

for \( i \geq n \geq 0 \). We will only need the classes

\[
c_i := c_{i+1,1} : K_1(X) \to A^i(X, K_{i+1}).
\]

There is the following product formula (see [21]):

**Proposition 3.1.** If \( x \in K_0(X) \) is the class of a line bundle \( L \) and \( y \in K_1(X) \), we have

\[
c_i(xy) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} h^j \cup c_{i-j}(y),
\]
where \( h \) is the first (classical) Chern class of \( L \) in \( A^1(X, K_1) = \text{CH}^1(X) \).

Let \( E \to X \) be a vector bundle of rank \( n \) and \( \text{SL}(E) \) the group scheme over \( Z \) of determinant 1 automorphisms of \( E \). We will be using the following result due to Suslin [22, Th. 4.2].

**Proposition 3.2.** If \( X \) is a smooth variety, the ring \( A^*(\text{SL}(E), K_n) \) is almost exterior algebra over \( A^*(X, K_1) \) with generators \( c_1(\beta), c_2(\beta), \ldots, c_{n-1}(\beta) \), where \( \beta \in K_1(\text{SL}(E)) \) is the generic element. In particular,

\[
\text{CH}(\text{SL}(E)) \simeq \text{CH}(X).
\]

4. **Severi-Brauer varieties**

Let \( A \) be a central simple algebra of degree \( n \) over \( F \), \( X = \text{SB}(A) \) the Severi-Brauer variety of rank \( n \) right ideals of \( A \). If \( A \) is split, i.e., \( A = \text{End}(V) \) for a vector space of dimension \( n \), the variety \( X \) is isomorphic to the projective space \( \mathbb{P}(V) \).

The variety \( X \) has a point over a field extension \( L/F \) if and only if \( A \) is split over \( L \), i.e., \( A_L := A \otimes_F L \simeq M_n(L) \).

Write \( h \) for the class of a hyperplane section in \( \text{CH}^i(\mathbb{P}^{n-1}) \). The Chow group \( \text{CH}^i(\mathbb{P}^{n-1}) \) for \( i = 0, 1, \ldots, n-1 \) is infinite cyclic generated by \( h^i \).

In the general case, the kernel of the *degree* homomorphism

\[
\deg : \text{CH}^i(X) \to \text{CH}^i(X_{\text{sep}}) = \mathbb{Z}h^i
\]
coincides with the torsion part of \( \text{CH}^i(X) \). The group \( \text{CH}_0(X) \) is torsion free (see [16] or [1, Corollary 7.3]). Therefore, the classes in \( \text{CH}_0(X) \) of every two points of the same degree are equal.

If \( A = M_n(B) \) for a central simple algebra \( B \) over \( F \) and \( S = \text{SB}(B) \), then \( S \) is a closed subvariety of \( X = \text{SB}(A) \). Moreover, the Chow motive \( M(X) \) of \( X \) is isomorphic to the direct sum \( M(S) \oplus M(S)\{k\} \oplus \cdots \oplus M(S)\{(m-1)k\} \), where \( k = n/m \).

Let \( I \to X \) be the *tautological* rank \( n \) vector bundle. The fiber of this bundle over a right ideal in \( A \), a point of \( X \), is the ideal itself. In the split case \( A = \text{End}(V) \), where \( V \) is a vector space of dimension \( n \), a line \( l \subset V \) as a point of \( X = \mathbb{P}(V) \) corresponds to the right ideal \( \text{Hom}(V, l) = V^\vee \otimes l \). Therefore, \( I = V^\vee \otimes L_t \), where \( L_t \) is the tautological line bundle over \( \mathbb{P}(V) \).

The *canonical* bundle \( J \) over \( X \), the dual of \( I \), is equal then to \( V \otimes L_c \), where \( L_c \) is the canonical line bundle, dual of \( L_t \). We have in the split case

\[
X \times X = X \times \mathbb{P}(V) = \mathbb{P}_X(V) = \mathbb{P}_X(V \otimes L_c) = \mathbb{P}_X(J).
\]

Note that the projective linear group \( \text{PGL}(V) \) acts on \( \mathbb{P}(V) \) and the vector bundles \( I \) and \( J \). In the general case, twisting by the \( \text{PGL}(V) \)-torsor corresponding to the algebra \( A \), we get an isomorphism

\[
X \times X \simeq \mathbb{P}_X(J),
\]

i.e., \( X \times X \) is a projective vector bundle of \( J \) over \( X \) (with respect to the first of the two projections \( q_1, q_2 : X \times X \to X \)).
The tautological line bundle $\mathcal{L}$ over $X \times X = \mathbb{P}_X(J)$ is the sub-bundle $q_1^*(L_1) \otimes q_2^*(L_2)$ of the bundle $q_1^*(J) = V \otimes q_2^*(L_2)$ in the split case. Therefore,

\begin{equation}
\mathcal{L}_c = q_1^*(L_c) \otimes q_2^*(L_t),
\end{equation}

where $\mathcal{L}_c$ is the canonical bundle over over $X \times X$.

**Lemma 4.2.** Let $x \in X$ be a closed point. Then the push-forward homomorphism $Z = \text{CH}(X_{F(x)}) \to \text{CH}(X \times X)$ for the closed embedding $i : X_{F(x)} = X \times \text{Spec} F(x) \hookrightarrow X \times X$

depends only on the degree of $x$.

**Proof.** The canonical line bundle $L$ over the projective space is the pull-back of the canonical bundle $\mathcal{L}$ on $X \times X$. Hence the class $h_1 = c_1(L)$ is equal to $i^*(h)$, where $h = c_1(\mathcal{L})$. By the projection formula,

\[i_*(h_1^{\text{t}}) = i_*(i^*(h^\text{t})) = i_*(1) \cdot h^\text{t} = [X_{F(x)}] \cdot h^\text{t}.\]

The class of $X_{F(x)}$ in $\text{CH}(X \times X)$ is the image of $[X] \times [x]$ under the exterior product map

\[\text{CH}(X) \otimes \text{CH}_0(X) \to \text{CH}(X \times X).\]

Finally, the class of $x$ in $\text{CH}_0(X)$ depends only on the degree of $x$. \qed

Choose a splitting field extension $L/F$ of the smallest degree $\text{ind}(A)$. We have $X_L \simeq \mathbb{P}^{n-1}_L$. Let $l_i \in \text{CH}_i(X_L)$ be the class of a projective linear subspace of dimension $i$ and $e_i = e_i(A)$ the image of $l_i$ under the norm homomorphism

\[N_{L/F} : \text{CH}_i(X_L) \to \text{CH}_i(X).\]

Then $e_i$ is independent of the choice of $L$. Indeed, choose a closed point $x \in X$ such that $F(x) \simeq L$. Then $e_i$ is the image of $l_i$ under the composition

\[\text{CH}_i(X_L) = \text{CH}_i(X_{F(x)}) \to \text{CH}_i(X \times X) \to \text{CH}_i(X),\]

where the last map is induced by the first projection. By Lemma 4.2, the composition does not depend on the choice of $x$.

The proof of Lemma 4.2 shows that for every closed point $x \in X$, we have

\begin{equation}
N_{F(x)/F}(l_i) = \frac{\text{deg}(x)}{\text{ind}(A)} e_i(A).
\end{equation}

**Lemma 4.4.** If $K/F$ is a finite extension, then

\[N_{K/F}(e_i(A_K)) = \frac{|K : F| \text{ind}(A_K)}{\text{ind}(A)} e_i(A).\]

**Proof.** Let $L/K$ be a splitting field of $A_K$ of degree $\text{ind}(A_K)$. Choose an $L$-point $\text{Spec}(L) \to X$. Let $\{x\}$ be the image of this morphism. We have by

\begin{equation}
N_{K/F}(e_i(A_K)) = N_{L/F}(e_i(A_L)) = \frac{|L : F(x)| \cdot N_{F(x)/F}(e_i(A_L))}{\text{ind}(A)} e_i(A) = \frac{|K : F| \text{ind}(A_K)}{\text{ind}(A)} e_i(A). \quad \Box
\end{equation}
Proposition 4.6. Let $p$ be a prime integer and $A$ a central simple $F$-algebra of $p$-primary degree, $X = \text{SB}(A)$ the Severi-Brauer variety of $A$. Then $\text{CH}_i(X) = \mathbb{Z}e_i$ for $i = 0, 1, \ldots, p - 2$. In particular, these groups have no torsion.

Proof. If $D$ is a division algebra Brauer equivalent to $A$, the Severi-Brauer variety $Y = \text{SB}(D)$ is a closed subscheme of $X$. The push-forward map $\text{CH}_i(Y) \to \text{CH}_i(X)$ is an isomorphism for $i \leq \dim(Y)$ taking $e_i(D)$ to $e_i(A)$. Thus, in the proof of the proposition it suffices to assume that $A$ is a division algebra.

We prove the proposition by induction on $\text{ind}(A)$. The case $\text{ind}(A) = p$ was considered in [12, Corollary 8.7.2]. A standard restriction-corestriction argument reduces the proof to the case when $F$ is a $p$-special field, i.e., the degree of every finite field extension of $F$ is a power of $p$.

Let $A$ be a central division algebra of $p$-primary degree $n$ and $L \subset A$ a maximal subfield (of degree $n$ over $F$). The torus $T = R_{L/F}(\mathbb{G}_m)/\mathbb{G}_m$ acts naturally on $X$ making $X$ a toric variety. Write $U$ for the open $T$-invariant orbit and $Z$ for $X \setminus U$. Thus, $U$ is a $T$-torsor over $\text{Spec}(F)$.

Conversely, let $U$ be a $T$-torsor over $\text{Spec}(F)$ and let $A$ be a central simple algebra degree $n$ over $F$ with class in the relative Brauer group $\text{Br}(L/F) = H^1(F, T)$ corresponding to the class of $U$. Then $U$ is the open orbit of the $T$-action on $\text{SB}(A)$.

In the split case, $X = \mathbb{P}^{n-1}$ and $T$ is the torus of invertible diagonal matrices modulo the scalar matrices. Then $U$ consists of all points in $\mathbb{P}^{n-1}$ with all coordinates $\neq 0$. The $T$-orbits are the subsets in $\mathbb{P}^{n-1}$ with zeros on the fixed set of coordinates.

Let $\Sigma$ be the set of all $n$ primitive idempotents of $L \otimes_F F_{\text{sep}} = F_{\text{sep}} \times \cdots \times F_{\text{sep}}$. Every $\sigma \in \Sigma$ yields a co-character $\chi_{\sigma} : \mathbb{G}_{m, F_{\text{sep}}} \to T_{\text{sep}}$ which belongs to an edge (1-dimensional cone) in the fan of the toric variety $X_{\text{sep}}$. Moreover, the correspondence $\sigma \mapsto \chi_{\sigma}$ yields a bijection between the set of nonempty subsets in $\Sigma$ and the set of cones in the fan (or the set of $T$-orbits in $X_{\text{sep}}$). The absolute Galois group $\Gamma = \text{Gal}(F_{\text{sep}}/F)$ of $F$ acts transitively on the set $\Sigma$.

Lemma 4.7. We have $\text{CH}_i(U) = 0$ for $i = 0, 1, \ldots, p - 2$.

Proof. If $\text{ind}(A) = p$, every cycle $c$ in $\text{CH}_i(U)$ comes by restriction from $\text{CH}_i(X) = p\mathbb{Z}$ and therefore, by the norm, comes from $\text{CH}_i(X_L)$. Hence $c$ comes by the norm from $\text{CH}_i(U_L)$. But $U_L \simeq T_L$, hence $\text{CH}_i(U_L) = 0$.

In the general case, since $F$ is a $p$-special field, there is a subfield $K \subset L$ of degree $p$ over $F$. Consider the subtorus $S := R_{K/F}(\mathbb{G}_m)/\mathbb{G}_m$ of $T$, the $S$-torsor

$$f : U \to X := U/S$$

and Rost’s spectral sequence for $f$ converging to $\text{CH}_i(U)$. On the zero diagonal, we have the groups $\prod_{x \in X_{(j)}} \text{CH}_k(f^{-1}(x))$ with $j + k = i$. Note that $f^{-1}(x)$ is an $S$-torsor over $\text{Spec} F(x)$. Since $k \leq i \leq p - 2$, by the first part of the proof, $\text{CH}_k(f^{-1}(x)) = 0$. \qed
The $T$-orbits in $Z_{\text{sep}}$ correspond to proper subsets of the set of $\Sigma$. No such subset is fixed by $\Gamma$, hence no orbit in $Z_{\text{sep}}$ is fixed by $\Gamma$.

We have a sequence of closed $T$-invariant subsets
\begin{equation}
Z = Z_0 \supset Z_1 \supset \cdots \supset Z_m \supset Z_{m+1} = \emptyset
\end{equation}
such that every variety $(Z_j \setminus Z_{j+1})_{\text{sep}}$ is the disjoint union of $T$-orbits of the same dimension which are permuted by $\Gamma$. It follows that each $Z_j \setminus Z_{j+1}$ is a disjoint union of varieties defined over finite separable field extensions $K/F$ corresponding to the stabilizers $\Gamma' \subset \Gamma$ of $T$-orbits. The group $\Gamma'$ does not act transitively on the set $\Sigma$, hence $L \otimes_F K$ is not a field and therefore, $A_K$ is not a division algebra, i.e., $\text{ind}(A_K) < \text{ind}(A)$.

If $W$ is a scheme over a finite separable field extension $K/F$, the norm map $\text{CH}(W \otimes_F K) \to \text{CH}(W)$ is surjective, since $K$ is a direct factor of $K \otimes_K F$.

Fix an integer $i = 0, 1, \ldots, p - 2$. We say that a variety $W$ over $F$ satisfies the condition (⋆) if $\text{CH}_i(W)$ is generated by the images of the norm maps $\text{CH}_i(W) \to \text{CH}_i(W)$ over finite field extensions $K/F$ with $\text{ind}(A_K) < \text{ind}(A)$. We have proved that all the differences $Z_j \setminus Z_{j+1}$ satisfy (⋆).

Let $W'$ be a closed subvariety of $W$. The exactness of the localization sequence
\[
\text{CH}_i(W') \to \text{CH}_i(W) \to \text{CH}_i(W \setminus W') \to 0
\]
shows that if $W'$ and $W \setminus W'$ satisfy (⋆), then so does $W$. It follows from (4.8) that $Z$ satisfies (⋆). By Lemma 4.7, $U$ satisfies (⋆), hence so does $X$.

By the induction hypothesis, $\text{CH}_i(X_K)$ for $K$ as above, is generated by $e_i(A_K)$. By Lemma 4.4, $\text{CH}_i(X)$ is generated by $e_i(A)$.

**Corollary 4.9.** The degree map $\text{CH}_i(X) \to \text{CH}_i(X_{\text{sep}}) = \mathbb{Z}l_i$ is injective, it takes $e_i$ to $\text{ind}(A)l_i$. Thus, $\text{CH}_i(X)$ is identified with the subgroup $\text{ind}(A)\mathbb{Z}l_i$ in $\mathbb{Z}l_i$.

By the Projective Bundle Theorem, for every $j \geq 0$, we have
\[
\text{CH}^{d-j}(X \times X) \simeq \text{CH}^{d-j}(X) \oplus \text{CH}^{d-j-1}(X)h \oplus \cdots \oplus \text{CH}^0(X)h^{d-j},
\]
where $h \in \text{CH}^1(X \times X)$ is the first Chern class of the canonical line bundle $\mathcal{L}_e$ over $X \times X$. The element $\lambda_j := h^{d-j}$ can be viewed as a degree $j$ correspondence from $X$ to itself and hence $\lambda_j$ yields the homomorphism (see Section 2):
\[
\lambda_j^* : \text{CH}_0(X) \to \text{CH}_j(X).
\]

**Lemma 4.10.** The maps $\lambda_j^*$ are isomorphisms for $j = 0, 1, \ldots, p - 2$, taking $e_0(A)$ to $e_j(A)$.

**Proof.** By (4.1), in the split case, $h = h_2 - h_1$, where $h_i$ are the pull-backs to $X \times X$ of the classes of the hyperplanes in $X$, hence $\lambda_j = (h_2 - h_1)^{d-j}$. Therefore, $\lambda_j^*$ takes the generator $l_0$ of the infinite cyclic group $\text{CH}_0(X)$ to the generator $l_j$ of $\text{CH}_j(X)$.

By Proposition 4.6, in the general case, the degree map $\text{CH}_j(X) \to \text{CH}(X_{\text{sep}}) = \mathbb{Z}l_j$ identifies the group $\text{CH}_j(X)$ with $\text{ind}(A)l_j$ by Corollary 4.9. The result follows. \qed
5. Two cycle modules

Let $A$ be a central simple algebra over $F$. The first cycle module $K^QA_n$ is defined by

$$K^QA_n(L) = K_n(A_L)$$

for a field extension $L/F$. The reduced norm map $\text{Nrd} : K_n(A_L) \to K_n(L)$ is defined for $n = 0, 1, 2$ (see [12, §6]).

Let $G = \text{SL}_1(A)$ be the algebraic group of reduced norm 1 elements in $A$. There is a canonical isomorphism (see [6, Proposition 7.3])

$$A^1(G, K_2) \simeq \mathbb{Z}.$$  

The group $A^1(G, K_2)$ does not change under field extensions.

In particular, we have a homomorphism

$$\text{Nrd}^QA : A^1(G, K_2^QA) \to A^1(G, K_2) = \mathbb{Z}.$$  

Let $X$ be the Severi-Brauer variety of $A$ of dimension $d$. We will be using another cycle module $K^A_n$ over $F$ defined by

$$K^A_n(L) = A^d(X_L, K^d+n).$$

The push-forward homomorphism for the morphism $X_L \to \text{Spec}(L)$ yields a map $A^d(X_L, K^d+n) \to K_n(L)$ and therefore, a morphism of cycle modules $K^A_n \to K_*$. In particular, we have a homomorphism

$$\text{Nrd}^A : A^1(G, K_2^A) \to A^1(G, K_2) = \mathbb{Z}.$$  

There is a natural homomorphism $A^d(X_L, K^d+n) \to K_n(A_L)$ which is an isomorphism for $n = 0$ and 1 (see [14]). Thus, we have a morphism of cycle modules $K^A_* \to K^{QA}_*$ that is isomorphism in degree 0 and 1. It follows that the images of the maps $\text{Nrd}^QA$ and $\text{Nrd}^A$ coincide.

If $A$ is split, $K^{QA}_* = K^A_* = K_*$.  

6. A reduction

Recall that $G = \text{SL}_1(A)$ for a central simple algebra $A$ of degree $n$ over $F$. For every commutative $F$-algebra $R$ there is a natural composition

$$G(R) \hookrightarrow A^\times_R \to K_1(A_R),$$

where $A_R = A \otimes_F R$.

Consider the generic point $\xi \in G(F[G])$ and its image $\xi_{F(G)}$ in $G(F(G))$. Let $\alpha$ be the image of $\xi$ under the map

$$G(F[G]) \to K_1(A_{F[G]}),$$

and let $\alpha_{F(G)}$ be the image of $\xi_{F(G)}$ under the map

$$G(F(G)) \to K_1(A_{F(G)}).$$

We will prove that $\alpha_{F(G)}$ is nontrivial in $K_1(A_{F(G)})$ when $A$ is a central simple algebra with $\text{ind}(A)$ not square-free.
Filtering the category of coherent $A \otimes_F \mathcal{O}_G$-modules by codimension of support as in [18, §7.5], we get the Brown-Gersten-Quillen spectral sequence (see [18, §7])

$$E_1^{r,s} = \prod_{g \in G^{(r)}} K_{r-s}(A_{F(g)}) \Rightarrow K_{r-s}(A_{F[G]}),$$

where the limit is the $K$-group of the category of coherent $A \otimes_F \mathcal{O}_G$-modules equipped with the topological filtration (by codimension of support). In particular,

$$E_2^{r,s} = A^1(G, K^Q_2)$$

and the first term of the topological filtration on $K_1(A_{F[G]})$ is equal to

$$K_1(A_{F[G]})^{(1)} = \text{Ker} \left( K_1(A_{F[G]}) \to K_1(A_{F[G]}) \right).$$

The spectral sequence gives then a homomorphism

$$\varepsilon : K_1(A_{F[G]})^{(1)} \to A^1(G, K^Q_2).$$

If $\alpha_{F[G]}$ is trivial in $K_1(A_{F[G]})$, then $\alpha \in K_1(A_{F[G]})^{(1)}$. Therefore, we have an element $\varepsilon(\alpha) \in A^1(G, K^Q_2)$.

We compute $\varepsilon(\alpha)$ in the split case. We have $G = \text{SL}_n$ and

$$\alpha \in K_1(A_{F[G]}) = K_1(F[G]) = K_1(G).$$

By [22, Th. 2.7], the first Chern class $c_1(\alpha)$ of $\alpha$ generates the group $A^1(G, K^Q_2) = A^1(G, K_2) = \mathbb{Z}$.

**Lemma 6.1.** In the split case, $\varepsilon(\alpha) = c_1(\alpha)$.

**Proof.** Let $H := \text{GL}_n$ and $\beta \in K_1(H)$ be the element given by the generic matrix. By [22, Th. 3.10], $\gamma_{i+1}(\beta) \in K_1(H)^{(i)}$ for all $i \geq 0$, where $\gamma$ is the gamma operation, and the image of $-\gamma_2(\beta)$ under the canonical homomorphism

$$K_1(H)^{(1)} \to A^1(H, K_2)$$

is equal to $c_1(\beta)$. On the other hand, the sum of $\gamma_i(\beta)$ for all $i \geq 1$ coincides with $\Lambda^a(\beta) = \text{det}(\beta)$ by [22, p. 65]. Hence $-\gamma_2(\beta) \equiv \beta - \text{det}(\beta)$ modulo $K_1(H)^{(2)}$.

Pulling back with respect to the embedding of $G$ into $H$ we have $-\gamma_2(\alpha) \equiv \alpha$ modulo $K_1(G)^{(2)}$ since $\text{det}(\alpha)$ is trivial and therefore, the image of $\alpha$ under the homomorphism $K_1(G)^{(1)} \to A^1(G, K_2)$ is equal to $c_1(\alpha)$. \qed

Let $L/F$ be a splitting field of $A$. We have a commutative diagram

$$\begin{array}{ccc}
K_1(A_{F[G]})^{(1)} & \xrightarrow{\varepsilon} & A^1(G, K^Q_2) \\
\downarrow & & \downarrow \\
K_1(A_{L[G]})^{(1)} & \xrightarrow{\varepsilon} & A^1(G_L, K^Q_2) \equiv \mathbb{Z}.
\end{array}$$

The right vertical homomorphism factors as follows:

$$A^1(G, K^Q_2) \xrightarrow{\text{Nrd}^Q_2} A^1(G, K_2) \xrightarrow{\sim} A^1(G_L, K_2) = A^1(G_L, K^Q_2).$$
Assume that $\alpha_{F(G)}$ is trivial in $K_1(A_{F(G)})$, hence $\alpha \in K_1(A_{F(G)})^{(1)}$. By Lemma 6.1, $\epsilon(\alpha)_L$ in $A^1(G_L, K_2^{QA}) = \mathbb{Z}$ is a generator. It follows that the image of $\epsilon(\alpha)$ under the map $\text{Nrd}^{QA} : A^1(G, K_2^{QA}) \to A^1(G, K_2) = \mathbb{Z}$ is equal to $\pm 1$, hence $\text{Nrd}^{QA}$ is surjective.

We have proved:

**Proposition 6.2.** Suppose that the map $\text{Nrd}^{QA} : A^1(G, K_2^{QA}) \to A^1(G, K_2) = \mathbb{Z}$ is not surjective. Then Suslin’s Conjecture holds for $A$.

Let $A$ be a central simple $F$-algebra such that $\text{ind}(A)$ is not square-free, i.e., $\text{ind}(A)$ is divisible by $p^2$ for a prime integer $p$. We want to prove that $\text{SK}_1(A)$ is nontrivial generically. Replacing $F$ by a field extension over which $A$ has index exactly $p^2$ and replacing $A$ by a Brauer equivalent division algebra, we may assume that $A$ is a division algebra of degree $p^2$. Moreover, an application of the index reduction formula shows that we may assume that $A$ is decomposable, i.e., $A$ is a tensor product of two algebras of degree $p$ (see [19, Theorem 1.20]).

We will prove that if $A$ is a decomposable division algebra of degree $p^2$, then the map $\text{Nrd}^{QA}$ is not surjective. Recall that the maps $\text{Nrd}^{QA}$ and $\text{Nrd}^A$ have the same images. Therefore, it suffices to prove that the map $\text{Nrd}^A : A^1(G, K_2^A) \to A^1(G, K_2) = \mathbb{Z}$ is not surjective.

### 7. A Spectral Sequence

Let $A$ be a central simple $F$-algebra of degree $p^2$ and $X$ the Severi-Brauer variety of $A$ with $\dim(X) = d = p^2 - 1$. We would like to find a reasonable description the group $A^1(G, K_2^A)$ via algebraic cycles on $G \times X$.

Consider the spectral sequence associated with the projection $q : G \times X \to G$ (see Section 2):

\[
E_{r,s}^1 = E_0^{r,s}(q, d + 2) = \prod_{g \in G^{(r)}} A^s(X_{F(g)}, K_{d+2-r}) \Rightarrow A^{r+s}(G \times X, K_{d+2}).
\]

We have $E_1^{r,s} = 0$ if $s > d$ and

\[
E_2^{r,d} = A^r(G, K_2^A).
\]

There are no nontrivial differentials arriving at $E_{*}^{r,d}$.

**Proposition 7.2.** We have $E_2^{d+i, d-i} = 0$ for $i = 2, 3, \ldots, p$. In particular,

\[
A^1(G, K_2^A) = E_2^{1,d} = E_3^{1,d} = \cdots = E_p^{1,d}.
\]

**Proof.** Let $j = i - 2$ and $\lambda_j$ be the correspondence on $X \times X$ of degree $j$ considered in Section 4. By Lemma 4.10, the maps

\[
\lambda_j^* : \text{CH}_0(X_L) \to \text{CH}_j(X_L).
\]

are isomorphisms for $j = 0, 1, \ldots, p - 2$ and every field extension $L/F$. 

Consider the spectral sequence

\[(7.3) \quad \hat{E}_1^{r,s} := E_1^{r,s}(q, d + i) = \prod_{g \in G^{(r)}} A^s(X_{F(g)}, K_{d+i-r}) \Rightarrow A^{r+s}(G \times X, K_{d+i}).\]

The edge homomorphism

\[\text{CH}^{d+i}(G \times X) = A^{d+i}(G \times X, K_{d+i}) \to \hat{E}_2^{i,d}\]

is surjective. By Proposition 3.2,

\[\text{CH}^{d+i}(G \times X) = 0\]

since \(d + i > \dim(X)\). It follows that \(\hat{E}_2^{i,d} = 0\).

By Proposition 7.2, we have a differential

\[A^1(G, K^A) = E_1^{1,d} \delta \to E_1^{p+1,d+1-p}.\]

**Proposition 7.4.** If \(\text{ind}(A) = p\), the image of \(\text{Ker}(\delta)\) under the homomorphism

\[\text{Nrd}^A : A^1(G, K^A) \to A^1(G, K^2) = \mathbb{Z}\]

is equal to \(p\mathbb{Z}\).

We will prove this proposition in Section 10.

Let \(A\) be a division algebra of degree \(p^2\) over \(F\). Choose a field extension 
\(K/F\) such that \(\text{ind}(A_K) = p\) and set \(A = A_K, \ X = X_K, \ \tilde{G} = G_K\) and write \(\tilde{E}_r^{s} \) for the terms of the spectral sequence associated with the projection 
\(\tilde{G} \times \tilde{X} \to \tilde{G}\). We have the following commutative diagram

\[
\begin{array}{ccc}
A^1(G, K^A) & \xrightarrow{\delta} & E_1^{1,d} \\
\downarrow & & \downarrow \\
A^1(\tilde{G}, K^{\tilde{A}}) & \xrightarrow{\delta} & \tilde{E}_1^{1,d}
\end{array}
\]

where \(\delta\) and \(\tilde{\delta}\) are the differentials in the \(p\)-th pages of the spectral sequences.

**Proposition 7.5.** If \(A\) is decomposable degree \(p^2\) division algebra, then \(\kappa : E_1^{p+1,d+1-p} \to \tilde{E}_1^{p+1,d+1-p}\) is the zero map.

We will prove this proposition in Section 11.
8. Main theorem

We deduce the following theorem from Propositions 7.4 and 7.5.

**Theorem 8.1.** Let $A$ be a central simple $F$-algebra. If $\text{ind}(A)$ is not square-free, then there is a field extension $L/F$ such that $\text{SK}_1(A_L) \neq 0$.

**Proof.** We may assume that $A$ is a decomposable division algebra of degree $p^2$ for a prime integer $p$. Note that $\text{ind}(A) = p$.

By Propositions 7.4 (applied to the algebra $A$) and 7.5, the image of the composition $E_1^1, d \to E_1^1, d = A^1(G, K_2) \xrightarrow{\text{Nrd}^d} A^1(G, K_2) = \mathbb{Z}$
is contained in $p\mathbb{Z}$. On the other hand, this composition coincides with $E_1^1, d = A^1(G, K_2) \xrightarrow{\text{Nrd}^d} A^1(G, K_2) = \mathbb{Z}$.

Therefore, the norm homomorphism $\text{Nrd}^d : A^1(G, K_2) \to A^1(G, K_2)$ is not surjective and this finishes the proof by Proposition 6.2 since $\text{Im}(\text{Nrd}^d) = \text{Im}(\text{Nrd}^d)$.

An irreducible variety $Z$ over $F$ is called a retract rational variety if there exist rational morphisms $\alpha : Z \to \mathbb{P}^m$ and $\beta : \mathbb{P}^m \to Z$ for some $m$ such that the composition $\beta \circ \alpha$ is defined and equal to the identity of $Z$.

**Corollary 8.2.** Let $A$ be a central simple algebra over $F$. Then the following are equivalent:

1. The group $\text{SL}_1(A)$ is a retract rational variety;
2. $\text{SK}_1(A_L) = 0$ for every field extension $L/F$;
3. The index $\text{ind}(A)$ is square-free.

**Proof.** (1) $\Rightarrow$ (2): If $G := \text{SL}_1(A)$ is a retract rational variety, then $G_L$ is so for every field extension $L/F$. By [2, Proposition 11], the group of $R$-equivalence classes $G(L)/R$ is trivial. But $G(L)/R$ is isomorphic to $\text{SK}_1(A_L)$ by [24, §18.2].

(2) $\Rightarrow$ (1): This is proved in [5, Proposition 2.4] and [10, Proposition 5.1].

(2) $\Leftrightarrow$ (3): This is Theorem 8.1.

9. Chow ring of $G$

Let $G = \text{SL}_1(A)$ for a central simple algebra $A$ of $p$-primary degree.

**Lemma 9.1.** The Chow groups $\text{CH}^i(G)$ are trivial for $i = 1, 2, \ldots, p$ and $p \cdot \text{CH}^{p+1}(G) = 0$.

**Proof.** Since $\text{CH}^i(G_{\text{sep}}) = \mathbb{Z}$ by [22, Theorem 2.7], the groups $\text{CH}^i(G)$ are $p$-primary torsion if $i > 0$. As $K_0(G) = \mathbb{Z}$ (see [22, Theorem 4.1]), by [4, Example 15.3.6], we have $(i - 1)! \text{CH}^i(G) = 0$ for $i > 0$. The result follows.
Consider the Brown-Gersten-Quillen spectral sequence
\[ E_2^{r,s} = A^r(G, K_{-s}) \Rightarrow K_{-r-s}(G). \]
It follows from Lemma 9.1 that
\[ A^1(G, K_2) = E_2^{1,-2} = E_3^{1,-2} = \cdots = E_p^{1,-2}. \]
Moreover, by [8, §3],
\[ \text{CH}^{p+1}(G) = E_2^{p+1,-p-1} = E_3^{p+1,-p-1} = \cdots = E_p^{p+1,-p-1}. \]
We have then a differential
\[ \delta : A^1(G, K_2) = E_2^{1,-2} \rightarrow E_2^{p+1,-p-1} = \text{CH}^{p+1}(G). \]
Write \( h \in \text{CH}^{p+1}(G) \) for the image under \( \delta \) of the canonical generator of the group \( A^1(G, K_2) = \mathbb{Z} \).

**Proposition 9.2.** Suppose that \( \text{ind}(A) = p \), i.e., \( A = M_n(B) \) and \( G = \text{SL}_n(B) \) for some \( n \) and a central division algebra \( B \) of degree \( p \). Then
\[ \text{CH}^*(G) = \mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}. \]

**Proof.** Induction on \( n \). The case \( n = 1 \) is done in [8, Theorem 9.7].

Let \( H = \text{SL}_{n-1}(B) \). We view \( H \) as a subgroup of \( G \) with respect to the embedding \( x \mapsto \text{diag}(1, x) \). Consider the closed subvariety \( V \) of the affine space \( B^{2n} \) consisting of tuples \((b_1, \ldots, b_n, c_1, \ldots, c_n)\) such that \( \sum b_i c_i = 1 \). Define the morphism
\[ f : G \rightarrow V, \quad a = (a_{ij}) \mapsto (a_{11}, \ldots, a_{1n}, a'_{11}, \ldots, a'_{nn}), \]
where \((a'_{ij}) = a^{-1}\). Clearly, \( f \) is an \( H \)-torsor over \( V \). For any field extension \( L/F \), in the exact sequence of Galois cohomology
\[ G(L) \xrightarrow{f(L)} V(L) \rightarrow H^1(L, H) \xrightarrow{r} H^1(L, G) \]
the map \( r \) is a bijection (both sets are identified with \( L^\times / \text{Nrd}(B^+_L) \)) and \( r \) is the identity map by [11, Cor. 2.9.4)]. Hence \( f \) is surjective on \( L \)-points.

Let \( W \) be the open subset of the affine space \( B^a \) consisting of all tuples \((b_1, \ldots, b_n)\) such that \( \sum b_i B = B \). We have \( \text{CH}^i(W) = 0 \) for \( i > 0 \). The obvious projection \( V \rightarrow W \) is an affine bundle, hence by the homotopy invariance property,
\[ \text{CH}^i(V) \simeq \text{CH}^i(W) = 0 \]
for every \( i > 0 \).

For every \( m \), consider the spectral sequence associated with the morphism \( f \):
\[ E_1^{r,s} = E_1^{r,s}(f, m) = \prod_{v \in V^{(r)}} A^s(f^{-1}(v), K_{m-r}) \Rightarrow A^{r+s}(G, K_m). \]
Since \( f \) is surjective on \( L \)-points, \( f^{-1}(v) \simeq H_{F(v)} \).

We claim that \( E_2^{r,s} = 0 \) if \( r + s = m \) and \( r > 0 \). By induction, the group \( A^s(G_v, K_s) = \text{CH}^s(G_v) \) is trivial unless \( s = (p + 1)i \) for \( i = 0, 1, \ldots, p-1 \). In
the latter case the map $\text{CH}^r(V) \to E_2^{r,s}$ of multiplication by $h^r$ is surjective by the induction hypothesis. The claim follows from the triviality of $\text{CH}^r(V)$ for $r > 0$.

By the claim, $\text{CH}(G) \simeq \text{CH}(H_{F(V)})$. The statement of the proposition follows by induction.

**Corollary 9.4.** Let $A$ be a central simple algebra of degree $p^2$. Then for every field extension $L/F$ such that $\text{ind}(A_L) \leq p$, the map

$$\text{CH}(G) \to \text{CH}(G_L)$$

is surjective.

**Proof.** The element $h$ belongs to $\text{CH}^{r+1}(G)$. As $\text{ind}(A_L) \leq p$, by Proposition 9.2, the element $h_L$ generates the ring $\text{CH}(G_L)$, whence the result. □

**10. Proof of Proposition 7.4**

In this section, $A$ is a central simple algebra of degree $p^2$ and index $p$, so that $A = M_p(B)$, where $B$ is a division algebra of degree $p$. We write $S$ for the Severi-Brauer variety $SB(B)$ of dimension $p^2 - 1$. Recall that the variety $S$ can be viewed as a closed subvariety of $X$. Moreover, the Chow motive $M(X)$ of $X$ is isomorphic to $M(S) \oplus M(S)\{p\} \oplus \cdots \oplus M(S)\{(p-1)p\}$.

Consider the spectral sequence associated with the projection $t : G \times S \to G$:

$$\hat{E}_r^{s,p} := E_r^{s,p}(t, p+1) = \prod_{g \in G(r)} A^s(S_{F(g)}, K_{p+1-r}) \Rightarrow A^{p+q}(G \times S, K_{p+1}).$$

The embedding of $S$ into $X$ induces the push-forward morphisms between the spectral sequences (10.1) and (7.1). Moreover, (10.1) is a direct summand of (7.1). More precisely, the maps

$$\hat{E}_r^{s,p} \to E_r^{s+p+1-p}$$

are isomorphisms for $s = 0, 1, \ldots, p - 1$.

By Proposition 7.2, we have $\hat{E}_2^{s,d+2-i} = 0$ for $i = 2, 3, \ldots, p$. It follows that $\hat{E}_2^{s,p+1-r} = 0$ for $i = 2, 3, \ldots, p$, i.e., all the terms but $\hat{E}_2^{p+1,0}$ on the diagonal $r + s = p + 1$ on page $\hat{E}_s^{s,p}$ are zero. Moreover,

$$E^{p+1,d+1-p}_2 = \hat{E}_2^{p+1,0} = \text{CH}^{p+1}(G).$$

It follows that

$$A^1(G, K_2^A) = \hat{E}_2^{1,p-1} = \hat{E}_3^{1,p-1} = \cdots = E_p^{1,p-1}$$

and the only potentially nonzero differential starting in $\hat{E}_2^{1,p-1}$ appears on page $p$:

$$A^1(G, K_2^A) = \hat{E}_p^{1,p-1} \Rightarrow \hat{E}_p^{p+1,0}.$$

The spectral sequence (10.1) yields then an exact sequence

$$A^1(G \times S, K_{p+1}) \to \hat{E}_p^{1,p-1} \Rightarrow \hat{E}_p^{p+1,0}.$$
The differential $\delta: E_p^{1,d} \to E_p^{1,d+1-p}$ in (10.1) is identified with the differential $\hat{\delta}: \hat{E}_p^{1,p-1} \to \hat{E}_p^{p+1,0}$ in (7.1). Thus, to prove the proposition, it suffices to show that the image of the composition

$$A^p(G \times S, K_{p+1}) \to \hat{E}_p^{1,p-1} = A^1(G, K^2_p) \xrightarrow{\text{Nrd}_A} A^1(G, K_2) = \mathbb{Z}$$

is equal to $p\mathbb{Z}$.

This composition is the push-forward homomorphism

$$t_*: A^p(G \times S, K_{p+1}) \to A^1(G, K_2) = \mathbb{Z}$$

with respect to the projection $t: G \times S \to G$.

Over $S$, the algebra $A$ is isomorphic to $\mathcal{E}nd_{O_S}(J^p)$, where $J$ is the canonical vector bundle over $S$ of rank $p$. By Proposition 3.2,

$$A^p(G \times S, K_{p+1}) = \text{CH}^{p-1}(S) \cdot c_1(\beta) \oplus \text{CH}^{p-2}(S) \cdot c_2(\beta) \oplus \cdots \oplus \text{CH}^0(S) \cdot c_p(\beta),$$

where $\beta \in K_1(G \times S)$ is the generic element.

Since the group $A^1(G, K_2)$ does not change under field extensions, it is sufficient to compute the image over a field extension $L/F$ splitting $A$. Over such a field extension the group $G_L$ is isomorphic to $\text{SL}_{p^2}$. Let $\beta' \in K_1(G_L)$ be the class of the generic matrix, so that $\beta = [L_c] \cdot t^*(\beta')$, where $L_c$ is the canonical line bundle over $X$. By Proposition 3.1,

$$c_i(\beta) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} h^j c_{i-j}(t^*\beta')$$

for every $i = 1, 2, \ldots, p$, where $h \in \text{CH}^1(S_L)$ is the first Chern class of $L_c$. Note that $c_1(\beta')$ is the canonical generator of $A^1(G_L, K_2)$. By the projection formula, the image of $t_*$ is the sum of the subgroups

$$\binom{i}{j} \cdot t_* [\text{CH}^{p-i}(S) \cdot h^j] \cdot c_{i-j}(\beta')$$

over all $i = 1, 2, \ldots, p$ and $j = 0, 1, \ldots, i$. By dimension consideration, the subgroup is trivial if $j \neq i - 1$. Consider the case $j = i - 1$. If $p - i > 0$, then the image of $\text{CH}^{p-i}(S)$ in $\mathbb{Z}$ (when splitting $S$) is equal to $p\mathbb{Z}$. Finally, if $i = p$, the multiple $\binom{i}{j}$ is equal to $p$. The proposition is proved.

11. Proof of Proposition 7.5

In this section we assume that $A$ is a decomposable division algebra of degree $p^2$.

Since for every $i$ and $j$ with $i + j = d + 2$ the natural homomorphism $E^{i,j}_2 \to E^{j,i}_p$ is surjective, it is sufficient to prove that the homomorphism

$$E^{p+1,d+1-p}_2 \to \bar{E}^{p+1,d+1-p}_2$$

is trivial.
Let $L/F$ be a field extension. Considering $X$ over a separable closure of $L$ we get the homomorphisms

$$A^i(X_L, K_{i+n}) \rightarrow A^i(X_{L_{sep}}, K_{i+n}) = K_n(L)$$

for $i = d + 1 - p$ and $n = 0, 1$. These homomorphisms induce the vertical maps in the following commutative diagram

$$
\begin{array}{ccc}
E_2^{p+1,d+1-p} & \longrightarrow & \bar{E}_2^{p+1,d+1-p} \\
\varphi \downarrow & & \downarrow \bar{\varphi} \\
CH^{p+1}(G) & \longrightarrow & CH^{p+1}(\tilde{G}).
\end{array}
$$

Since $\text{ind}(\tilde{A}) = p$, it follows from (10.2) (applied to $\tilde{A}$) that $\bar{\varphi}$ is an isomorphism. Thus, it is sufficient to prove that $\varphi = 0$.

Recall that $A$ is a decomposable algebra. By a theorem of Karpenko [9, Th. 1],

$$(11.1) \quad \text{Im}(CH^{d+1-p}(X_{F(g)}) \xrightarrow{\deg} \mathbb{Z}) = \begin{cases} p\mathbb{Z}, & \text{if } \text{ind} A_{F(g)} = p^2; \\ \mathbb{Z}, & \text{if } \text{ind} A_{F(g)} \leq p. \end{cases}$$

Let $Y = SB(p, A)$ be the generalized Severi-Brauer variety and set $m := \dim(Y) = p^3 - p^2$. Consider the cycle module $M_\ast$ over $F$ defined by

$$M_\ast(L) = A^m(Y_L, K_{n+m}).$$

There is the norm morphism $N : M_\ast \rightarrow K_\ast$ well defined. The variety $Y$ has a point over a field extension $L/F$ if and only if $\text{ind}(A_L) \leq p$. It follows that

$$\text{Im}(A^m(Y_L, K_m) \xrightarrow{N} \mathbb{Z}) = \begin{cases} p\mathbb{Z}, & \text{if } \text{ind} A_L = p^2; \\ \mathbb{Z}, & \text{if } \text{ind} A_L \leq p. \end{cases}$$

Therefore the image of $\varphi$ coincide with the image of the map

$$\psi : A^{p+1}(G, M_{p+1}) \rightarrow A^{p+1}(G, K_{p+1}) = CH^{p+1}(G)$$

induced by the norm map $N$. It is sufficient to prove that $\psi = 0$.

The spectral sequence for the projection $G \times Y \rightarrow G$,

$$E_1^{r,s} = \coprod_{g \in G^{(r)}} A^s(Y_{F(g)}, K_{m+p+1-r}) \Rightarrow A^{r+s}(G \times Y, K_{m+p+1})$$

yields a surjective homomorphism $\text{CH}^{m+p+1}(G \times Y) \rightarrow A^{p+1}(G, M_{p+1})$. The composition

$$\text{CH}^{m+p+1}(G \times Y) \rightarrow A^{p+1}(G, M_{p+1}) \xrightarrow{\psi} A^{p+1}(G, K_{p+1}) = CH^{p+1}(G)$$

is the push-forward homomorphism with respect to the projection $G \times Y \rightarrow G$. Thus, it is sufficient to show that the push-forward homomorphism

$$\text{CH}^{m+p+1}(G \times Y) \rightarrow CH^{p+1}(G)$$

is zero.
Since $\text{ind}(A_{F(y)}) \leq p$ for every $y \in Y$, it follows from Corollary 9.4 that the map $\text{CH}(G) \to \text{CH}(G_{F(y)})$ is surjective. Then the proof of [3, Lemma 88.5] yields the following lemma.

**Lemma 11.2.** The product homomorphism

$$\text{CH}(G) \otimes \text{CH}(Y) \to \text{CH}(G \times Y)$$

is surjective.

**Lemma 11.3.** For every closed point $y \in Y$, the norm homomorphism

$$N_{F(y)} : \text{CH}^{p+1}(G_{F(y)}) \to \text{CH}^{p+1}(G)$$

is trivial.

**Proof.** The first map in the composition

$$\text{CH}^{p+1}(G) \to \text{CH}^{p+1}(G_{F(y)}) \xrightarrow{N_{F(y)/F}} \text{CH}^{p+1}(G)$$

is surjective by Corollary 9.4 since $\text{ind}(A_{F(y)}) \leq p$. The composition is multiplication by $\text{deg}(y)$. Note that $\text{deg}(y)$ is divisible by $p$ since $\text{ind}(A) = p^2$. The result follows from Lemma 9.1. \hfill \Box

**Proposition 11.4.** If $A$ is a division algebra, the push-forward homomorphism

$$\text{CH}^{m+p+1}(G \times Y) \to \text{CH}^{p+1}(G)$$

is trivial.

**Proof.** By Lemma 11.2, it is sufficient to show that for every closed point $y \in Y$ the norm homomorphism $\text{CH}^{p+1}(G_{F(y)}) \to \text{CH}^{p+1}(G)$ is trivial. This is proved in Lemma 11.3. \hfill \Box

**References**


Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA

E-mail address: merkurev@math.ucla.edu