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Essential dimension

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Abstract. We review and slightly generalize some definitions and results on the essential dimension.

The notion of essential dimension of an algebraic group was introduced by Buhler and Reichstein in [**6**] and [**21**]. Informally speaking, essential dimension $\mathrm{ed}(G)$ of an algebraic group G over a field F is the smallest number of algebraically independent parameters required to define a G-torsor over a field extension of F. Thus, the essential dimension of G measures complexity of the category of G-torsors.

More generally, the essential dimension of a functor from the category Fields/F of field extensions of F to the category Sets of sets was discussed in [**2**].

Let p be a prime integer. Essential p-dimension $\mathrm{ed}_p(G)$ of an algebraic group was introduced in [**22**]. The integer $\mathrm{ed}_p(G)$ is usually easier to calculate than $\mathrm{ed}(G)$, and it measures the complexity of the category of G-torsors modulo "effects of degree prime to p".

In the present paper we study essential dimension and p-dimension of a functor $\mathit{Fields}/F \to \mathit{Sets}$ in a uniform way (Section 1). We also introduce essential p-dimension of a class of field extensions of F, or equivalently, of a *detection* functor $T : \mathit{Fields}/F \to \mathit{Sets}$, i.e., a functor T with $T(L)$ consisting of at most one element for every L.

For every functor $T : \mathit{Fields}/F \to \mathit{Sets}$, we associate the class of field extensions L/F such that $T(L) \neq \emptyset$. The essential p-dimension of this class is called *canonical p-dimension of T*. Note that canonical p-dimension of a detection functor was introduced in [**16**] with the help of so-called generic fields that are defined in terms of places of fields. We show that this notion of the canonical p-dimension coincides with ours under a mild assumption (Theorem 1.16).

In Section 2, we introduce essential p-dimension of a presheaf of sets S on the category Var/F of algebraic varieties over F. We associate a functor $\widetilde{S} : \mathit{Fields}/F \to \mathit{Sets}$ to every such an S, and show that $\mathrm{ed}_p(S) = \mathrm{ed}_p(\widetilde{S})$ (Proposition 2.6). In practice, many functors $\mathit{Fields}/F \to \mathit{Sets}$ are of the form $\widetilde{S}$ for some presheaf of sets S. This setting allows us to define *p-generic* elements $a \in S(X)$ for S and show that $\mathrm{ed}_p(S) = \mathrm{ed}_p(a)$ (Theorem 2.9). Thus, to determine $\mathrm{ed}_p(S)$ or $\mathrm{ed}_p(\widetilde{S})$ it is sufficient to compute the essential p-dimension of a single generic element.

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Following the approach developed by Brosnan, Reichstein and Vistoli in [**3**], in Section 3 we define essential p-dimension of a fibered category over Var/F. In Section 4, we consider essential dimension of an algebraic group scheme and in Section 5 the essential p-dimension of finite groups. Technical results used in the paper are summarized in the Appendix.

We use the following notation:

We write Fields/F for the category of finitely generated field extensions over F and field homomorphisms over F. For any $L \in \mathsf{Fields}/F$, we have $\operatorname{tr.deg}_F(L) < \infty$.

In the present paper, the word "scheme" over a field F means a separated scheme of finite type over F and a "variety" over F is an integral scheme over F. Note that by definition, every variety is nonempty.

The category of *algebraic varieties* over F is denoted by Var/F. For any $X \in \mathsf{Var}/F$, the function field $F(X)$ is an object of Fields/F and $\operatorname{tr.deg} F(X) = \dim(X)$.

Let $f : X \dashrightarrow Y$ be a rational morphism of varieties over F of the same dimension. The *degree* $\deg(f)$ of f is zero if f is not dominant and is equal to the degree of the field extension $F(X)/F(Y)$ otherwise.

An algebraic group scheme over F in the paper is a group scheme of finite type over F.

If R is a ring, we write $\mathsf{M}(R)$ for the category of finitely generated right R-modules.

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1. Definition of the essential p-dimension

The letter p in the paper denotes either a prime integer or 0. An integer k is said to be *prime to p* when k is prime to p if $p > 0$ and $k = 1$ if $p = 0$.

1.1. Essential p-dimension of a functor.

Let $T : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. Let $\alpha \in T(L)$ and $f : L \to L'$ a field homomorphism over F. The field L' can be viewed as an extension of L via f. Abusing notation we shall write $\alpha_{L'}$ for the image of α under the map $T(f) : T(L) \to T(L')$.

Let $K, L \in \mathsf{Fields}/F$, $\beta \in T(K)$ and $\alpha \in T(L)$. We write $\alpha \succ_p \beta$ if there exist a finite field extension L' of L of degree prime to p and a field homomorphism $K \to L'$ over F such that $\alpha_{L'} = \beta_{L'}$. In the case $p = 0$, the relation $\alpha \succ_p \beta$ will be written as $\alpha \succ \beta$ and simply means that L is an extension of K with $\alpha = \beta_L$.

LEMMA 1.1. *The relation $\succ_p$ is transitive.*

PROOF. Let $\alpha \in T(L)$, $\beta \in T(K)$ and $\gamma \in T(J)$. Suppose $\alpha \succ_p \beta$ and $\beta \succ_p \gamma$, i.e., there exist finite extensions K' of K and L' of L, both of degree prime to p and F-homomorphisms $J \to K'$ and $K \to L'$ such that $\alpha_{L'} = \beta_{L'}$ and $\beta_{K'} = \gamma_{K'}$. By Lemma 6.1, there is a field extension L''/L' of degree prime to p and a field homomorphism $K' \to L''$ extending $K \to L'$. We have $\alpha_{L''} = \beta_{L''} = \gamma_{L''}$ and $[L'' : L]$ is prime to p, hence $\alpha \succ_p \gamma$. □

Let $K, L \in \mathsf{Fields}/F$. An element $\alpha \in T(L)$ is said to be *p-defined over K* and *K is a field of p-definition of* α if $\alpha \succ_p \beta$ for some $\beta \in T(K)$. In the case $p = 0$, we say that α is *defined over K* and *K is a field of definition of α*. The latter means that L is an extension of K and $\alpha = \beta_L$ for some $\beta \in T(K)$.

The *essential p-dimension of* α, denoted $\operatorname{ed}_p(\alpha)$, is the least integer $\operatorname{tr.deg}_F(K)$ over all fields of p-definition K of α. In other words,

$$\operatorname{ed}_p(\alpha) = \min\{\operatorname{tr.deg}_F(K)\}$$

where the minimum is taken over all fields K/F such that there exists an element $\beta \in T(K)$ with $\alpha \succ_p \beta$.

The *essential p-dimension of the functor* T is the integer

$$\operatorname{ed}_p(T) = \sup\{\operatorname{ed}_p(\alpha)\}$$

where the supremum is taken over all $\alpha \in T(L)$ and fields $L \in \mathsf{Fields}/F$.

We write $\operatorname{ed}(T)$ for $\operatorname{ed}_0(T)$ and simply call $\operatorname{ed}(T)$ the *essential dimension of* T. Clearly, $\operatorname{ed}(T) \geq \operatorname{ed}_p(T)$ for all p.

Informally speaking, the essential dimension of T is the smallest number of algebraically independent parameters required to define T.

An element $\alpha \in T(L)$ is called *p-minimal* if $\operatorname{ed}_p(\alpha) = \operatorname{tr.deg}_F(L)$, i.e., whenever $\alpha \succ_p \beta$ for some $\beta \in T(K)$, we have $\operatorname{tr.deg}_F(K) = \operatorname{tr.deg}_F(L)$. By Lemma 1.1, for every $\alpha \in T(L)$ there is a p-minimal element $\beta \in T(K)$ with $\alpha \succ_p \beta$. It follows that $\operatorname{ed}_p(T)$ is the supremum of $\operatorname{ed}_p(\alpha)$ over all p-minimal elements α.

1.2. Essential p-dimension of a scheme. Let X be a scheme over F. We can view X as a functor from Fields/F to Sets taking a field extension L/F to the set of L-points $X(L) := \operatorname{Mor}_F(\operatorname{Spec} L, X)$.

PROPOSITION 1.2. *For any scheme X over F, we have* $\operatorname{ed}_p(X) = \dim(X)$ *for all p.*

PROOF. Let $\alpha : \operatorname{Spec} L \to X$ be a point over a field $L \in \mathsf{Fields}/F$ with image $\{x\}$. Every field of p-definition of α contains an image of the residue field $F(x)$. Moreover, α is p-defined over $F(x)$ hence $\operatorname{ed}_p(\alpha) = \operatorname{tr.deg}_F F(x) = \dim(x)$. It follows that $\operatorname{ed}_p(X) = \dim(X)$. □

1.3. Classifying variety of a functor. Let $f : S \to T$ be a morphism of functors from Fields/F to Sets. We say that f is *p-surjective* if for any field $L \in \mathsf{Fields}/F$ and any $\alpha \in T(L)$, there is a finite field extension L'/L of degree prime to p such that $\alpha_{L'}$ belongs to the image of the map $S(L') \to T(L')$.

PROPOSITION 1.3. *Let $f : S \to T$ be a p-surjective morphism of functors from Fields/F to Sets. Then $ed_p(S) \geq \operatorname{ed}_p(T)$.*

PROOF. Let $\alpha \in T(L)$ for a field $L \in \mathsf{Fields}/F$. By assumption, there is a finite field extension L'/L of degree prime to p and an element $\beta \in S(L')$ such that $f(\beta) = \alpha_{L'}$ in $T(L')$. Let K be a field of p-definition of β, i.e., there is a field extension L''/L' of degree prime to p, an F-homomorphism $K \to L''$ and an element $\gamma \in S(K)$ such that $\beta_{L''} = \gamma_{L''}$. It follows from the equality

$$f(\gamma)_{L''} = f(\gamma_{L''}) = f(\beta_{L''}) = f(\beta)_{L''} = \alpha_{L''}$$

that α is p-defined over K, hence $\operatorname{ed}_p(\beta) \geq \operatorname{ed}_p(\alpha)$. The result follows. □

Let $T : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor. A scheme X over F is called *p-classifying for T* if there is p-surjective morphism of functors $X \to T$.

Propositions 1.2 and 1.3 yield:

COROLLARY 1.4. *Let $T : \mathsf{Fields}/F \to \mathsf{Sets}$ be a functor and let X be a p-classifying scheme for T. Then* $\dim(X) \geq \operatorname{ed}_p(T)$.

1.4. Restriction. Let $K \in \mathit{Fields}/F$ and $T : \mathit{Fields}/F \to \mathit{Sets}$ a functor. The *restriction* T_K of the functor T is the composition of T with the natural functor $\mathit{Fields}/K \to \mathit{Fields}/F$ that is the identity on objects.

PROPOSITION 1.5. *Let $K \in \mathit{Fields}/F$ and let $T : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. Then for every p, we have:*

(1) $\operatorname{ed}_p(T_K) \le \operatorname{ed}_p(T)$.
(2) *If $[K : F]$ is finite and relatively prime to p, then $\operatorname{ed}_p(T_K) = \operatorname{ed}_p(T)$.*

PROOF. (1): Let $\alpha \in T_K(L)$ for a field $L \in \mathit{Fields}/K$. We write α' for the element α considered in the set $T(L)$. Every field of p-definition of α is also a field of p-definition of α', hence $\operatorname{ed}_p(\alpha) \le \operatorname{ed}_p(\alpha')$ and $\operatorname{ed}_p(T_K) \le \operatorname{ed}_p(T)$.

(2): Let $\alpha \in T(L)$ for some $L \in \mathit{Fields}/F$. By Lemma 6.1, there is a field extension L'/L of degree prime to p and an F-homomorphism $K \to L'$. As $L' \in \mathit{Fields}/K$, there is a field extension L''/L' of degree prime to p, a subfield $K' \subset L''$ in Fields/K and an element $\beta \in T(K')$ with $\beta_{L''} = \alpha_{L''}$ and $\operatorname{tr.deg}_F(K') = \operatorname{tr.deg}_K(K') \le \operatorname{ed}_p(T_K)$. Hence α is p-defined over K'. It follows that $\operatorname{ed}_p(\alpha) \le \operatorname{ed}_p(T_K)$ and $\operatorname{ed}_p(T) \le \operatorname{ed}_p(T_K)$. □

1.5. Essential p-dimension of a class of field extensions. In this section we introduce essential p-dimension of a class of fields and relate it to the essential p-dimension of certain functors.

Let L and K be in Fields/F. We write $L \succ_p K$ if there is a finite field extension L'/L of degree prime to p and a field homomorphism $K \to L'$ over F. In particular, $L \succ_p K$ if $K \subset L$. The relation $\succ_p$ coincides with the relation introduced in Section 1.1 for the functor $T : \mathit{Fields}/F \to \mathit{Sets}$ defined by $T(L) = \{L\}$ (one-element set). It follows from Lemma 1.1 that this relation is transitive.

Let $\mathcal{C}$ be a class of fields in Fields/F closed under extensions, i.e., if $K \in \mathcal{C}$ and $L \in \mathit{Fields}/K$, then $L \in \mathcal{C}$. For any $L \in \mathcal{C}$, let $\operatorname{ed}_p^{\mathcal{C}}(L)$ be the least integer $\operatorname{tr.deg}_F(K)$ over all fields $K \in \mathcal{C}$ with $L \succ_p K$. The *essential p-dimension of the class $\mathcal{C}$* is the integer

$$\operatorname{ed}_p(\mathcal{C}) := \sup\{\operatorname{ed}_p^{\mathcal{C}}(L)\}$$

over all fields $L \in \mathcal{C}$. We simply write $\operatorname{ed}(\mathcal{C})$ for $\operatorname{ed}_p(\mathcal{C})$ with $p = 0$.

Essential p-dimensions of classes of fields and functors are related as follows. Let $\mathcal{C}$ be a class of fields in Fields/F closed under extensions. Consider the functor $T_{\mathcal{C}} : \mathit{Fields}/F \to \mathit{Sets}$ defined by

$$T_{\mathcal{C}}(L) = \begin{cases} \{L\}, & \text{if } L \in \mathcal{C}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

By the definition of the essential p-dimension, we have

$$\operatorname{ed}_p(\mathcal{C}) = \operatorname{ed}_p(T_{\mathcal{C}}).$$

Recall that a field $L \in \mathcal{C}$, considered as an elements of $T_{\mathcal{C}}(L)$, is called *p-minimal* if $\operatorname{ed}_p^{\mathcal{C}}(L) = \operatorname{tr.deg}_F(L)$. In other words, L is p-minimal if for any $K \in \mathcal{C}$ with $L \succ_p K$ we have $\operatorname{tr.deg}_F(L) = \operatorname{tr.deg}_F(K)$. It follows from the definition that

$$\operatorname{ed}_p(\mathcal{C}) = \sup\{\operatorname{tr.deg}_F(L)\}$$

over all p-minimal fields in $\mathcal{C}$.

The functor $T_{\mathcal{C}}$ is a *detection* functor, i.e., a functor T such that the set $T(L)$ has at most one element for every L. The correspondence $\mathcal{C} \mapsto T_{\mathcal{C}}$ is a bijection between classes of field extensions closed under extensions and detection functors.

1.6. Canonical p-dimension of a functor. Let $T : \mathit{Fields}/F \to \mathit{Sets}$ be a functor. Write $\mathcal{C}_T$ for the class of all fields $L \in \mathit{Fields}/F$ such that $T(L) \neq \emptyset$. The *canonical p-dimension* $\operatorname{cdim}_p(T)$ of the functor T is the integer $\operatorname{ed}_p(\mathcal{C}_T)$. Equivalently, $\operatorname{cdim}_p(T) = \operatorname{ed}_p(T_{\mathcal{C}})$ for the detection functor $T_{\mathcal{C}}$ with $\mathcal{C} = \mathcal{C}_T$.

In more details, for a field $L \in \mathit{Fields}/F$ satisfying $T(L) \neq \emptyset$ we have $\operatorname{ed}_p^{\mathcal{C}}(L)$ is the least integer $\operatorname{tr.deg}_F K$ over all fields K with $L \succ_p K$ and $T(K) \neq \emptyset$. Then

$$\operatorname{cdim}_p(T) = \sup\{\operatorname{ed}_p^{\mathcal{C}}(L)\}$$

over all fields $L \in \mathit{Fields}/F$ satisfying $T(L) \neq \emptyset$.

Note that the canonical dimension (respectively, canonical p-dimension) of a functor to the category of pointed sets was defined in [**1**] (respectively, [**16**]) by means of generic splitting fields. We consider a relation to generic fields in Section 1.7.

Proposition 1.6. *For a functor $T : \mathit{Fields}/F \to \mathit{Sets}$, we have* $\operatorname{cdim}_p(T) \leq \operatorname{ed}_p(T)$. *If T is a detection functor, then* $\operatorname{cdim}_p(T) = \operatorname{ed}_p(T)$.

Proof. There is a (unique) natural surjective morphism $T \to T_{\mathcal{C}}$ with $\mathcal{C} = \mathcal{C}_T$. It follows from Proposition 1.3 that $\operatorname{cdim}_p(T) = \operatorname{ed}_p(T_{\mathcal{C}}) \leq \operatorname{ed}_p(T)$. □

Let X be a scheme over F. Viewing X as a functor from Fields/F to Sets, we have the *canonical p-dimension* $\operatorname{cdim}_p(X)$ of X defined. In other words, $\operatorname{cdim}_p(X)$ is the essential p-dimension of the class

$$\mathcal{C}_X := \{L \in \mathit{Fields}/F \quad \text{such that} \quad X(L) \neq \emptyset\}.$$

By Propositions 1.2 and 1.6, $\operatorname{cdim}_p(X) \leq \operatorname{ed}_p(X) = \dim(X)$.

Proposition 1.7. *Let X be a smooth complete variety over F. Then* $\operatorname{cdim}_p(X)$ *is the least dimension of the image of a morphism $X' \to X$, where X' is a variety over F admitting a dominant morphism $X' \to X$ of degree prime to p. In particular,* $\operatorname{cdim}(X)$ *is the least dimension of the image of a rational morphism $X \dashrightarrow X$.*

Proof. Let $Z \subset X$ be a closed subvariety and let $X' \to X$ and $X' \to Z$ be dominant morphisms with the first one of degree prime to p. Replacing X' by the closure of the graph of the diagonal morphism $X' \to X \times Z$ we may assume that X' is complete.

Let L be in Fields/F with $X(L) \neq \emptyset$ and $f : \operatorname{Spec} L \to X$ a morphism over F. Let $\{x\}$ be the image of f. As x is non-singular, there is a geometric valuation v of $F(X)$ over F with center x and $F(v) = F(x) \subset L$ (cf. Lemma 6.6). We view $F(X)$ as a subfield of $F(X')$. As $F(X')/F(X)$ is a finite extension of degree prime to p, by Lemma 6.4 there is an extension v' of v on $F(X')$ such that $F(v')/F(v)$ is a finite extension of degree prime to p. Let x' be the center of v' on X' and z the image of x' in Z. As $F(x') \subset F(v')$, the extension $F(x')/F(x)$ is finite of degree prime to p. Since $L \succ_p F(x) \succ_p F(z)$, we have $L \succ_p F(z)$ by Lemma 1.1. Therefore,

$$\operatorname{ed}_p^{\mathcal{C}}(L) \leq \operatorname{tr.deg}_F F(z) \leq \dim(Z),$$

where $\mathcal{C} = \mathcal{C}_X$ and hence $\operatorname{cdim}_p(X) \leq \dim(Z)$.

Conversely, note that X has a point over the field $F(X)$. Choose a finite extension $L'/F(X)$ of degree prime to p and a subfield $K \subset L'$ such that $X(K) \neq \emptyset$ and $\operatorname{tr.deg}_F(K) = \operatorname{ed}_p^{\mathcal{C}}\big(F(X)\big)$. Let Z be the closure of the image of a point $\operatorname{Spec} K \to X$. We have $\dim(Z) \leq \operatorname{tr.deg}_F(K)$. The compositions $\operatorname{Spec} L' \to \operatorname{Spec} F(X) \to X$ and $\operatorname{Spec} L' \to \operatorname{Spec} K \to Z$ yield a model X' of L' and two dominant morphisms $X' \to X$ of degree prime to p and $X' \to Z$ (cf. Appendix 6.1). We have

$$\operatorname{cdim}_p(X) \geq \operatorname{ed}_p^{\mathcal{C}}\big(F(X)\big) = \operatorname{tr.deg}_F(K) \geq \dim(Z). \qquad \square$$

As we noticed above, one has $\operatorname{cdim}_p(X) \leq \dim(X)$ for every scheme X. We say that a scheme X over F is *p-minimal* if $\operatorname{cdim}_p(X) = \dim(X)$. A scheme X is *minimal* if it is p-minimal with $p = 0$. Every p-minimal scheme is minimal.

Proposition 1.7 then yields:

Corollary 1.8. *Let X be a smooth complete variety over F. Then*

(1) *X is p-minimal if and only if for any variety X' over F admitting a surjective morphism $X' \to X$ of degree prime to p, every morphism $X' \to X$ is dominant.*
(2) *X is minimal if and only if every rational morphism $X \dashrightarrow X$ is dominant.*

Let X and Y be varieties over F and $d = \dim(X)$. A *correspondence from X to Y*, denoted $\alpha\colon X \rightsquigarrow Y$, is an element $\alpha \in \operatorname{CH}_d(X \times Y)$. If $\dim(Y) = d$, we write $\alpha^t : Y \rightsquigarrow X$ for the image of α under the exchange isomorphism $\operatorname{CH}_d(X \times Y) \simeq \operatorname{CH}_d(Y \times X)$.

Let $\alpha\colon X \rightsquigarrow Y$ be a correspondence. Assume that Y is complete. The projection morphism $p\colon X \times Y \to X$ is proper and hence the push-forward homomorphism

$$p_*\colon \operatorname{CH}_d(X \times Y) \to \operatorname{CH}_d(X) = \mathbb{Z} \cdot [X]$$

is defined [**11**, § 1.4]. The integer $\operatorname{mult}(\alpha) \in \mathbb{Z}$ such that $p_*(\alpha) = \operatorname{mult}(\alpha) \cdot [X]$ is called the *multiplicity* of α. For example, if α is the the class of the closure of the graph of a rational morphism $X \dashrightarrow Y$ of varieties of the same dimension, then $\operatorname{mult}(\alpha) = 1$ and $\operatorname{mult}(\alpha^t) = \deg(f)$.

Proposition 1.9. *Let X be a complete variety of dimension d over F. Suppose that for a prime integer p and every correspondence $\alpha \in \operatorname{CH}_d(X \times X)$ one has $\operatorname{mult}(\alpha) \equiv \operatorname{mult}(\alpha^t)$ modulo p. Then X is p-minimal.*

Proof. Let f and $g : X' \to X$ be morphisms from a complete variety X' of dimension d and let $\alpha \in \operatorname{CH}_d(X \times X)$ be the class of the closure of the image of $(f, g) : X' \to X \times X$. Then $\operatorname{mult}(\alpha) = \deg(f)$ and $\operatorname{mult}(\alpha^t) = \deg(g)$. Hence by assumption, $\deg(f) \equiv \deg(g)$ modulo p. If $\deg(f)$ is relatively prime to p, then so is $\deg(g)$. In particular, g is dominant. By Corollary 1.8(1), X is p-minimal. $\square$

Example 1.10. Let q be a non-degenerate anisotropic quadratic form on a vector space V over F of dimension at least 2 and let X be the associated quadric hypersurface in $\mathbb{P}(V)$ (cf. [**9**, §22]). The *first Witt index $i_1(q)$ of q* is the Witt index of q over the function field $F(X)$. It is proved in [**15**, Prop. 7.1] that the condition of Proposition 1.9 holds for X and $p = 2$ if and only if $i_1(q) = 1$. In this case X is 2-minimal. It follows that $\operatorname{cdim}_2(X) = \operatorname{cdim}(X) = \dim(X)$ if $i_1(q) = 1$. In general, $\operatorname{cdim}_2(X) = \operatorname{cdim}(X) = \dim(X) - i_1(q) + 1$ (cf. [**15**, Th. 7.6]).

Example 1.11. Let A be a central simple algebra over F of dimension n^2 and $X = \mathrm{SB}(A)$ the *Severi-Brauer variety* of right ideals in A of dimension n. In is shown in [**15**, Th. 2.1] that if A is a division algebra of dimension a power of a prime integer p, then the condition of Proposition 1.9 holds for X and p. In particular, X is p-minimal. It follows that for any central simple algebra A of p-primary index, we have $\mathrm{cdim}_p(X) = \mathrm{cdim}(X) = \mathrm{ind}(A) - 1$. Moreover, the equality $\mathrm{cdim}_p(X) = \mathrm{ind}_p(A) - 1$, where $\mathrm{ind}_p(A)$ is the largest power of p dividing $\mathrm{ind}_p(A)$, holds for every central simple algebra A.

This example can be generalized as follows.

Example 1.12. Let p be a prime integer and D a (finite) p-subgroup of the Brauer group $\mathrm{Br}(F)$ of a field F. Let $A_1, A_2, \dots, A_s$ be central simple F-algebras whose classes in $\mathrm{Br}(F)$ generate D. Let $X = X_1 \times \cdots \times X_s$, where $X_i = \mathrm{SB}(A_i)$ for every $i = 1, \dots, s$. Suppose that $\dim(X)$ is the smallest possible (over all choices of the generators). Then the condition of Proposition 1.9 holds for X and p (cf. [**14**, Cor. 2.6, Rem. 2.9]) and hence X is p-minimal.

Let A be a central simple F-algebra of degree n. Consider the class $\mathcal{C}_A$ of all splitting fields of A in *Fields*$/F$. Let $X = \mathrm{SB}(A)$, so $\dim(X) = n - 1$. We write $\mathrm{cdim}_p(A)$ for $\mathrm{cdim}_p(X)$ and $\mathrm{cdim}(A)$ for $\mathrm{cdim}(X)$. Since A is split over a field extension E/F if and only if $X(E) \neq \emptyset$, we have

$$\mathrm{cdim}_p(A) = \mathrm{cdim}_p(\mathcal{C}_A) = \mathrm{cdim}_p(X)$$

for every $p \geq 0$. Write $n = q_1 q_2 \cdots q_r$ where the q_i are powers of distinct primes. Then A is a tensor product $A_1 \otimes A_2 \otimes \ldots \otimes A_r$, where A_i is a central division F-algebra of degree q_i. A field extension E/F splits A if and only if E splits A_i for all i. In other words, X has an E-point if and only if the variety $Y = \mathrm{SB}(A_1) \times \mathrm{SB}(A_2) \times \cdots \times \mathrm{SB}(A_r)$ has an E-point. Hence

$$\mathrm{cdim}(A) = \mathrm{cdim}(X) = \mathrm{cdim}(Y) \leq \dim(Y) = \sum_{i=1}^{r} (q_i - 1). \tag{1}$$

It was conjectured in [**8**] that the inequality in (1) is actually an equality. This is proved in [**15**, Th. 2.1] (see also [**1**, Th. 11.4]) in the case when $r = 1$, i.e., when $\deg(A)$ is power of a prime integer. The case $n = 6$ was settled in [**8**].

1.7. Canonical dimension and generic fields. Let F be a field and let $\mathcal{C}$ be a class of fields in *Fields*$/F$. A field $L \in \mathcal{C}$ is called *p-generic in $\mathcal{C}$* if for any field $K \in \mathcal{C}$ there is a geometric F-place $L \rightharpoonup K'$, where K' is a finite extension of K of degree prime to p (cf. Appendix 6.2). In the case $p = 0$ we simply say that L is *generic in $\mathcal{C}$*. Clearly, if L is generic, then it is p-generic for all p.

Example 1.13. If X is a smooth variety, then by Lemma 6.6, the function field $F(X)$ is generic.

Lemma 1.14. *If L is a p-generic field in $\mathcal{C}$ and $L \succ_p M$ with $M \in \mathcal{C}$, then M is p-generic.*

Proof. Take any $K \in \mathcal{C}$. There are field extensions K'/K and L'/L of degree prime to p, a geometric F-place $L \rightharpoonup K'$ and an F-homomorphism $M \to L'$. By Lemma 6.5, there is a field extension K''/K' of degree prime to p and a geometric

F-place $L' \rightharpoonup K''$ extending the place $L \rightharpoonup K'$. The composition $M \to L' \rightharpoonup K''$ is a geometric place and K''/K is an extension of degree prime to p. Hence M is p-generic. □

We say that a class $\mathcal{C}$ is *closed under specializations*, if for any F-place $L \rightharpoonup K$ with $L \in \mathcal{C}$ we have $K \in \mathcal{C}$. Clearly if $\mathcal{C}$ is closed under specializations, then $\mathcal{C}$ is closed under extensions.

Example 1.15. If a variety X is complete, then the class $\mathcal{C}_X$ is closed under specializations. Indeed, let $L \rightharpoonup K$ be an F-place with $X(L) \neq \emptyset$. If $R \subset L$ is the valuation ring of the place, then $X(R) \neq \emptyset$ as X is complete. It follows that $X(K) \neq \emptyset$ since there is an F-homomorphism $R \to K$.

Theorem 1.16. *Let $\mathcal{C}$ be a class of fields in Fields/F and $p \geq 0$ satisfying:*

1. *$\mathcal{C}$ has a p-generic field.*
2. *$\mathcal{C}$ is closed under specializations.*

Then $\operatorname{ed}_p(\mathcal{C})$ is the least $\operatorname{tr.deg}_F(L)$ over all p-generic fields $L \in \mathcal{C}$.

Proof. Let $L \in \mathcal{C}$ be a p-generic field with the least $\operatorname{tr.deg}_F(L)$. By Lemma 1.14, any field $M \in \mathcal{C}$ with $L \succ_p M$ is also p-generic. Hence L is p-minimal. It follows that $\operatorname{tr.deg}_F(L) \leq \operatorname{ed}_p(\mathcal{C})$.

Let $L \in \mathcal{C}$ be a p-generic field and $K \in \mathcal{C}$ an arbitrary p-minimal field. There is a place $L \rightsquigarrow K'$ over F, where K' is an extension of K of degree prime to p. Let $K'' \subset K'$ be the image of the place. As $\mathcal{C}$ is closed under specializations, we have $K'' \in \mathcal{C}$. Since $K \succ_p K''$ and K is p-minimal, we have $\operatorname{tr.deg}_F(K'') = \operatorname{tr.deg}_F(K)$. Hence

$$\operatorname{tr.deg}_F(L) \geq \operatorname{tr.deg}_F(K'') = \operatorname{tr.deg}_F(K).$$

Therefore, $\operatorname{tr.deg}_F(L) \geq \operatorname{ed}_p(\mathcal{C})$. □

Remark 1.17. By Examples 1.13 and 1.15, for a smooth complete variety X over F, the class $\mathcal{C}_X$ satisfies the conditions of the theorem. In particular, for such an X, the integer $\operatorname{cdim}_p(X)$ coincides with the canonical p-dimension introduced in [**16**].

Example 1.18. Let G be either a (finite) étale or a split (connected) reductive group over F. Let B be a Borel subgroup in G and E a G-torsor over a field extension L of F. Then E has an L-point if and only if E/B has an L-point. As E/B is a smooth complete variety, the class the class $\mathcal{C}_E$ satisfies the conditions of Theorem 1.16, hence $\operatorname{cdim}_p(E)$ can be computed using p-generic splitting fields as in [**16**].

2. Essential p-dimension of a presheaf of sets

By a *presheaf of sets on Var/F* we mean a functor $S : (\mathsf{Var}/F)^{op} \to \mathsf{Sets}$. If $f : X' \to X$ is a morphism in Var/F and $a \in S(X)$, then abusing notation we shall often write $a_{X'}$ for the image of a under the map $S(f) : S(X) \to S(X')$.

Definition 2.1. Let S be a presheaf of sets on Var/F. Let $X, Y \in \mathsf{Var}/F$ and $a \in S(X)$, $b \in S(Y)$. We write $a >_p b$ if there is a variety $X' \in \mathsf{Var}/F$, a morphism $g : X' \to Y$ and a dominant morphism $f : X' \to X$ of degree prime to p such that $a_{X'} = b_{X'}$ in $S(X')$.

Let S be a presheaf of sets on Var/F and $a \in S(X)$ for some $X \in \mathsf{Var}/F$. The *essential dimension of* a, denoted $\operatorname{ed}_p(a)$, is the least $\dim(Y)$ over all elements $b \in S(Y)$ for a variety Y with $a >_p b$. As $a >_p a$, we have $\operatorname{ed}_p(a) \leq \dim(X)$.

The *essential p-dimension of the functor* S is the integer

$$\operatorname{ed}_p(S) = \sup\{\operatorname{ed}_p(a)\}$$

over all $a \in S(X)$ and varieties $X \in \mathsf{Var}/F$. We also write $\operatorname{ed}(S)$ for $\operatorname{ed}_p(S)$ if $p = 0$.

The relation $>_p$ is not transitive in general. We refine this relation as follows. We write $a \rhd_p b$ if $a >_p b$ and in addition, in Definition 2.1, the morphism g is dominant. We also write $a \blacktriangleright_p b$ if $a >_p b$ and in addition, in Definition 2.1, the morphism f satisfies the following condition: for every point $x \in X$, there is a point $x' \in X'$ with $f(x') = x$ and $[F(x') : F(x)]$ prime to p.

Lemma 2.2. *Let S be a presheaf of sets on Var/F, $a \in S(X)$, $b \in S(Y)$ and $c \in S(Z)$.*

1. *If $a >_p b$ and $b \blacktriangleright_p c$, then $a >_p c$.*
2. *If $a \rhd_p b$ and $b >_p c$, then $a >_p c$.*

Proof. In the definition of $a >_p b$, let $f : X' \to X$ be a dominant morphism of degree prime to p and $g : X' \to Y$ a morphism. In the definition of $b >_p c$, let $h : Y' \to Y$ be a dominant morphism of degree prime to p and $k : Y' \to Z$ a morphism. Let $y \in Y$ be the image of the generic point of X' under g. In the case (1), there is an $y' \in Y'$ such that $f(y') = y$ and $[F(y') : F(y)]$ is prime to p. In the case (2), y is the generic point of Y. If y' is the generic point of Y', then $[F(y') : F(y)]$ is prime to p. Thus in any case, $[F(y') : F(y)]$ is prime to p. Hence by Lemma 6.3, there is a commutative square of morphisms of varieties

$$\begin{tikzcd}
X'' \arrow[r, "m"] \arrow[d, "l"'] & X' \arrow[d, "g"'] \\
Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

with m dominant of degree prime to p. Then the compositions $f \circ m$ and $k \circ l$ yield $a >_p c$. □

Let $a \in S(X)$ and $V \subset X$ a subvariety. We write $a|_V$ for the restriction of a on V.

Lemma 2.3. *Let S be a presheaf of sets on Var/F, $a \in S(X)$ and $b \in S(Y)$. Suppose that $a >_p b$. Then:*

1. *There is an open subvariety $U \subset X$ such that $(a|_U) \blacktriangleright_p b$.*
2. *There is a closed subvariety $Z \subset Y$ such that $a \rhd_p (b|_Z)$.*

Proof. Choose a variety $X' \in \mathsf{Var}/F$, a morphism $g : X' \to Y$ and a dominant morphism $f : X' \to X$ of degree prime to p such that $a_{X'} = b_{X'}$ in $S(X')$.

(1): By Lemma 6.2, there exists a nonempty open subset $U \subset X$ such that for every $x \in U$ there is a point $x' \in X'$ with $f(x') = x$ and the degree $[F(x') : F(x)]$ prime to p. Then the restrictions $f^{-1}(U) \to U$ and $f^{-1}(U) \to Y$ yield $(a|_U) \blacktriangleright_p b$.

(2): Let Z be the closure of the image of g. We have $a \rhd_p (b|_Z)$. □

Corollary 2.4. *Let S be a presheaf of sets on Var/F and $a \in S(X)$. Then there is an element $b \in S(Y)$ such that $\operatorname{ed}_p(a) = \dim(Y)$ and $a \rhd_p b$.*

Proof. By the definition of the essential p-dimension, there is $b \in S(Y)$ such that $\operatorname{ed}_p(a) = \dim(Y)$ and $a >_p b$. By Lemma 2.3, there is a closed subvariety $Z \subset Y$ such that $a \rhd_p (b|_Z)$. In particular, $a >_p (b|_Z)$. As $\dim(Y)$ is the smallest integer with the property that $a >_p b$, we must have $\dim(Z) = \dim(Y)$, i.e., $Z = Y$. It follows that $a \rhd_p b$. $\square$

2.1. The associated functor $\widetilde{S}$. Let S be a presheaf of sets on Var/F. We define a functor $\widetilde{S} : \mathsf{Fields}/F \to \mathsf{Sets}$ as follows. Let $L \in \mathsf{Fields}/F$. The sets $S(X)$ over all models X of L form a direct system with respect to morphisms of models (cf. Appendix 6.1). Set

$$\widetilde{S}(L) = \operatorname{colim} S(X).$$

In particular, for any $X \in \mathsf{Var}/F$, we have a canonical map $S(X) \to \widetilde{S}(L)$ with $L = F(X)$. We write $\tilde{a} \in \widetilde{S}(L)$ for the image of an element $a \in S(X)$. For every $L \in \mathsf{Fields}/F$, any element of $\widetilde{S}(L)$ is of the form $\tilde{a}$ for some $a \in S(X)$, where X is a model of L.

An F-homomorphism of fields $L \to L'$ yields a morphism $X' \to X$ of the corresponding models and hence the maps of sets $S(X) \to S(X')$ and $\widetilde{S}(L) \to \widetilde{S}(L')$ making $\widetilde{S}$ a functor.

Recall that we have the relations $>_p$ and $\succ_p$ defined for the functors S and $\widetilde{S}$ respectively.

Lemma 2.5. *Let S be a presheaf of sets on Var/F, $X \in \mathsf{Var}/F$, $K \in \mathsf{Fields}/F$ $a \in S(X)$ and $\beta \in \widetilde{S}(K)$. Then $\tilde{a} \succ_p \beta$ if and only if there is a model Y of K and an element $b \in S(Y)$ such that $\tilde{b} = \beta$ and $a \rhd_p b$.*

Proof. $\Rightarrow$: There is a finite field extension $L'/F(X)$ of degree prime to p and an F-homomorphism $K \to L'$ such that $\tilde{a}_{L'} = \beta_{L'}$. One can choose a model X' of L' and Y of K together with two dominant morphisms $X' \to X$ and $X' \to Y$, the first of degree prime to p, that induce field homomorphisms $F(X) \to L'$ and $K \to L'$ respectively. Replacing Y and X' by open subvarieties, we may assume that there is $b \in S(Y)$ with $\tilde{b} = \beta$. The elements $a_{X'}$ and $b_{X'}$ may not be equal in $S(X')$ but they coincide when restricted to an open subvariety $U \subset X'$. Replacing X' by U, the variety Y by an open subvariety W in the image of U and b by $b|_W$ we get the $a \rhd_p b$.

$\Leftarrow$: Choose a variety $X' \in \mathsf{Var}/F$, a dominant morphism $g : X' \to Y$ and a dominant morphism $f : X' \to X$ of degree prime to p such that $a_{X'} = b_{X'}$ in $S(X')$. Then $F(Y)$ and $F(X')$ are subfields of $F(X')$, the degree $[F(X') : F(X)]$ is prime to p and $\tilde{a}_{F(X')} = \tilde{b}_{F(X')} = \beta_{F(X')}$, hence $\tilde{a} \succ_p \beta$. $\square$

Proposition 2.6. *Let S be a presheaf of sets on Var/F, $X \in \mathsf{Var}/F$ and $a \in S(X)$. Then $\operatorname{ed}_p(a) = \operatorname{ed}_p(\tilde{a})$ for all p. Moreover, $\operatorname{ed}_p(S) = \operatorname{ed}_p(\widetilde{S})$.*

Proof. By Corollary 2.4, there is $b \in S(Y)$ such that $\operatorname{ed}_p(a) = \dim(Y)$ and $a \rhd_p b$. It follows from Lemma 2.5 that $\tilde{a} \succ_p \tilde{b}$. Hence

$$\operatorname{ed}_p(\tilde{a}) \le \operatorname{tr.deg}_F F(Y) = \dim(Y) = \operatorname{ed}_p(a).$$

Let $\beta \in \widetilde{S}(L)$ be so that $\tilde{a} \succ_p \beta$ and $\operatorname{ed}_p(\tilde{a}) = \operatorname{tr.deg}_F(L)$. By Lemma 2.5, we can choose a model Y of L and an element $b \in S(Y)$ so that $\tilde{b} = \beta$ and $a >_p b$. Hence

$$\operatorname{ed}_p(a) \le \dim(Y) = \operatorname{tr.deg}_F(L) = \operatorname{ed}_p(\tilde{a}). \qquad \square$$

2.2. Generic elements. Let S be a presheaf of sets on Var/F and $X \in \mathsf{Var}/F$. An element $a \in S(X)$ is called *p-generic for S* if for any open subvariety $U \subset X$ and any $b \in S(Y)$ with the infinite field $F(Y)$ we have $b >_p (a|_U)$. Note that $F(Y)$ is infinite if either F is infinite or $\dim(Y) > 0$. We say that a is *generic* if a is p-generic for $p = 0$. If a is generic, then a is p-generic for all p.

Generic elements provide an upper bound for the essential dimension.

PROPOSITION 2.7. *Let S be a presheaf of sets on Var/F and $a \in S(X)$ a p-generic element for S. Then $\mathrm{ed}_p(S) \le \dim(X)$.*

PROOF. Let $b \in S(Y)$. If the field $F(Y)$ is finite, we have $\mathrm{ed}_p(b) = 0$. If $F(Y)$ is infinite, $b >_p a$ since a is p-generic. By the definition of the essential p-dimension, in any case, $\mathrm{ed}_p(b) \le \dim(X)$, hence $\mathrm{ed}_p(S) \le \dim(X)$. $\square$

Clearly, if a is p-generic, then so is the restriction $a|_U \in S(U)$ for any open subvariety $U \subset X$. This can be generalized as follows.

PROPOSITION 2.8. *Let S be a presheaf of sets on Var/F, $X, Y \in \mathsf{Var}/F$, $a \in S(X)$ and $b \in S(Y)$. Suppose that $a >_p b$ and a is p-generic. Then b is also p-generic for S.*

PROOF. Let $c \in S(Z)$ with the field $F(Z)$ infinite and $V \subset Y$ an open subvariety. Clearly, $a >_p (b|_V)$. By Lemma 2.3(1), we have $(a|_U) \blacktriangleright_p (b|_V)$ for an open subvariety $U \subset X$. Since a is p-generic, we have $c >_p (a|_U)$. By Lemma 2.2(1), $c >_p (b|_V)$, hence b is p-generic. $\square$

THEOREM 2.9. *Let S be a presheaf of sets on Var/F. If $a \in S(X)$ is a p-generic element for S, then*

$$\mathrm{ed}_p(S) = \mathrm{ed}_p(\widetilde{S}) = \mathrm{ed}_p(\tilde{a}) = \mathrm{ed}_p(a).$$

PROOF. In view of Proposition 2.6, it suffices to prove that $\mathrm{ed}_p(S) \le \mathrm{ed}_p(a)$. Choose an element $c \in S(Z)$ such that $a >_p c$ and $\mathrm{ed}_p(a) = \dim(Z)$. By Lemma 2.3(1), there is an open subvariety $U \subset X$ such that $(a|_U) \blacktriangleright_p c$.

Let $Y \in \mathsf{Var}/F$ and let $b \in S(Y)$ be any element. If the field $F(Y)$ is finite, we have $\mathrm{ed}_p(b) = 0$. Otherwise, as a is p-generic, we have $b >_p (a|_U)$. It follows from Lemma 2.2(1) that $b >_p c$. Hence, in any case, $\mathrm{ed}_p(b) \le \dim(Z) = \mathrm{ed}_p(a)$ and therefore, $\mathrm{ed}_p(S) \le \mathrm{ed}_p(a)$. $\square$

Let S be a presheaf of sets on Var/F. An element $\alpha \in \widetilde{S}(L)$ is called *p-generic for $\widetilde{S}$* is $\alpha = \tilde{a}$ for a p-generic element a for S.

EXAMPLE 2.10. One can view a scheme X over F as a presheaf of sets on Var/F by $X(Y) := \mathrm{Mor}_F(Y, X)$ for every $Y \in \mathsf{Var}/F$. Then the functor $\widetilde{X} : \mathsf{Fields}/F \to \mathsf{Sets}$ coincides with the one in Proposition 1.2. It follows from Theorem 2.9 that $\mathrm{ed}_p(X) = \dim(X)$ for all p.

By Proposition 2.7, for a p-generic element $a \in S(X)$, one has $\mathrm{ed}_p(S) \le \dim(X)$. The following proposition asserts that $\mathrm{ed}_p(S)$ is equal to the dimension of a closed subvariety of X with a certain property.

PROPOSITION 2.11. *Let S be a presheaf of sets on Var/F and $a \in S(X)$ a p-generic element for S. Suppose that either F is infinite or $\mathrm{ed}_p(S) > 0$. Then $\mathrm{ed}_p(S) = \min \dim(Z)$ over all closed subvarieties $Z \subset X$ such that $a >_p (a|_Z)$.*

Proof. For any closed subvariety $Z \subset X$ with $a >_p (a|_Z)$ one has $\operatorname{ed}_p(S) = \operatorname{ed}_p(a) \leq \dim(Z)$. We shall show that the equality holds for some $Z \subset X$.

By Corollary 2.4, there is $b \in S(Y)$ with $\dim(Y) = \operatorname{ed}_p(a) = \operatorname{ed}_p(S)$ and $a \rhd_p b$. By assumption, the field $F(Y)$ is infinite. As a is p-generic, we have $b >_p a$. By Lemma 2.3(2), there is a closed subvariety $Z \subset X$ such that $b \rhd_p (a|_Z)$. It follows that $\dim(Z) \leq \dim(Y) = \operatorname{ed}_p(S)$. By Lemma 2.2(2), $a >_p (a|_Z)$. □

Remark 2.12. The assumption in the proposition can not be dropped (cf. Remark 4.7).

An element $a \in S(X)$ is called *p-minimal* if $\operatorname{ed}_p(a) = \dim(X)$, i.e., whenever $\alpha >_p \beta$ for some $\beta \in S(Y)$, we have $\dim(X) \leq \dim(Y)$. By Lemma 2.2(2) and Corollary 2.4, for every $a \in S(X)$, there is a p-minimal $b \in S(Y)$ such that $\operatorname{ed}_p(a) = \dim(Y)$ and $a \rhd_p b$. It follows that $\operatorname{ed}_p(S)$ is the maximum of $\operatorname{ed}_p(\alpha)$ over all p-minimal elements α.

A p-minimal element with $p = 0$ is called *minimal*.

If $a \in S(X)$ is p-generic p-minimal, then $\operatorname{ed}_p(S) = \dim(X)$.

If $a \in S(X)$ is a p-generic element for S and $b \in S(Y)$ is a p-minimal element satisfying $a \rhd_p b$, then by Proposition 2.8, b is also p-generic, and hence $\operatorname{ed}_p(S) = \dim(Y)$.

The following statement gives a characterization of p-generic p-minimal elements.

Proposition 2.13. *Let S be a presheaf of sets on Var/F and $a \in S(X)$ a p-generic element for S. Suppose that either F is infinite or $\operatorname{ed}_p(S) > 0$. Then a is p-minimal if and only if for any two morphisms f and g from a variety X' to X such that $S(f)(a) = S(g)(a)$ with f dominant of degree prime to p, the morphism g is also dominant.*

Proof. Suppose a is p-minimal and let f and g be morphisms in the statement of the proposition. Let Z be the closure of the image of g, so $a >_p (a|_Z)$. By Proposition 2.11, $\dim(X) = \operatorname{ed}_p(S) \leq \dim(Z)$, hence $Z = X$ and g is dominant.

Suppose a is not p-minimal. By Proposition 2.11, there is a proper closed subvariety $Z \subset X$ such that $a >_p (a|_Z)$, i.e., there are morphisms $f : X' \to X$ and $g' : X' \to Z$ such that $S(f)(a) = S(g')(a|_Z)$ and f is dominant of degree prime to p. If $g : X' \to X$ if the composition of g' with the embedding of Z into X, then $S(f)(a) = S(g)(a)$ and g is not dominant. □

Specializing to the case $p = 0$ we have:

Corollary 2.14. *In the conditions of the proposition, a is minimal if and only if for any two morphisms f and g from a variety X' to X such that $S(f)(a) = S(g)(a)$ with f a birational isomorphism, the morphism g is dominant.*

3. Essential p-dimension of fibered categories

The notion of the essential p-dimension can be defined for fibered categories over Var/F or Fields/F as follows (cf. [**3**]).

Let $\mathcal{A}$ be a category and $\varphi : \mathcal{A} \to \mathsf{Var}/F$ a functor. For a variety $Y \in \mathsf{Var}/F$, we write $\mathcal{A}(Y)$ for the *fiber category* of all objects ξ in $\mathcal{A}$ with $\varphi(\xi) = Y$ and morphisms over the identity of Y. We assume that the category $\mathcal{A}(Y)$ is essentially small for all Y, i.e., the isomorphism classes of objects form a set.

Suppose that $\mathcal{A}$ is a *fibered category over* Var/F (cf. [**26**]). In particular, for any morphism $f : Y \to Y'$ in Var/F, there is a *pull-back functor* $f^* : \mathcal{A}(Y') \to \mathcal{A}(Y)$ such that for any two morphisms $f : Y \to Y'$ and $g : Y' \to Y''$ in Var/F, the composition $f^* \circ g^*$ is isomorphic to $(g \circ f)^*$.

Let $\mathcal{A}$ be a fibered category over Var/F. For any $Y \in \mathsf{Var}/F$, we write $S_{\mathcal{A}}(Y)$ for the set of isomorphism classes of objects in the category $\mathcal{A}(Y)$. The functor f^* for a morphism $f : Y \to Y'$ in Var/F induces a map of sets $S_{\mathcal{A}}(Y') \to S_{\mathcal{A}}(Y)$ making $S_{\mathcal{A}}$ a presheaf of sets on Var/F. We call $S_{\mathcal{A}}$ the *presheaf of sets associated with* $\mathcal{A}$. The *essential p-dimension* $\operatorname{ed}_p(\mathcal{A})$ *of* $\mathcal{A}$ (respectively, the *canonical p-dimension* $\operatorname{cdim}_p(\mathcal{A})$ *of* $\mathcal{A}$) is defined as $\operatorname{ed}_p(S_{\mathcal{A}})$ (respectively, $\operatorname{cdim}_p(S_{\mathcal{A}})$).

Remark 3.1. In a similar fashion, one can define the essential p-dimension for fibered categories over Fields/F. This notion agrees with the one given above in view of Theorem 2.9.

Example 3.2. Let X be a scheme over F. Consider the category Var/X of varieties over X, i.e., morphisms $Y \to X$ for a variety Y over F. Morphisms in Var/X are morphisms of varieties over X. The functor $\mathsf{Var}/X \to \mathsf{Var}/F$ taking $Y \to X$ to Y together with the obvious pull-back functors f^* make Var/X a fibered category. For any variety Y, the fiber category over Y is equal to the set $\operatorname{Mor}_F(Y, X)$. Hence the associated presheaf of sets on Var/F coincides with X viewed as a presheaf as in Example 2.10. It follows that $\operatorname{ed}_p(\mathsf{Var}/X) = \dim(X)$ for all p.

Example 3.3. Let G be an algebraic group scheme over a field F. The *classifying space* $\mathrm{B}G$ *of the group* G is the category with objects (right) G-torsors $q : E \to Y$ with $Y \in \mathsf{Var}/F$ and morphisms between G-torsors $q : E \to Y$ and $q' : E' \to Y'$ given by commutative diagrams

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E' \arrow[d] \\
Y \arrow[r] & Y'
\end{tikzcd}$$

with the top arrow a G-equivariant morphism. For every $Y \in \mathsf{Var}/F$, the fiber category $\mathrm{B}G(Y)$ is the category of G-torsors over Y. We write $\operatorname{ed}_p(G)$ for $\operatorname{ed}_p(\mathrm{B}G)$ and call this integer the *essential p-dimension of* G. Equivalently, by Proposition 2.6, $\operatorname{ed}_p(G)$ is the essential p-dimension of the functor $\mathsf{Fields}/F \to \mathsf{Sets}$ taking a field L to the set of isomorphism classes of G-torsors over L.

Example 3.4. We can generalize the previous example as follows. Let an algebraic group scheme G act on a scheme X over F. We define the fibered category X/G as follows. An object in X/G over a variety Y is a diagram

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "q"'] & X \\
Y &
\end{tikzcd}$$

where q is a G-torsor and f is a G-equivariant morphism. Morphisms of diagrams in X/G are defined in the obvious way. The functor $X/G \to \mathsf{Var}/F$ takes the diagram to the scheme Y. The set $S_{X/G}(Y)$ consists of all isomorphism classes of the diagrams above. For any field $L \in \mathit{Fields}/F$, an element of the set $\widetilde{S}_{X/G}(L)$ is given by the diagram

$$\begin{tikzcd}
E' \arrow[r, "f'"] \arrow[d, "q'"'] & X \\
\operatorname{Spec} L &
\end{tikzcd}$$

where q' is a G-torsor and f' is a G-equivariant morphism.

Note that if X is a G-torsor over a scheme Y, then $X/G \simeq Y$, and if $X = \operatorname{Spec} F$, then $X/G = \mathrm{B}G$.

3.1. Gerbes. Let C be a commutative algebraic group scheme over F. There is the notion of a *gerbe banded by* C (cf. [**19**, p. 144], [**13**, IV.3.1.1], see also examples below). There exists a bijection between the flat cohomology group $H^2(F, C) := H^2_{fppf}(\operatorname{Spec} F, C)$ and the set of isomorphism classes of gerbes banded by C. The trivial element in $H^2(F, C)$ corresponds to the classifying space $\mathrm{B}C$, so $\mathrm{B}C$ is a *trivial* (*split*) gerbe banded by C. In general, a gerbe banded by C can be viewed as a "twisted form" of $\mathrm{B}C$.

Example 3.5. Let

$$1 \to C \to G \to H \to 1$$

be an exact sequence of algebraic group schemes with C a commutative group and $E \to \operatorname{Spec} F$ an H-torsor. The group G acts on E via the map $G \to H$. The category E/G is a gerbe banded by C. The corresponding element in $H^2(F, C)$ is the image of the class of E under the connecting map

$$H^1(F, H) \to H^2(F, C).$$

Example 3.6. (Gerbes banded by μ_n) Let A be a central simple F-algebra and n an integer with $[A] \in \mathrm{Br}_n(F) = H^2(F, \mu_n)$. Let X be the Severi-Brauer variety of A. Denote by $\mathcal{X}_A$ the gerbe banded by μ_n corresponding to $[A]$. It is shown in [**3**] that if n is a power of a prime integer p, then

$$\mathrm{ed}_p(\mathcal{X}_A) = \mathrm{ed}(\mathcal{X}_A) = \mathrm{cdim}_p(\mathcal{X}_A) + 1 = \mathrm{cdim}(\mathcal{X}_A) + 1 = \mathrm{ind}(A).$$

Example 3.7. One can generalize the previous example as follows. Let p be a prime integer and C a diagonalizable algebraic group scheme of rank s and exponent p over F. In other words, C is isomorphic to the product of s copies of μ_p. An element $\theta \in H^2(F, C)$ determines a gerbe $\mathcal{X}$ banded by C. Consider the homomorphism $\beta : C^* \to \mathrm{Br}(F)$ taking a character $\chi \in C^*$ to the image of θ under the map $H^2(F, C) \to H^2(F, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}(F)$ induced by χ. It follows from [**14**, 3.1] that

$$\mathrm{ed}_p(\mathcal{X}) = \mathrm{ed}(\mathcal{X}) = \mathrm{cdim}_p(\mathcal{X}) + s = \mathrm{cdim}(\mathcal{X}) + s. \tag{2}$$

For a generating set $\chi_1, \chi_2, \dots, \chi_s$ of C^*, let $A_1, A_2, \dots, A_s$ be central division F-algebras such that $[A_i] = \beta(\chi_i)$. Set $X_i = \mathrm{SB}(A_i)$ and $X = X_1 \times \cdots \times X_s$. Clearly, the gerbe $\mathcal{X}$ is split over a field extension L of F if and only if all the algebras A_i are split over L if and only if X has a point over L. It follows that $\mathrm{cdim}_p(\mathcal{X}) = \mathrm{cdim}_p(X)$.

By Example 1.12, any basis of $\mathrm{Ker}(\beta)$ over $\mathbb{Z}/p\mathbb{Z}$ can be completed to a basis $\chi_1, \chi_2, \ldots, \chi_s$ of C^* such that X is p-minimal, i.e.,

$$\mathrm{cdim}_p(X) = \dim(X) = \sum_{i=1}^{s} \big(\mathrm{ind}(A_i) - 1\big) = \sum_{i=1}^{s} \big(\mathrm{ind}\,\beta(\chi_i) - 1\big).$$

It follows from (2) that

$$\mathrm{ed}_p(\mathcal{X}) = \sum_{i=1}^{s} \mathrm{ind}\,\beta(\chi_i).$$

4. Essential p-dimension of algebraic group schemes

Let G be an algebraic group scheme over a field F. A *G-space* is a finite dimensional vector space V with a (right) linear G-action. (Equivalently, the natural map $G \to \mathbf{GL}(V)$ is a finite dimensional representation of G.) We say that *G acts on V generically freely* (or *V is generically free*) if there is a nonempty open G-invariant subset $V' \subset V$ and a G-torsor $V' \to X$ for some scheme X over F (cf. [**2**, Def. 4.8 and 4.10]).

One can construct G-spaces V with generically free action as follows. Embed G into $\mathbf{GL}(W)$ as a subgroup for some vector space W of finite dimension and set $V = \mathrm{End}(W)$. We view V as a G-space via right multiplications. Then $\mathbf{GL}(W)$ is an open G-invariant subset in V and the natural morphism $\mathbf{GL}(W) \to \mathbf{GL}(W)/G$ is a G-torsor.

Theorem 4.1. (cf. [**22**, Lemma 6.6], [**12**, Example 5.4]) *Let G be an algebraic group scheme over a field F and V a G-space. Suppose that G acts on V generically freely, i.e., there is a nonempty open subset $V' \subset V$ and a G-torsor $a : V' \to X$ for some scheme X. Then the torsor a is p-generic for all p.*

Proof. Let $b : E \to Y$ be a G-torsor with the infinite field $F(Y)$. Let $U \subset X$ be an open subvariety. We need to show that $b >_p (a|_U)$. Replacing X by U and V' by $a^{-1}(U)$ we may assume that $U = X$. We shall show that $b >_p a$.

The morphism $a \times b : V' \times E \to X \times Y$ is a $(G \times G)$-torsor. Considering G as a diagonal subgroup of $G \times G$ we have a G-torsor $c : V' \times E \to Z$ and a commutative diagram

$$\begin{tikzcd}
V' \arrow[d, "a"'] & V' \times E \arrow[l] \arrow[r] \arrow[d, "c"'] & E \arrow[d, "b"'] \\
X & Z \arrow[l, "g"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

with the projections in the top row. The scheme $V' \times E$ is an open subset of the (trivial) vector bundle $V \times E$ over E. By descent, Z is an open subset of a vector bundle over Y. Therefore, the generic fiber of f is an open set of a vector space over the infinite field $F(Y)$ and hence it has a point over $F(Y)$, i.e., the generic fiber of f has a splitting. It follows that there is an open subvariety $W \subset Y$ such that f has a splitting $h : W \to Z$ over W.

Set $E' := W \times_Z (V' \times E)$. In the commutative diagram with fiber product squares

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & V' \times E \arrow[r] \arrow[d, "c"'] & E \arrow[d, "b"'] \\
W \arrow[r, "h"] & Z \arrow[r, "f"] & Y
\end{tikzcd}$$

the composition in the bottom row is the inclusion morphism. Hence $E' = E|_W$ and the left vertical arrow coincides with $b|_W$. The commutative diagram

$$\begin{tikzcd}
E & E|_W \arrow[l] \arrow[r] \arrow[d, "b|_W"'] & V' \arrow[d, "a"'] \\
Y & W \arrow[l] \arrow[r, "gh"] & X
\arrow[from=1-1, to=2-1, "b"']
\end{tikzcd}$$

then yields $b >_p a$ for all p. □

Corollary 4.2. (cf. [**2**, Prop. 4.11]) *Let G be an algebraic group scheme over a field F. Then* $\mathrm{ed}_p(G) \leq \dim(V) - \dim(G)$ *for every generically free G-space V.*

Corollary 4.3. *Let G be an algebraic group scheme over a field F and H a subgroup of G. Then* $\mathrm{ed}_p(G) + \dim(G) \geq \mathrm{ed}_p(H) + \dim(H)$.

Proof. Let $a : V' \to X$ be the p-generic G-torsor as in Theorem 4.1. Since H acts on V generically freely, there is a p-generic H-torsor $b : V' \to Y$. Let $a >_p c$ for a G-torsor $c : E \to Z$ with $\dim(Z) = \mathrm{ed}_p(G)$. Let $d : E \to S$ be the H-torsor associated to c. As $a >_p c$, we have $b >_p d$ and hence

$$\begin{aligned}\mathrm{ed}_p(H) &\leq \dim(S) = \dim(E) - \dim(H) \\ &= \dim(Z) + \dim(G) - \dim(H) = \mathrm{ed}_p(G) + \dim(G) - \dim(H).\end{aligned}$$ □

4.1. Torsion primes and special groups.

For a scheme X over F we let n_X denote the $\gcd \deg(x)$ over all closed points $x \in X$.

Let G be an algebraic group scheme over F. A prime integer p is called a *torsion prime for G* if p divides n_E for a G-torsor $E \to \operatorname{Spec} L$ over a field extension L/F (cf. [**24**, Sec. 2.3]).

An algebraic group scheme G over F is called *special* if for any field extension L/F, every G-torsor over $\operatorname{Spec} L$ is trivial. Clearly, special group schemes have no torsion primes.

The last statement of the following proposition was proven in [**21**, Prop. 5.3] in the case of algebraically closed field F.

Proposition 4.4. *Let G be an algebraic group scheme over F. Then a prime integer p is a torsion prime for G if and only if* $\mathrm{ed}_p(G) \neq 0$. *An algebraic group scheme G is special if and only if* $\mathrm{ed}(G) = 0$.

Proof. Let $p \geq 0$. Suppose that p is not a torsion prime for G if $p > 0$ or G is special if $p = 0$. Let $E \to \operatorname{Spec} L$ be a G-torsor over $L \in \mathit{Fields}/F$. As p is relatively prime to n_E, there is a finite field extension E'/E such that the G-torsor $E_{L'}$ is split and hence comes from a trivial G-torsor over F. It follows that $\mathrm{ed}_p(E) = 0$ and hence $\mathrm{ed}_p(G) = 0$.

Conversely, suppose that $\mathrm{ed}_p(G) = 0$ for $p \geq 0$. Assume that F is infinite. Choose a p-minimal p-generic G-torsor $E \to X$. We claim that n_E is relatively prime to p. Since $\dim(X) = \mathrm{ed}_p(G) = 0$, we have $X = \operatorname{Spec} L$ for a finite field extension L/F. Let E' be a trivial G-torsor over F. As E is generic and the field F is infinite, we have $E' >_p E$, i.e., there is a finite field extension L'/L of degree prime to p such that $E_{L'} \simeq E'_{L'}$. Thus $E_{L'}$ is trivial and hence n_E is relatively prime to p as n_E divides $[L' : L]$.

Let $\gamma : I \to \operatorname{Spec} K$ be a G-torsor over a field extension K/F. We need to show that n_I is relatively prime to p. We may assume that $K \in \mathit{Fields}/F$. Choose a model $c : J \to Z$ of γ, i.e., a G-torsor c with Z a model of K and γ the generic fiber of c. As a is generic, we have $c >_p a$, i.e., a fiber product diagram

$$\begin{tikzcd} J & J' \arrow[l] \arrow[r] \arrow[d, "c'"'] & E \arrow[d, "a"'] \\ Z \arrow[from=1-1, "c"'] & Z' \arrow[l, "f"'] \arrow[r] & X. \end{tikzcd}$$

with f a dominant morphism of degree prime to p and a G-torsor c'. Let $I' \to \operatorname{Spec} K'$ be the generic fiber of c'. Since $n_{I'}$ divides n_E and n_E is relatively prime to p, the integer $n_{I'}$ is also relatively prime to p. It follows that n_I is relatively prime to p since n_I divides $[K' : K]n_{I'}$.

Now let F be a finite field and $\operatorname{ed}_p(G) = 0$. If G is smooth and connected, then G is special (cf. [**25**]). In general, if G° is the connected component of the identity and $G' = G/G^\circ$, then the categories BG and BG' are equivalent, in particular, $\operatorname{ed}_p(G) = \operatorname{ed}_p(G')$ and G and G' have the same torsion primes. Thus, we may assume that $G = G'$ is an étale group scheme. Let K/F be a finite splitting field of G, i.e., G_K is a finite constant group. Every torsion prime of G_K is a torsion prime of G and $\operatorname{ed}_p(G_K) = 0$ by Proposition 1.5(1), so we may assume that G is a constant group.

We claim that the order of G is relatively prime to p. If not, let H be a finite subgroup of G of order p if $p > 0$ and of any prime order if $p = 0$. We have $\operatorname{ed}_p(G) \geq \operatorname{ed}_p(H) > 0$ by Corollary 4.3, a contradiction. Thus, $|G|$ is relatively prime to p. Then every G-torsor E (a Galois G-algebra) is split by a finite field extension of degree prime to p, i.e., n_E is relatively prime to p and p is not a torsion prime of G. □

Theorem 4.5. *Let G be an algebraic group scheme. Assume that either G is not special or F is infinite. Let $a : E \to X$ be a generic G-torsor and let d be the smallest dimension of the image of a rational G-equivariant morphism $E \dashrightarrow E$. Then* $\operatorname{ed}(G) = d - \dim(G)$.

Proof. Let $f : E \dashrightarrow E$ be a rational G-equivariant morphism. Denote by $f' : X \dashrightarrow X$ the corresponding rational morphism. Let Z be the closure of the image of f', so dimension of the image of f is equal to $\dim(Z) + \dim(G)$. There are morphisms $g : X' \to X$ and $h : X' \to Z$ with g a birational isomorphism such that $g^*(E) \simeq h^*(E|_Z)$, i.e., $a > (a|_Z)$. The statement of the theorem follows now from Proposition 2.11. □

Corollary 4.6. *Let G be an algebraic group scheme. Assume that either G is not special or F is infinite. Let $a : E \to X$ be a generic G-torsor. Then a is minimal if and only if every rational G-equivariant morphism $E \dashrightarrow E$ is dominant.*

Remark 4.7. Corollary 4.6 fails for special groups over a finite field. Indeed, let G be the trivial group over a finite field and let X be the affine line with all rational points removed. Since X has no rational points, every rational morphism $X \dashrightarrow X$ is dominant. But the identity morphism of X, which is obviously a generic G-torsor, is not a minimal G-torsor as $\operatorname{ed}(G) = 0$.

4.2. A lower bound.

The following statement was proven in [**3**].

Theorem 4.8. *Let $f : G \to H$ be a homomorphism of algebraic group schemes. Then for any H-torsor E over F, we have* $\operatorname{ed}_p(G) \geq \operatorname{ed}_p(E/G) - \dim(H)$.

Proof. Let L/F be a field extension and let $x = (J, q, \alpha)$ be an object of E/G over $\operatorname{Spec}(L)$. Let $\beta : f_*(J) \to E$ be the isomorphism of H-torsors induced by α. Choose a field extension L'/L of degree prime to p and a subfield $K \subset L'$ over F such that $\operatorname{tr.deg}_F(K) = \operatorname{ed}_p(J)$ and there is a G-torsor I over K with $I_{L'} \simeq J_{L'}$.

We shall write Z for the scheme of isomorphisms $\mathbf{Iso}_K\big(f_*(J), E_K\big)$ of H-torsors over K. Clearly, Z is a torsor over K for the twisted form $\mathbf{Aut}_K\big(f_*(J)\big)$ of H, so $\dim_K(Z) = \dim(H)$. The image of the morphism $\operatorname{Spec} L' \to Z$ over K representing the isomorphism $\beta_{L'}$ is a one-point set $\{z\}$ of Z. Therefore, $\beta_{L'}$ and hence $x_{L'}$ are defined over $K(z)$. It follows that

$$\operatorname{ed}_p(J) + \dim(H) = \operatorname{tr.deg}_F(K) + \dim_K(Z) \geq \operatorname{tr.deg}_F\big(K(z)\big) \geq \operatorname{ed}_p(x).$$

Hence

$$\operatorname{ed}_p(G) \geq \operatorname{ed}_p(J) \geq \operatorname{ed}_p(x) - \dim(H),$$

and $\operatorname{ed}_p(G) \geq \operatorname{ed}_p(E/G) - \dim(H)$. □

4.3. Essential dimension of spinor groups. Let $\mathbf{Spin}_n$, $n \geq 3$, be the split spinor group over a field of characteristic 2. The following inequalities are proved in [**5**, Th. 3.3] if $n \geq 15$:

$$\begin{aligned}
&\operatorname{ed}_2(\mathbf{Spin}_n) \geq 2^{(n-1)/2} - n(n-1)/2 && \text{if } n \text{ is odd}\\
&\operatorname{ed}_2(\mathbf{Spin}_n) \geq 2^{(n-2)/2} - n(n-1)/2 && \text{if } n \equiv 2 \pmod 4\\
&\operatorname{ed}_2(\mathbf{Spin}_n) \geq 2^{(n-2)/2} + 1 - n(n-1)/2 && \text{if } n \equiv 0 \pmod 4
\end{aligned}$$

Moreover, if $\operatorname{char}(F) = 0$, then

$$\begin{aligned}
&\operatorname{ed}_2(\mathbf{Spin}_n) = \operatorname{ed}(\mathbf{Spin}_n) = 2^{(n-1)/2} - n(n-1)/2 && \text{if } n \text{ is odd}\\
&\operatorname{ed}_2(\mathbf{Spin}_n) = \operatorname{ed}(\mathbf{Spin}_n) = 2^{(n-2)/2} - n(n-1)/2 && \text{if } n \equiv 2 \pmod 4\\
&\operatorname{ed}_2(\mathbf{Spin}_n) \leq \operatorname{ed}(\mathbf{Spin}_n) \leq 2^{(n-2)/2} + n - n(n-1)/2 && \text{if } n \equiv 0 \pmod 4
\end{aligned}$$

We improve the lower bound for $\operatorname{ed}_2(\mathbf{Spin}_n)$ in the case $n \equiv 0 \pmod 4$.

Theorem 4.9. *Let n be a positive integer divisible by 4 and $\mathbf{Spin}_n$ the split spinor group over a field F of characteristic different from 2. Let 2^k be the largest power of 2 dividing n. Then*

$$\operatorname{ed}_2(\mathbf{Spin}_n) \geq 2^{(n-2)/2} + 2^k - n(n-1)/2.$$

Proof. The center C of the group $G = \mathbf{Spin}_n$ is isomorphic to $\mu_2 \times \mu_2$. The factor group $H = G/C$ is the special projective orthogonal group (cf. [**17**]). An H-torsor over a field extension L/F determines a central simple algebra A with an orthogonal involution σ of trivial discriminant. The image of the map $C^* \to \operatorname{Br}(L)$ is equal to $\{0, [A], [C^+], [C^-]\}$, where C^+ and C^- are simple components of the Clifford algebra $C(A, \sigma)$. By [**18**], there is a field extension L/F and an H-torsor E over L such that $\operatorname{ind}(C^+) = \operatorname{ind}(C^-) = 2^{(n-2)/2}$ and $\operatorname{ind}(A) = 2^k$, the largest power of 2 dividing n. By Example 3.7,

$$\operatorname{ed}_2(E/G) = \operatorname{ind}(A) + \operatorname{ind}(C^+) = 2^{(n-2)/2} + 2^k.$$

It follows from Theorem 4.8 that

$$\operatorname{ed}_2(\mathbf{Spin}_n) \geq \operatorname{ed}_2(E/G) - \dim(H) = 2^{(n-2)/2} + 2^k - n(n-1)/2.$$ □

COROLLARY 4.10. *If n is a power of 2 and $\mathrm{char}(F) = 0$ then*

$$\mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n) = 2^{(n-2)/2} + n - n(n-1)/2.$$

Below is the table of values $d_n := \mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n)$ over a field of characteristic zero (cf. [**5**]):

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
d_n	0	0	0	0	4	5	5	4	5	6	6	7	23	24	120	103	341

The torsors for $\mathbf{Spin}_n$ are essentially the isomorphism classes of quadratic forms in I^3, where I is the fundamental ideal in the Witt ring of F. A jump of the value of $\mathrm{ed}(\mathbf{Spin}_n)$ when $n > 14$ is probably related to the fact that there is no simple classification of quadratic forms in I^3 of dimension greater than 14.

5. Essential p-dimension of finite groups

Let G be a finite group. We consider G as a constant algebraic group over a field F. A G-torsor E over $\mathrm{Spec}(L)$ for a field extension L/F is of the form $E = \mathrm{Spec}(A)$, where A is a Galois G-algebra over L. Thus, the fibered category $\mathrm{B}G$ is equivalent to the category of Galois G-algebras over field extensions of F.

A generically free G-space is the same as a faithful G-space, i.e., a G-space V such that the group homomorphism $G \to \mathbf{GL}(V)$ is injective. By Corollary 4.2, $\mathrm{ed}(G) \leq \dim(V)$ for any faithful G-space V. The essential dimension $\mathrm{ed}(G)$ can be smaller than dimension of every any faithful G-space V. For example, for the symmetric group S_n one has $\mathrm{ed}(S_n) \leq n-2$ if $n \geq 3$ (cf. [**6**, Th. 6.5]), whereas the least dimension of a faithful S_n-space is equal to $n-1$. Note that the value of $\mathrm{ed}(S_n)$ is unknown for $n \geq 7$.

Computation of the essential p-dimension of a finite group G for $p > 0$ is somewhat simpler. The following proposition shows that G can be replaced by a Sylow p-subgroup.

PROPOSITION 5.1. *Let G be a finite group and $H \subset G$ a Sylow p-subgroup. Then $\mathrm{ed}_p(G) = \mathrm{ed}_p(H)$.*

PROOF. By Corollary 4.3, $\mathrm{ed}_p(G) \geq \mathrm{ed}_p(H)$. Let A be a Galois G-algebra over a field $L \in \mathit{Fields}/F$. Then the subalgebra A^H of H-invariant elements is an étale L-algebra of rank prime to p. Let $e \in A^H$ be an idempotent such that $K = A^H e$ is a field extension of L of degree prime to p. Then Ae is a Galois H-algebra over K. Choose a field extension K'/K of degree prime to p and a subfield $M \subset K$ over F such that there is a Galois H-algebra B over M with $B \otimes_M K' \simeq Ae \otimes_K K'$ and $\mathrm{ed}_p(Ae) = \mathrm{tr.deg}_F(M) \leq \mathrm{ed}_p(H)$.

For any Galois H-algebra C we write $\overline{C}$ for the algebra $\mathrm{Map}_H(G, C)$ of H-equivariant maps $G \to C$. Clearly, $\overline{C}$ has structure of a Galois G-algebra. Considering A as a Galois H-algebra over A^H, we have an isomorphism of Galois G-algebras

$$A \otimes_L (A^H) \to \overline{A}$$

taking $a \otimes a'$ to the map $f : G \to A$ defined by $f(g) = g(a)a'$. It follows that

$$\overline{B} \otimes_M K \simeq \overline{Ae} \otimes_K K' \simeq \overline{A}e \otimes_K K' \simeq A \otimes_L (A^H e) \otimes_K K' = A \otimes_L K'.$$

Hence, A is p-defined over M and the essential p-dimension of the Galois G-algebra A is at most $\mathrm{tr.deg}_F(M) \leq \mathrm{ed}_p(H)$. It follows that $\mathrm{ed}_p(G) \leq \mathrm{ed}_p(H)$. □

By Proposition 1.5(2), the integer $\mathrm{ed}_p(G)$ does not change under field extensions of F of degree prime to p. It follows then from Proposition 5.1 that $\mathrm{ed}_p(G) \leq \dim(V)$ for any faithful H-space V for a Sylow p-subgroup H of G over the field $F(\xi_p)$, where ξ_p is a primitive p-th root of unity.

The following statement was proven in [**14**, Th. 4.1, Rem. 4.8].

Theorem 5.2. *Let p be a prime integer and let F be a field of characteristic different from p. Then the essential p-dimension $\mathrm{ed}_p(G)$ over F of a finite group G is equal to the least dimension of a faithful H-space of a Sylow p-subgroup H of G over the field $F(\xi_p)$.*

Proof. By Propositions 1.5 and 5.1, we may assume that G is a p-group and F contains a primitive p-th root of unity.

By Corollary 4.2, it suffices to find a faithful G-space V with $\mathrm{ed}_p(G) \geq \dim(V)$.

Denote by C the subgroup of all central elements of G of exponent p and set $H = G/C$, so we have an exact sequence

$$1 \to C \to G \to H \to 1. \tag{3}$$

Let $E \to \operatorname{Spec} F$ be an H-torsor over F and let C^* denote the character group $\mathrm{Hom}(C, \mathbf{G}_{\mathrm{m}})$ of C. The H-torsor E over F yields the homomorphism

$$\beta^E : C^* \to \mathrm{Br}(F) \tag{4}$$

taking a character $\chi : C \to \mathbf{G}_{\mathrm{m}}$ to the image of the class of E under the composition

$$H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi_*} H^2(F, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}(F),$$

where ∂ is the connecting map for the exact sequence (3). Note that as $\mu_p \subset F^\times$, we can identify C with $(\boldsymbol{\mu}_p)^s$, i.e., C is a diagonalizable group of exponent p.

Consider the gerbe E/G banded by C. The class of E/G in $H^2(F, C)$ coincides with the image of the class of E under ∂.

By Example 3.7, there is a basis $\chi_1, \chi_2, \dots, \chi_s$ of C^* such that

$$\mathrm{ed}_p(E/G) = \sum_{i=1}^{s} \operatorname{ind} \beta^E(\chi_i). \tag{5}$$

Now we choose a specific E, namely a generic H-torsor over a field extension L of F. Let $\chi : C \to \mathbf{G}_{\mathrm{m}}$ be a character and $\mathrm{Rep}^{(\chi)}(G)$ the category of all G-spaces such that $v^c = \chi(c)v$ any $c \in C$ and $v \in V$. By Theorem 6.7,

$$\operatorname{ind} \beta^E(\chi) = \gcd \dim(V) \tag{6}$$

over all G-spaces V in $\mathrm{Rep}^{(\chi)}(G)$. Note that dimension of every irreducible G-space is a power of p. Indeed, let q be the order of G. By [**23**, Th. 24], every irreducible G-space is defined over the field $K = F(\mu_q)$. Since F contains p-th roots of unity, the degree $[K : F]$ is a power of p. Let V be an irreducible G-space over F. Write V_K as a direct sum of irreducible G-spaces V_j over K. As each V_j is absolutely irreducible, $\dim(V_j)$ divides q and hence $\dim(V_j)$ is a power of p. The group $\Gamma = \mathrm{Gal}(K/F)$ permutes transitively the V_j. As $|\Gamma|$ is a power of p, the number of the V_i's is also a power of p.

Hence, the gcd in (6) can be replaced by min. Therefore, for any character $\chi \in C^*$, there is a G-space $V_\chi \in \mathrm{Rep}^{(\chi)}(G)$ such that $\operatorname{ind} \beta^E(\chi) = \dim(V_\chi)$. Let V be the direct sum of the V_{χ_i} for $i = 1, \dots, s$. It follows from (5) that

$$\mathrm{ed}_p(E/G) = \dim(V).$$

Applying Proposition 1.5(1) and Theorem 4.8 for the gerbe E/G over the field L, we get the inequality

$$\mathrm{ed}_p(G) \geq \mathrm{ed}_p(G_L) \geq \mathrm{ed}_p(E/G) = \dim(V).$$

It suffices to show that V is a faithful G-space. Since the χ_i form a basis of C^*, the C-space V is faithful. Let N be the kernel of V. We have $N \cap C = \{1\}$. As every nontrivial normal subgroup of G intersects C nontrivially, it follows that $N = \{1\}$, i.e., the G-space V is faithful. $\square$

Corollary 5.3. *Let G be a p-group and let F be a field containing p-th roots of unity. Then* $\mathrm{ed}(G)$ *coincides with* $\mathrm{ed}_p(G)$ *and is equal to the least dimension of a faithful G-space over F.*

Proof. Let V be a faithful G-space of the least dimension. Then by Theorem 5.2 and Corollary 4.2,

$$\dim(V) = \mathrm{ed}_p(G) \leq \mathrm{ed}(G) \leq \dim(V). \qquad \square$$

The case of a cyclic group was considered in [**10**]:

Corollary 5.4. *Let G be a cyclic group of a primary order p^n and let F be a field containing p-th roots of unity. Then* $\mathrm{ed}(G) = \mathrm{ed}_p(G) = [F(\xi_{p^n}) : F]$.

Proof. The G-space $F(\xi_{p^n})$ with a generator of G acting by multiplication by ξ_{p^n} is a faithful irreducible G-space of the least dimension. $\square$

6. Appendix

6.1. Models. For any $X \in \mathsf{Var}/F$, the field $F(X)$ lies in Fields/F. Conversely, let $L \in \mathsf{Fields}/F$. A *model of* L is a variety $X \in \mathsf{Var}/F$ together with an isomorphism $F(X) \simeq L$ over F. A morphism of two models X and X' of L is a (unique) birational isomorphism between X and X' preserving the identifications of the field $F(X)$ and $F(X')$ with L.

Let $K \subset L$ be a subfield and Y a model of K, so we have a morphism $\operatorname{Spec} L \to Y$. Then there is a model X of L and a dominant morphism $f : X \to Y$ inducing the field embedding $K \hookrightarrow L$. Indeed, we can start with any model X of L and then replace it by the graph of the corresponding rational morphism $X \dashrightarrow Y$. The morphism f is called a *model of the morphism* $\operatorname{Spec} L \to Y$.

Let p be a prime integer.

Lemma 6.1 (cf. [**14**, Lemma 3.3]). *Let K be an arbitrary field, K'/K a finite field extension of degree prime to p, and $K \to L$ a field homomorphism. Then there exists a field extension L'/L of degree prime to p and a field homomorphism $K' \to L'$ extending $K \to L$.*

Proof. We may assume that K' is generated over K by one element. Let $f(t) \in F[t]$ be its minimal polynomial. Since the degree of f is prime to p, there exists an irreducible divisor $g \in L[t]$ of f over L such that $\deg(g)$ is prime to p. We set $L' = L[t]/(g)$. $\square$

Lemma 6.2. *Let $f : X' \to X$ be a morphism of varieties over F of degree prime to p. Then there is an open subvariety $U \subset X$ such that for every $x \in U$ there exists a point $x' \in X'$ with $f(x') = x$ and the degree $[F(x') : F(x)]$ prime to p.*

PROOF. Let $U \subset X$ be an open subvariety such that the restriction $f^{-1}(U) \to U$ of f is flat of degree d (prime to p). Then for every $x \in U$, the fiber $f^{-1}(x)$ is a finite scheme over $F(x)$ of degree d, i.e., $f^{-1}(x) = \operatorname{Spec} A$ for an $F(x)$-algebra A of dimension d. The artinian ring A is a product of local rings A_i with maximal ideals P_i. We have

$$d = \sum \dim(A_i) = \sum \dim(A_i/P_i) \cdot l(A_i),$$

where $l(A_i)$ is the length of the A-module A_i and dimension is taken over $F(x)$. As d is prime to p, there is an i such that $\dim(A_i/P_i)$ is prime to p. The corresponding point $x' \in f^{-1}(x)$ satisfies the required conditions. □

LEMMA 6.3. *Let $g : X \to Y$ and $h : Y' \to Y$ be morphisms of varieties over F. Let $y \in Y$ be the image of the generic point of X. Suppose that there is a point $y' \in Y'$ such that $h(y') = y$ and $[F(y') : F(y)]$ is prime to p. Then there exists a commutative square of morphisms of varieties*

$$\begin{tikzcd} X' \arrow[r, "m"] \arrow[d, "l"'] & X \arrow[d, "g"'] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

with m dominant of degree prime to p.

PROOF. We view the residue field $F(y)$ as a subfield of the fields $F(X)$ and $F(y')$. By Lemma 6.1, there is a field extension L of $F(X)$ and $F(y')$ such that $[L : F(X)]$ is prime to p. The natural morphisms $\operatorname{Spec} L \to X$ and $\operatorname{Spec} L \to Y'$ yield a morphism $\operatorname{Spec} L \to X \times_Y Y'$. Clearly, a model $X' \to X \times_Y Y'$ of this morphism together with the projections $m : X' \to X$ and $l : X' \to Y'$ fit in the required diagram. □

6.2. Valuations and places. A *geometric valuation* of a field $L \in \mathsf{Fields}/F$ is a valuation v of L over F with residue field $F(v)$ such that $\operatorname{rank}(v) = \operatorname{tr.deg}_F(L) - \operatorname{tr.deg}_F F(v)$. The residue field of a geometric valuation is necessarily finitely generated over F (cf. [**27**]).

Let L and K be field extensions of F. An *F-place* $\pi : K \rightharpoonup L$ is a local ring homomorphism $R \to K$ of a valuation ring R in L containing F. The ring R is called the *valuation ring of* π. We say that π is *geometric* is the valuation of R is geometric.

If $\pi : L \rightharpoonup K$ and $\rho : M \rightharpoonup L$ are two places, then the *composition of places* $\pi \circ \rho : M \rightharpoonup K$ is defined. If π and ρ are geometric, then so is $\pi \circ \rho$.

A geometric place is a composition of places with discrete geometric valuation rings.

LEMMA 6.4. *Let $L \in \mathsf{Fields}/F$, let v be a geometric valuation of L over F and let L'/L be a finite field extension of degree prime to p. Then there exists a geometric valuation v' of L' extending v such that the degree of the residue field extension $F(v')/F(v)$ is prime to p.*

PROOF. If L'/L is separable and $v_1, \dots, v_k$ are all the extensions of v on L', then $[L' : L] = \sum e_i [F(v_i) : F(v)]$ where e_i is the ramification index (cf. [**27**, Ch. VI, Th. 20 and p. 63]). It follows that the integer $[F(v_i) : F(v)]$ is prime to p for some i.

If L'/L is purely inseparable of degree q, then the valuation v' of L' defined by $v'(x) = v(x^q)$ satisfies the desired properties. The general case follows. $\square$

This lemma translates to the language of place as follows:

Lemma 6.5. [**16**, Lemma 3.2] *Let $L \in \mathsf{Fields}/F$, let $\rho : L \rightharpoonup K$ be a geometric F-place and let L'/L be a field extension of degree prime to p. Then there exists a field extension K'/K of degree prime to p and an extension $L' \rightharpoonup K'$ of the place ρ.*

Lemma 6.6. *Let X be an algebraic variety over F and $x \in X$ a nonsingular point. Then there is a geometric valuation of $F(X)$ with center x and residue field $F(x)$.*

Proof. Choose a regular system of parameters $a_1, a_2, \dots, a_n$ in the regular local ring $R = O_{X,x}$. Let M_i be the ideal of R generated by $a_1, \dots, a_i$. Set $R_i = R/M_i$ and $P_i = M_{i+1}/M_i$. Denote by F_i the quotient field of R_i, in particular, $F_0 = F(X)$ and $F_n = F(x)$. The localization ring $(R_i)_{P_i}$ is a discrete geometric valuation ring with quotient field F_i and residue field F_{i+1}, therefore it determines a geometric place $F_i \rightharpoonup F_{i+1}$. The valuation corresponding to the composition of places

$$F(X) = F_0 \rightharpoonup F_1 \rightharpoonup \dots \rightharpoonup F_n = F(x)$$

is a geometric valuation satisfying the required conditions. $\square$

6.3. Indices of algebras. Let G be a finite group and C a central subgroup. We set $H = G/C$. Let W be a faithful H-space and W' an open subset of the affine space of W where H acts freely, so that there is an H-torsor $\pi : W' \to Y$. Let E be the generic fiber of the H-torsor π. It is a generic H-torsor over the function field $L = F(Y)$. Consider the homomorphism $\beta^E : C^* \to \mathrm{Br}(F)$ defined in (4).

Let $\chi : C \to \mathbf{G}_{\mathrm{m}}$ be a character and let $\mathrm{Rep}^{(\chi)}(G)$ be the category of all G-spaces such that $v^c = \chi(c)v$ any $c \in C$ and $v \in V$.

Theorem 6.7. *Let G be a finite group and let C be a central subgroup of G. Assume that $|C|$ is not divisible by* $\operatorname{char} F$. *Set $H = G/C$ and let E be a generic H-torsor. Then for any character $\chi \in C^*$, we have* $\operatorname{ind} \beta^E(\chi) = \gcd \dim(V)$ *over all G-spaces V in* $\mathrm{Rep}^{(\chi)}(G)$.

In the rest of the section we give a proof of this theorem.

Let S be a commutative ring and H a finite group acting on S (on the right) by ring automorphisms. Set

$$R = S^H := \{s \in S \quad \text{such that} \quad s^h = s \quad \text{for all} \quad h \in H\}$$

and denote by $S * H$ the crossed product with trivial factors. Precisely, $S * H$ consists of formal sums $\sum_{h \in H} hs_h$ with $s_h \in S$. The product is given by the rule $(hs)(h's') = (hh')(s^{h'}s')$.

Let M be a (right) S-module. Suppose that H acts on M on the right such that $(ms)^h = m^h s^h$. Then M is a right $S*H$-module by $m(hs) = m^h s$. Conversely, a right $S*H$-module is a right S-module together with a right H-action as above. If M is a right $S*H$-module, then the subset M^H of H-invariant elements in M is an R-module. We have a natural S-module homomorphism $M^H \otimes_R S \to M$, $m \otimes s \mapsto ms$.

We say that S is a *Galois H-algebra over R* is the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is an H-torsor.

PROPOSITION 6.8. (cf. [**7**]) *The following are equivalent:*

1. *S is an Galois H-algebra over R.*
2. *The morphism* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is an H-torsor.*
3. *For any $h \in H$, $h \neq 1$, the elements $s^h - s$ with $s \in S$ generate the unit ideal in S.*
4. *For every left $S{*}H$-module M, the natural map $M^H \otimes_R S \to M$ is an isomorphism.*

COROLLARY 6.9. *Let S be an Galois H-algebra over R. Then the functors between the categories of finitely generated right modules*

$$\mathcal{M}(R) \to \mathcal{M}(S{*}H), \qquad N \mapsto N \otimes_R S$$
$$\mathcal{M}(S{*}H) \to \mathcal{M}(R), \qquad M \mapsto M^H$$

are equivalences inverse to each other.

PROOF OF THEOREM 6.7. Let W be a faithful H-space. Let S denote the symmetric algebra of the dual space W^*. The group H acts on S. Set $R = S^H$, $Y = \operatorname{Spec}(R)$ and $L = F(Y)$ the quotient field of R.

For any $h \in H$, $h \neq 1$, there is a linear form $\varphi_h \in W^*$ satisfying $(\varphi_h)^h \neq \varphi_h$. Set

$$r = \prod_{h,h' \in H, h \neq 1} \big((\varphi_h)^{hh'} - (\varphi_h)^{h'}\big)$$

in S. We have $r \in R$ and $r \neq 0$. For any $h \neq 1$, the element $(\varphi_h)^h - \varphi_h$ is invertible in the localization ring S_r. By Proposition 6.8, the localization ring S_r is a Galois H-algebra over R_r.

Let $\chi : C \to \mathbf{G}_{\mathrm{m}}$ be a character of C. Note that G acts upon S via the group homomorphism $G \to H$, so we have the ring $S{*}G$ defined. We write $\mathcal{M}^{(\chi)}(S{*}G)$ for the full subcategory of $\mathcal{M}(S{*}G)$ consisting of all modules M with $m^c = \chi(c)m$ for all $m \in M$ and $c \in C$. We also write $K_0^{(\chi)}(S{*}G)$ for the Grothendieck group of $\mathcal{M}^{(\chi)}(S{*}G)$. Note that $K_0^{(\chi)}(S{*}G)$ is a natural direct summand of $K_0(S{*}G)$.

Fix a G-space $U \in \operatorname{Rep}^{(\chi)}(G)$ and set $U_{S_r} = U \otimes_F S_r$. We have

$$\operatorname{End}(U) \otimes_F S_r \simeq \operatorname{End}_{S_r}\big(U_{S_r}\big).$$

The conjugation G-action on $\operatorname{End}(U)$ factors through an H-action. Consider the algebra $\mathcal{A} = \operatorname{End}_{S_r}\big(U_{S_r}\big)^H$ over R_r. By Proposition 6.8(4),

$$\mathcal{A} \otimes_{R_r} S_r \simeq \operatorname{End}_{S_r}\big(U_{S_r}\big),$$

hence $\mathcal{A}$ is an Azumaya R_r-algebra (by descent, as S_r is a faithfully flat R_r-algebra).

Recall that $L = F(Y)$ is the quotient field of R. Set

$$A = \mathcal{A} \otimes_{R_r} L.$$

Clearly, A is a central simple algebra over L of degree $\dim U$. We also have

$$A = \big(\operatorname{End}(U) \otimes_F L'\big)^H,$$

where L' is the quotient field of S. Moreover, $[A] = \beta^E(\chi)$ in $\operatorname{Br}(L)$.

The localization in algebraic K-theory provides a surjective homomorphism

$$K_0(\mathcal{A}) \to K_0(A). \tag{7}$$

By Corollary 6.9, the category of right $\mathcal{A}$-modules and right $\mathrm{End}_{S_r}\big(U_{S_r}\big)*H$-modules are equivalent. Thus the functor $M \mapsto M^H$ induces an isomorphism

$$K_0\big(\mathrm{End}_{S_r}(U_{S_r})*H\big) \xrightarrow{\sim} K_0(\mathcal{A}). \tag{8}$$

The category of right $\mathrm{End}_{S_r}\big(U_{S_r}\big)*H$-modules is equivalent to the subcategory of right $\mathrm{End}_{S_r}\big(U_{S_r}\big)*G$-modules with the group C acting trivially. Hence we have an isomorphism

$$K_0^{(1)}\big(\mathrm{End}_{S_r}(U_{S_r})*G\big) \xrightarrow{\sim} K_0\big(\mathrm{End}_{S_r}(U_{S_r})*H\big). \tag{9}$$

By Morita equivalence, the functors

$$\mathsf{M}(S_r*G) \to \mathsf{M}\big(\mathrm{End}_{S_r}(U_{S_r})*G\big), \qquad N \mapsto N \otimes_F U^*$$

$$\mathsf{M}\big(\mathrm{End}_{S_r}(U_{S_r})*G\big) \to \mathsf{M}(S_r*G), \qquad M \mapsto M \otimes_{\mathrm{End}(U)} U$$

are equivalences inverse to each other. Moreover, under these equivalences, the subcategory $\mathsf{M}^{(\chi)}(S_r*G)$ corresponds to $\mathsf{M}^{(1)}\big(\mathrm{End}_{S_r}(U_{S_r})*G\big)$. Hence we get an isomorphism

$$K_0^{(\chi)}\big(S_r*G\big) \xrightarrow{\sim} K_0^{(1)}\big(\mathrm{End}_{S_r}(U_{S_r})*G\big). \tag{10}$$

By localization, we have a surjection

$$K_0^{(\chi)}\big(S*G\big) \to K_0^{(\chi)}\big(S_r*G\big). \tag{11}$$

The ring S is graded with $S_0 = F$. We view the ring $B = S*G$ as a graded ring with $B_0 = F*G = FG$ (the group algebra). Note that B is a free left B_0-module. As the global dimension of the polynomial ring S is finite, we can choose a finite projective resolution $P^\bullet \to F$ of the S-module $F = S_0$. Since B is a free right S-module, $B \otimes_S P^\bullet \to B \otimes_S F$ is a finite projective resolution of the left B-module $B \otimes_S F = FG = B_0$. Hence B_0 has finite Tor-dimension as a left B-module.

Therefore, B satisfies the conditions of the following theorem:

Theorem 6.10. [**20**, Th. 7] *Let $B = \coprod_{i\geq 0} B_i$ be a graded Noetherian ring. Suppose:*

1. *B is flat as a left B_0-module,*
2. *B_0 is of finite Tor-dimension as a left B-module.*

Then the exact functor $\mathsf{M}(B_0) \to \mathsf{M}(B)$ taking an M to $M \otimes_{B_0} B$ yields an isomorphism

$$K_0(B_0) \xrightarrow{\sim} K_0(B).$$

By Theorem 6.10, applied to the graded ring $B = S*G$, there is a canonical isomorphism

$$K_0\big(\mathrm{Rep}(G)\big) = K_0(FG) = K_0(B_0) \xrightarrow{\sim} K_0(B) = K_0\big(S*G\big).$$

Moreover, this isomorphism takes $K_0\big(\mathrm{Rep}^{(\chi)}(G)\big)$ onto $K_0^{(\chi)}\big(S*G\big)$, i.e., we have an isomorphism

$$K_0\big(\mathrm{Rep}^{(\chi)}(G)\big) \xrightarrow{\sim} K_0^{(\chi)}\big(S*G\big). \tag{12}$$

The surjective composition $K_0\big(\mathrm{Rep}^{(\chi)}(G)\big) \to K_0(A)$ of the surjective maps (7)-(12) takes the class of a G-space $V \in \mathrm{Rep}^{(\chi)}(G)$ to the class of the right A-module

$$\big(V \otimes_F U^* \otimes_F L'\big)^H$$

of dimension $\dim(V) \cdot \dim(U)$ over the field L. On the other hand, the group $K_0(A)$ is infinite cyclic group generated by the class of a simple module of dimension $\operatorname{ind}(A) \cdot \dim(U)$ over L. The result follows. □

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