INVARIANTS OF ALGEBRAIC GROUPS AND RETRACT RATIONALITY OF CLASSIFYING SPACES

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1. Introduction

Let G be an algebraic group over a field F, V a generically free representation of G (i.e., the stabilizer of the generic point in V is trivial) and $U \subset V$ a G-equivariant open subset such that there is a G-torsor $f : U \longrightarrow U/G$. This is a *versal* G-torsor, in particular, every G-torsor over a field extension K/F with K infinite is isomorphic to the fiber of f over a K-point of U/G. Thus, the K-points of U/G parameterize all G-torsors over $\mathrm{Spec}(K)$.

We think of U/G as an approximation of the classifying space $\mathrm{B}G$. The stable birational and retract rational equivalence classes of U/G are independent of the choice of V and U. We simply say that $\mathrm{B}G$ is *stably rational* (respectively, *retract rational*) if so is U/G. In these cases all the G-torsors over field extensions of F can be parameterized by algebraically independent variables.

The stable (retract) non-rationality of $\mathrm{B}G$ can be detected by *cohomological invariants* which were introduced by J.-P. Serre in [23]. A cohomological invariant of an algebraic group G over a field F with coefficients in a Galois module M over F assigns naturally to every G-torsor over a field extension K/F a Galois cohomology class of K with coefficients in M. A cohomological invariant I is called *unramified* if all values of I over a field extension K/F are unramified with respect to all discrete valuations of K over F. A relation between retract rationality of $\mathrm{B}G$ and unramified invariants is given by the following statement:

If G admits a non-constant unramified cohomological invariant, then the classifying space $\mathrm{B}G$ *is not retract rational.*

For example, this was used by D. Saltman who disproved Noether's Conjecture on the rationality of classifying spaces of finite groups by showing that certain finite groups admit a non-constant degree 2 unramified cohomological invariant.

It is still an open problem whether there exists a connected algebraic group G over an algebraically closed field such that $\mathrm{B}G$ is not retract rational.

In the present paper we review and slightly improve some classical results on the properties of classifying spaces and cohomological invariants. We don't impose any restrictions on the ground field F. The coefficient module $\mathbb{Q}/\mathbb{Z}(j)$ includes nontrivial p-primary component if $p = \mathrm{char}(F) > 0$. In particular, Galois cohomology groups with values in $\mathbb{Q}/\mathbb{Z}(j)$ do not form a cycle module

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of M. Rost if $\mathrm{char}(F) > 0$ since the residue homomorphisms are not always defined. Moreover, the étale cohomology groups with coefficients in $\mathbb{Q}_p/\mathbb{Z}_p(j)$ are not homotopy invariant. All this makes some proofs more involved. We also present a new simpler proof of Rost's theorem on the determination of an invariant by its value at the generic torsor. In particular, non-smooth algebraic groups are allowed. We also consider retract rational varieties over arbitrary fields and give a classification of degree 1 invariants.

A *variety* over a field F in the paper is an integral separated scheme of finite type over F. If R is a commutative F-algebra and X a variety over F, we write $X(R)$ for the set $\mathrm{Mor}_F(\mathrm{Spec}\, R, X)$ of R-points of X.

An *algebraic group* is a linear group scheme of finite type over a field, not necessarily smooth or connected.

2. Galois cohomology

2.1. **The complexes $\mathbb{Q}/\mathbb{Z}(j)$.** For every $j \in \mathbb{Z}$, the complex $\mathbb{Q}/\mathbb{Z}(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\mathrm{Spec}\, F$ is defined as the direct sum of two complexes. The first complex is the sheaf

$$\operatorname*{colim}_n(\mu_n^{\otimes j})$$

placed in degree 0, where $\mu_n^{\otimes j}$ is the j^{th} tensor power of the Galois module μ_n of n^{th} roots of unity. The second complex is nontrivial only in the case $p = \mathrm{char}(F) > 0$ and it is defined via logarithmic de Rham-Witt differentials (see [14, I.5.7] or [15]). In particular, $\mathbb{Q}/\mathbb{Z}(0) = \mathbb{Q}/\mathbb{Z}$ and the p-part of $\mathbb{Q}/\mathbb{Z}(j)$ is trivial if $j < 0$.

For a scheme X over F, we write $H^n\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the degree n étale cohomology group of X with values in $\mathbb{Q}/\mathbb{Z}(j)$. For example,

$$H^2\big(X, \mathbb{Q}/\mathbb{Z}(1)\big) = H^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}(X)$$

is the (cohomological) Brauer group of X.

2.2. **Residue homomorphisms.** For a variety X over F and a closed subscheme $Z \subset X$ we write $H^n_Z\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology group of X with support in Z and values in $\mathbb{Q}/\mathbb{Z}(j)$ (see [19, Ch. III, §1]). Let $X^{(i)}$ be the set of points in X of codimension i. For a point $x \in X^{(1)}$ set

$$H^n_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) = \operatorname*{colim}_{x\in W} H^n_{\overline{\{x\}}\cap W}\big(W, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where the colimit is taken over all open subsets $W \subset X$ containing x. Write

$$\partial_x : H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

for the *residue* homomorphisms arising from the *coniveau spectral sequence* [6, §1.2].

Remark 2.1. If l is a prime integer different from $\mathrm{char}(F)$, then by purity [19, Chapter VI, §5], the primary l-component of $H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ is isomorphic to $H^{n-1}\big(F(x), \mathbb{Q}_l/\mathbb{Z}_l(j-1)\big)$ and the l-component of ∂_x is the standard residue homomorphism of a cycle module (see [20]).

If X is a smooth variety over F and $x \in X^{(1)}$, the sequence

$$0 \longrightarrow H^n\big(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial_x} H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is exact (see [6, Proposition 2.1.2]).

2.3. **The sheaves $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$.** Let X be a smooth variety over F. Write $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ for the Zariski sheaf on X associated with the presheaf

$$U \mapsto H^n\big(U, \mathbb{Q}/\mathbb{Z}(j)\big)$$

of the étale cohomology groups. Pulling back to the generic point of X yields an exact sequence

$$0 \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{x \in X^{(1)}} H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where $\partial = \coprod \partial_x$ (see [6, §2.1]).

2.4. **Unramified cohomology.** Let K/F be a field extension and v a discrete valuation of K over F with valuation ring R. Following [5] and [7], we say that an element $a \in H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ is *unramified with respect to* v if a belongs to the image of the map $H^n\big(R, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$. We write $H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the subgroup of all elements in $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ that are unramified with respect to all discrete valuations of K over F. We have a natural homomorphism

$$H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big). \tag{1}$$

Proposition 2.2. [2, Proposition 5.1] *Let K/F be a purely transcendental field extension. Then the homomorphism (1) is an isomorphism.*

Lemma 2.3. *Let K be an extension of an infinite field F and $a \in H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ such that $a_{K(t)} \in H^n_{\mathrm{nr}}\big(K(t), \mathbb{Q}/\mathbb{Z}(j)\big)$. Then $a \in H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$.*

Proof. We simply write $H(-)$ for $H^n\big(-, \mathbb{Q}/\mathbb{Z}(j)\big)$. Let v be a discrete valuation of K over F, $R \subset K$ the discrete valuation ring of v, M maximal ideal of R and $\widetilde{R}$ the localization of the polynomial ring $R[t]$ at the prime ideal $M[t]$. Then $\widetilde{R}$ is a discrete valuation ring with quotient field $F(t)$. By assumption, $a_{K(t)}$ belongs to the image of $H(\widetilde{R}) \longrightarrow H(K(t))$. As the étale cohomology commutes with colimits by [19, Ch. III, Lemma 1.16], there is a polynomial $f \in R[t] \setminus M[t]$ such that the image of a in $H\big(K[t]_f\big)$ belongs to the image of $\gamma^* : H\big(R[t]_f\big) \longrightarrow H\big(K[t]_f\big)$, where γ is the embedding of $R[t]_f$ into $K[t]_f$.

Since the residue field R/M contains F, it is infinite by the assumption. Therefore, there exists an element $r \in R$ such that $f(r) \in R^\times$. Consider the diagram

$$\begin{tikzcd}
R[t]_f \arrow[r, "\gamma"] \arrow[d, shift left, "\beta_R"] & K[t]_f \arrow[d, shift left, "\beta_K"] \\
R \arrow[r, "\delta"] \arrow[u, shift left, "\alpha_R"] & K \arrow[u, shift left, "\alpha_K"]
\end{tikzcd}$$

with the evaluation homomorphisms $t \mapsto r$ from the top row to the bottom row and the other homomorphisms the natural inclusions. We have $\alpha_K^*(a) = \gamma^*(b)$ for some $b \in H(R[t]_f)$ and therefore,

$$a = \beta_K^*(\alpha_K^*(a)) = \beta_K^*(\gamma^*(b)) = \delta^*(\beta_R^*(b)) \in \mathrm{Im}(\delta^*).$$

Hence a is unramified with respect to v. □

Let X be a smooth variety over F and $x \in X^{(1)}$. Then the local ring $O_{X,x}$ is a discrete valuation ring and let v_x be the corresponding discrete valuation on $F(X)$. It follows from the exact sequence in §2.2 that an element $a \in H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)$ is unramified with respect to the discrete valuation v_x if and only if $\partial_x(a) = 0$.

As shown in §2.3, $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ is identified with the subgroup of all elements in $H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)$ that are unramified with respect to v_x for all $x \in X^{(1)}$. Moreover, we have

$$H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \subset H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \subset H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big).$$

3. Retract rational varieties

The notion of a retract rational field was introduced in [22]. Retract rational varieties were defined in [8, §1]. We extend the definition and the properties of retract rational varieties to the case of an arbitrary base field (not necessarily infinite).

Let X be a variety over a field F. For a (commutative) local F-algebra R with residue field $\overline{R}$, we say that X *has the R-lifting property* if there is a nonempty open subset $W \subset X$ such that the map $W(R) \longrightarrow W(\overline{R})$ is surjective.

Note that if $W' \subset W$ are two open subsets of X and w in $W'(\overline{R})$ belongs to the image of $W(R) \longrightarrow W(\overline{R})$, then $w \in \mathrm{Im}\big(W'(R) \longrightarrow W'(\overline{R})\big)$ since R is a local ring.

It follows that if X has the R-lifting property and a variety Y is birationally isomorphic to X, then Y also has the R-lifting property.

Proposition 3.1. *Let X be a variety over F. Then the following conditions are equivalent:*

1. *X has the R-lifting property for all local F-algebras R.*
2. *X has the R-lifting property for all R with infinite residue field $\overline{R}$.*
3. *For every local F-algebra R with residue field $F(X)$, the generic point of X belongs to the image of the map $X(R) \longrightarrow X(F(X))$.*
4. *There is a morphism $f : Y \longrightarrow X$, where Y is an open subset of the affine space $\mathbb{A}^n_F$ for some n, and a rational morphism $g : X \dashrightarrow Y$ such that $f \circ g = 1_X$.*
5. *There is a morphism $f : Y' \longrightarrow W$, where Y' is an open subset of the affine space $\mathbb{A}^n_F$ for some n and $W \subset X$ a nonempty open subset such that the map $Y'(K) \longrightarrow W(K)$ is surjective for every field extension K/F.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): We may assume that $\overline{R} = F(X)$ is a finite field. In this case $X = \operatorname{Spec}(\overline{R})$. If $\overline{R} = F$, i.e., $X = \operatorname{Spec}(F)$, then X has the R-lifting property. We show that the case $\overline{R} \neq F$ does not occur. Let S be a localization of the polynomial ring $F[x_1, \dots, x_n]$ with respect to a prime ideal such that the residue field is isomorphic to $\overline{R}(t)$. If $\overline{R} \neq F$, i.e., $\overline{R}/F$ is a nontrivial finite field extension, we have $X(\overline{R}(t)) \neq \emptyset$ and $X(S) = \emptyset$ as F is algebraically closed in S. Hence X does not have the S-lifting property for the local ring S with infinite residue field, a contradiction.

(3) $\Rightarrow$ (4): Choose an F-algebra homomorphism $\alpha : F[x_1, \dots, x_n] \longrightarrow F(X)$ such that the quotient field of the image of α is equal to $F(X)$. Let $P = \operatorname{Ker}(\alpha)$. The F-algebra $R = F[x_1, \dots, x_n]_P$ is a local F-algebra with residue field $F(X)$.

Let $W \subset X$ be the open subset in the definition of the R-lifting property. By assumption, the generic point of X is in the image of the map $W(R) \longrightarrow W(F(X)) \subset X(F(X))$. Therefore, there exists a morphism $\beta : \operatorname{Spec}(R) \longrightarrow W$ such that the composition

$$\operatorname{Spec} F(X) \xrightarrow{\gamma} \operatorname{Spec}(R) \xrightarrow{\beta} W \hookrightarrow X$$

is the generic point of X. The map β yields a morphism

$$f : Y \longrightarrow W \hookrightarrow X,$$

where $Y = \operatorname{Spec} F[x_1, \dots, x_n]_h$ for some $h \in F[x_1, \dots, x_n] \setminus P$. The map γ gives a rational morphism $g : X \dashrightarrow Y$ with the required property.

(4) $\Rightarrow$ (1): Let $W \subset X$ be the domain of definition of g. The composition $W \xrightarrow{g} Y \xrightarrow{f} X$ is the inclusion. Let R be a local F-algebra. In the commutative diagram

$$\begin{tikzcd}
W(R) \arrow[r, "g"] \arrow[d, "a"'] & Y(R) \arrow[r, "f"] \arrow[d, "b"'] & X(R) \arrow[d, "c"'] \\
W(\overline{R}) \arrow[r, "g"] & Y(\overline{R}) \arrow[r, "f"] & X(\overline{R})
\end{tikzcd}$$

the map b is surjective since $\mathbb{A}^n_F(R) \longrightarrow \mathbb{A}^n_F(\overline{R})$ is surjective, Y is open in $\mathbb{A}^n_F$ and R is local. It follows that every point in $W(\overline{R})$ is in the image of c and hence is in the image of a since R is local.

(4) $\Rightarrow$ (5): Let W be the domain of definition of g and let $Y' = f^{-1}(W)$. Then the composition $W \longrightarrow Y' \longrightarrow W$ is the identity. It follows that the map $Y'(K) \longrightarrow W(K)$ is surjective for every field extension K/F.

(5) $\Rightarrow$ (4): A lift in $Y'(F(W))$ of the generic point from $W(F(W))$ yields a rational map $X \dashrightarrow Y'$ such that the composition $X \dashrightarrow Y' \longrightarrow W \hookrightarrow X$ is the identity. $\square$

A variety X is called *retract rational* if X satisfies the equivalent conditions of Proposition 3.1. If X is retract rational and a variety Y is birationally isomorphic to X, then Y is also retract rational.

Lemma 3.2. *A variety X over F is retract rational if and only if $X \times \mathbb{A}^1_F$ is retract rational.*

Proof. Suppose $X \times \mathbb{A}^1_F$ is retract rational and let $U \subset X \times \mathbb{A}^1_F$ be a nonempty subset such that the map $U(R) \longrightarrow U(\overline{R})$ is surjective for all local F algebras R. Let W be the image of U under the projection $X \times \mathbb{A}^1_F \longrightarrow X$. As the projection is a flat morphism, W is a nonempty open subset of X.

Let R be a local F-algebra with infinite residue field $\overline{R}$. The fiber of the projection $U \longrightarrow W$ over a point $x \in W(\overline{R})$ is a nonempty open subset of $\mathbb{A}^1_{\overline{R}}$. As the field $\overline{R}$ is infinite, the fiber has an $\overline{R}$-point. It follows that there is a point in $U(\overline{R})$ over x.

The top map in the diagram

$$\begin{tikzcd} U(R) \arrow[r] \arrow[d] & U(\overline{R}) \arrow[d] \\ W(R) \arrow[r] & W(\overline{R}) \end{tikzcd}$$

is surjective. It follows that x is in the image of the bottom map. By Proposition 3.1(2), X is retract rational. □

Corollary 3.3. *Let X be a retract rational variety and Y a variety stably birationally isomorphic to X. Then Y is also retract rational. A stably rational variety is retract rational.*

Proposition 3.4. *Let X be a smooth variety over F. Consider the following properties of X:*

1. *X is a rational variety.*
2. *X is a stably rational variety.*
3. *X is a retract rational variety.*
4. *The natural homomorphism*

$$H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)$$

is an isomorphism.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) is proved in Corollary 3.3.

(3) $\Rightarrow$ (4): Since X is retract rational, there is a (dominant) morphism $f : Y \longrightarrow X$, where Y is an open open subset of the affine space $\mathbb{A}^n_F$ for some n, and a rational morphism $g : X \dashrightarrow Y$ such that $f \circ g = 1_X$. Let $W \subset X$ be the domain of definition of g. We have the following commutative diagram with

all vertical maps injective:

$$\begin{tikzcd}
H^n_{\mathrm{nr}}\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[r, "\alpha"] \arrow[d, hook] & H^n_{\mathrm{nr}}\big(F(Y),\mathbb{Q}/\mathbb{Z}(j)\big) \arrow[d, hook] & \\
H^0_{\mathrm{Zar}}\big(X,\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \arrow[r, "\beta"] \arrow[d, hook] & H^0_{\mathrm{Zar}}\big(Y,\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \arrow[r] & H^0_{\mathrm{Zar}}\big(W,\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \arrow[d, hook] \\
H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big) & = & H^n\big(F(W),\mathbb{Q}/\mathbb{Z}(j)\big).
\end{tikzcd}$$

By diagram chase, β is injective. It follows that α is injective. By Proposition 2.2, $H^n_{\mathrm{nr}}\big(F(Y),\mathbb{Q}/\mathbb{Z}(j)\big) = H^n\big(F,\mathbb{Q}/\mathbb{Z}(j)\big)$. Therefore, $H^n_{\mathrm{nr}}\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big) = H^n\big(F,\mathbb{Q}/\mathbb{Z}(j)\big)$. □

4. Retract rational classifying spaces

4.1. **Versal torsors.** Let G be an algebraic group over a field F. A G-torsor E over a variety X is called *weakly versal* if every G-torsor $T \longrightarrow \mathrm{Spec}(K)$ for a field extension K/F with K infinite is isomorphic to the pull-back of E with respect to a point $\mathrm{Spec}(K) \longrightarrow X$, or equivalently, there is a fiber product square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & E \arrow[d] \\
\mathrm{Spec}(K) \arrow[r] & X
\end{tikzcd}$$

with the top G-equivariant morphism.

A G-torsor $E \longrightarrow X$ is called *versal* if for every nonempty open subset $W \subset X$, the restriction $E_W \longrightarrow W$ of the torsor $E \longrightarrow X$ is weakly versal.

4.2. **Standard torsors.** Let G be an algebraic group, V a generically free G-representation and $U \subset V$ a G-equivariant open subset together with a G-torsor $U \longrightarrow U/G$. We call this torsor a *standard* G-torsor.

By [11, Part 1, §5.4], a standard G-torsor $U \longrightarrow U/G$ is versal.

Example 4.1. Embed G as a subgroup into $\mathbf{GL}(W)$ for a finite dimensional vector space W over F. Then G acts generically freely on the affine space of $V := \mathrm{End}_F(W)$ and taking $U = \mathbf{GL}(W) \subset V$, we have $U/G = \mathbf{GL}(W)/G$.

We think of U/G as an approximation of the classifying space $\mathrm{B}G$ (which we don't define). The stable birational equivalence class (and hence retract rational equivalence class by Corollary 3.3) of U/G is well defined by the no-name Lemma [4]. We simply say that $\mathrm{B}G$ is *stably rational* (respectively, *retract rational*) if so is U/G.

The implication (2) ⇒ (1) in the following statement was proved in [8, Proposition 3.15].

Proposition 4.2. *Let G be an algebraic group over F. The following conditions are equivalent.*

(1) *The classifying space* $\mathrm{B}G$ *is retract rational.*
(2) *For every local* F*-algebra* R *with infinite residue field* $\overline{R}$*, every* G*-torsor over* $\overline{R}$ *can be lifted to a* G*-torsor over* R*.*
(3) *There is a weakly versal* G*-torsor over an open subset* $Y \subset \mathbb{A}^n_F$ *for some* n*.*

Proof. (1) ⇒ (3): Let $U \longrightarrow U/G$ be a standard G-torsor. As U/G is retract rational, there are nonempty open subsets $W \subset U/G$ and $Y \subset \mathbb{A}^n_F$, and morphisms

$$W \longrightarrow Y \longrightarrow U/G$$

with the composition the inclusion. Let $E \longrightarrow Y$ be the pull-back of the versal torsor $U \longrightarrow U/G$. The pull-back $J \longrightarrow W$ of $E \longrightarrow Y$ to W is the restriction $U_W \longrightarrow W$ of $U \longrightarrow U/G$ and hence is weakly versal. Therefore, $E \longrightarrow Y$ is also weakly versal.

(3) ⇒ (2): Let R be a local F-algebra with infinite residue field $\overline{R}$. The top map in the commutative diagram

$$\begin{array}{ccc} Y(R) & \longrightarrow & Y(\overline{R}) \\ \downarrow & & \downarrow \\ \mathrm{Tors}_G(R) & \longrightarrow & \mathrm{Tors}_G(\overline{R}) \end{array}$$

is surjective as Y is an open subset of an affine space. The right vertical map is surjective since $\overline{R}$ is infinite. Therefore, the bottom map is surjective.

(2) ⇒ (1): Embed G as a subgroup into $\mathbf{GL}_N$ for some N and consider the variety $X = \mathbf{GL}_N / G$. Let R be a local F-algebra with infinite residue field $\overline{R}$. We will show that X has the R-lifting property. Let $x \in X(\overline{R})$. By assumption, the corresponding G-torsor over $\overline{R}$ can be lifted to a G-torsor J over R. As R is local, by Hilbert Theorem 90, the map

$$X(R) \longrightarrow \mathrm{Tors}_G(R)$$

is surjective, therefore, there is $\tilde{x} \in X(R)$ mapping to J. The image $x' \in X(\overline{R})$ of $\tilde{x}$ and x give the same G-torsor over $\overline{R}$. Since

$$\mathrm{Tors}_G(\overline{R}) \simeq X(\overline{R}) / \mathrm{GL}_N(\overline{R}),$$

we have $x = gx'$ for some $g \in \mathrm{GL}_N(\overline{R})$. Since the map $\mathrm{GL}_N(R) \longrightarrow \mathrm{GL}_N(\overline{R})$ is surjective, there is $\tilde{g} \in \mathrm{GL}_N(R)$ over g. The image of $\tilde{g}\tilde{x}$ under the map $X(R) \longrightarrow X(\overline{R})$ is $gx' = x$. □

4.3. **Classifying spaces of spinor groups.** We consider an example here. Let F be a field of characteristic different from 2. For every n, consider the quadratic form

$$q_n = \begin{cases} m\mathbb{H}, & \text{if } n = 2m; \\ \langle 1 \rangle \perp m\mathbb{H}, & \text{if } n = 2m+1, \end{cases}$$

where $\mathbb{H}$ is the hyperbolic plane. Write $\mathbf{O}^+_n$, $\mathbf{Spin}_n$ and $\mathbf{\Gamma}^+_n$ for the (split) special orthogonal, spinor and even Clifford groups of q_n, respectively.

Let R be a local F-algebra. The set $\mathrm{Tors}_{\mathbf{O}_n^+}(R) = H^1_{\text{ét}}(R, \mathbf{O}_n^+)$ is identified with the set of isomorphism classes of non-degenerate quadratic forms of rank n over R and determinant $(-1)^m$, if $n = 2m$ or $n = 2m+1$. The connecting map $H^1_{\text{ét}}(R, \mathbf{O}_n^+) \longrightarrow H^2_{\text{ét}}(R, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}(R)$ for the exact sequence

$$1 \longrightarrow \mathbf{G}_{\mathrm{m}} \longrightarrow \mathbf{\Gamma}_n^+ \longrightarrow \mathbf{O}_n^+ \longrightarrow 1$$

takes a form q to the *Clifford invariant* of q which is the class of the Clifford algebra $C(q)$ in $\mathrm{Br}(R)$ if n is even, and to the class of the even Clifford algebra $C_0(q)$ if n is odd. It follows that the set $\mathrm{Tors}_{\mathbf{\Gamma}_n^+}(R) = H^1_{\text{ét}}(R, \mathbf{\Gamma}_n^+)$ is identified with the set of isomorphism classes of non-degenerate quadratic forms of rank n over R of determinant $(-1)^m$ and trivial Clifford invariant.

Lemma 4.3. *The space $\mathrm{B}\mathbf{\Gamma}_{2m}^+$ is retract rational if and only if $\mathrm{B}\mathbf{\Gamma}_{2m-1}^+$ is retract rational.*

Proof. Let R be a local F-algebra with infinite residue field. By Proposition 4.2, it suffice to show that $\mathrm{B}\mathbf{\Gamma}_{2m-1}^+$ has the R-lifting property if and only if so does $\mathrm{B}\mathbf{\Gamma}_{2m}^+$. Let $q \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m-1}^+}(\overline{R})$ be a non-degenerate quadratic forms of rank n over $\overline{R}$ of determinant $(-1)^{m-1}$ and trivial even Clifford invariant. Consider the form $q' = (-q) \perp \langle 1 \rangle$. We have $\det(q') = (-1)^m$ and by [10, Proposition 11.4],

$$[C(q')] = [C_0(q' \perp \langle -1 \rangle)] = [C_0(-q \perp \mathbb{H})] = [C_0(q)] = 0 \in \mathrm{Br}(\overline{R}).$$

It follows that $q' \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m}^+}(\overline{R})$. Let Q' be a lift of q' in $\mathrm{Tors}_{\mathbf{\Gamma}_{2m}^+}(R)$. Write $Q' = P \perp \langle r \rangle$ for some $r \in R^\times$ with $\bar{r} = 1$ and a $(2m-1)$-dimensional form P over R. Let $Q := -rP$. Then $Q \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m-1}^+}(R)$ and we have

$$(-Q) \perp \langle 1 \rangle = rP \perp \langle 1 \rangle = r(P \perp \langle r \rangle) = rQ'.$$

It follows that

$$(-\overline{Q}) \perp \langle 1 \rangle = \bar{r}\overline{Q}' = \overline{Q}' = q' = (-q) \perp \langle 1 \rangle.$$

Hence Q is a lift of q.

Now let $q' \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m}^+}(\overline{R})$. Write $q' = q \perp \langle a \rangle$ for some $a \in \overline{R}$ and q a $(2m-1)$-dimensional form over $\overline{R}$. Then $aq' = aq + \langle 1 \rangle$, hence $-aq \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m-1}^+}(\overline{R})$. Choose $A \in R^\times$ with $\overline{A} = a$. Let P be a lift in $\mathrm{Tors}_{\mathbf{\Gamma}_{2m-1}^+}(R)$ of $-aq$ and set $Q' := A(-P \perp \langle 1 \rangle)$. Then $Q' \in \mathrm{Tors}_{\mathbf{\Gamma}_{2m}^+}(R)$ and

$$\overline{Q}' = \overline{A}(-\overline{P} \perp \langle 1 \rangle) = a(aq \perp \langle 1 \rangle) = q \perp \langle a \rangle = q'.$$

It follows that Q' is a lift of q'. □

Theorem 4.4. (cf., [8, Theorem 4.15]) The classifying spaces $\mathrm{B}\mathbf{\Gamma}_n^+$ and $\mathrm{B}\mathbf{Spin}_n$ are retract rational if $n \leq 14$.

Proof. We first show that $\mathrm{B}\mathbf{\Gamma}_n^+$ is retract rational. By Lemma 4.3 it suffices to assume that n is even. We will proof the R-lifting property for a local F-algebra R. We use the classification of quadratic forms of dimension $n = 2m \leq 14$ of determinant $(-1)^m$ and trivial even Clifford invariant given in [12].

- If $n \le 6$, then $\operatorname{Tors}_{\mathbf{\Gamma}_n^+}(\overline{R})$ is trivial.
- $\operatorname{Tors}_{\mathbf{\Gamma}_8^+}(\overline{R})$ consists of all multiples of 3-fold Pfister forms.
- The map $\operatorname{Tors}_{\mathbf{\Gamma}_8^+}(\overline{R}) \longrightarrow \operatorname{Tors}_{\mathbf{\Gamma}_{10}^+}(\overline{R})$ taking q to $q \perp \mathbb{H}$ is a bijection.
- $\operatorname{Tors}_{\mathbf{\Gamma}_{12}^+}(\overline{R})$ consists of tensor products of a binary form and a 6-dimensional form of determinant -1.
- $\operatorname{Tors}_{\mathbf{\Gamma}_{14}^+}(\overline{R})$ consists of the corestriction (with respect to the trace map) of the form $\sqrt{d}\ \varphi'$ for a quadratic extension $\overline{R}(\sqrt{d})/\overline{R}$, where φ' is the 7-dimensional pure subform of a 3-fold Pfister form φ over $\overline{R}(\sqrt{d})$.

The proofs of the lifting property in all the cases are similar. We consider only the case $n = 14$. Consider a form q in $\operatorname{Tors}_{\mathbf{\Gamma}_{14}^+}(\overline{R})$. Let $D \in R^\times$ be a lift of d and set $S = R(\sqrt{D}) = R[t]/(t^2 - D)$. Let Φ be a 3-fold Pfister form over S lifting φ. Then the corestriction of the form $\sqrt{D}\ \Phi'$ in the extension S/R is a lift of q in $\operatorname{Tors}_{\mathbf{\Gamma}_{14}^+}(R)$. By Proposition 4.2, $\mathrm{B}\mathbf{\Gamma}_n^+$ is retract rational.

The exact sequence

$$1 \longrightarrow \mathbf{Spin}_n \longrightarrow \mathbf{\Gamma}_n^+ \xrightarrow{\mathrm{Sn}} \mathbf{G}_{\mathrm{m}} \longrightarrow 1,$$

where Sn is the spinor norm homomorphism, yields a $\mathbf{G}_{\mathrm{m}}$-torsor $\mathrm{B}\mathbf{Spin}_n \longrightarrow \mathrm{B}\mathbf{\Gamma}_n^+$. It follows that $\mathrm{B}\mathbf{Spin}_n$ is stably birational to $\mathrm{B}\mathbf{\Gamma}_n^+$ and hence is retract rational for $n \le 14$ by Corollary 3.3. □

Conjecture 4.5. *If $n \ge 15$, the space $\mathrm{B}\mathbf{Spin}_n$ is not retract rational.*

5. Cohomology of classifying spaces

Let E be a G-torsor over a variety X. Write E^n for the product of n copies of E. We have a G-torsor $E^n \to E^n/G$ for all n and therefore a simplicial scheme $E^\bullet/G$.

Let H be a contravariant functor from the category of smooth schemes over F to the category of abelian group. Set

$$H(E^\bullet/G) := \operatorname{Ker}(p_1^* - p_2^*) \subset H(X),$$

where p_1^* and p_2^* are induced by the two projections (the face maps of $E^\bullet/G$) $p_i : E^2/G \longrightarrow E/G = X$.

Remark 5.1. If H is a sheaf on the big Zariski site over F, the group $H(E^\bullet/G)$ coincides with the group of sections $H^0_{\mathrm{Zar}}(E^\bullet/G, H)$ of H over $E^\bullet/G$ (see [9, 5.1.3]).

Let $E' \to X'$ be another G-torsor and $f_1, f_2 : E' \longrightarrow E$ two G-equivariant morphisms. Then f_1 and f_2 induce morphisms of simplicial schemes $f_1^\bullet, f_2^\bullet : E'^\bullet/G \longrightarrow E^\bullet/G$.

Lemma 5.2. *The homomorphisms $f_1^*, f_2^* : H(E^\bullet/G) \longrightarrow H(E'^\bullet/G)$ are equal.*

Proof. It is standard that the morphisms $f_1^\bullet$ and $f_2^\bullet$ are (canonically) homotopic, and therefore, yield the same homomorphisms $f_1^* = f_2^*$. Precisely, f_1 and f_2 yield a morphism (homotopy) $h = (f_1, f_2) : X = E'/G \longrightarrow E^2/G$

such that $f_i = p_i \circ h$. Therefore, $f_1^* = h^* \circ p_1^*$ coincides with $f_2^* = h^* \circ p_2^*$ on $H(E^\bullet/G) = \operatorname{Ker}(p_1^* - p_2^*)$. □

Now let $E \longrightarrow X$ be a versal G-torsor and $E' \longrightarrow X'$ another G-torsor with the generic fiber $T \longrightarrow \operatorname{Spec} F(X')$. If $F(X')$ is infinite, by definition of the weak versality, there is a G-equivariant morphism $T \longrightarrow E$. Hence there is a nonempty open subset $W \subset X'$ and a G-equivariant morphism $E'_W \longrightarrow E$. It follows that there is a canonical homomorphism $H(E^\bullet/G) \longrightarrow H({E'_W}^\bullet/G)$.

Consider the following "purity condition" on a G-torsor $E \longrightarrow X$ (and the functor H):

(PC): *The restriction homomorphism* $H(E^\bullet/G) \longrightarrow H(E^\bullet_W/G)$ *is an isomorphism for every nonempty open subset* $W \subset X$.

Therefore, if we assume that $E \longrightarrow X$ is a versal G-torsor and the torsor $E' \longrightarrow X'$ satisfies the condition (PC), we get a canonical homomorphism

$$H(E^\bullet/G) \longrightarrow H(E'^\bullet/G). \tag{2}$$

Symmetrically, if we assume in addition that the torsor $E' \longrightarrow X$ is also versal and the torsor $E \longrightarrow X$ with $F(X)$ infinite satisfies the condition (PC), then the homomorphism (2) is an isomorphism, i.e., the group $H(E^\bullet/G)$ does not depend on the versal torsor $E \longrightarrow X$ up to canonical isomorphism.

Lemma 5.3. *The condition (PC) holds for every standard G-torsor $U \longrightarrow U/G$ and the functor $H(Y) = H^0_{\mathrm{Zar}}\big(Y, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ for every n and j.*

Proof. Let $X = U/G$ and $W \subset X$ be a nonempty open subset. By [2, Proposition A 9], we have $H(U^\bullet_W/G) \subset \operatorname{Ker}(\partial)$, where ∂ is the residue homomorphism in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H(X) \arrow[r] \arrow[d, "p_1^*-p_2^*"] & H(W) \arrow[r, "\partial"] \arrow[d, "p_1^*-p_2^*"] & \coprod\limits_{x\in X^{(1)}\setminus W^{(1)}} H^{n+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \\
0 \arrow[r] & H(U^2/G) \arrow[r] & H(U^2_W/G).
\end{tikzcd}$$

The rows are exact by §2.3. The statement follows by diagram chase. □

It follows from Lemma 5.3 that the group $H^0_{\mathrm{Zar}}\big(U^\bullet/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ does not depend on the choice of the standard G-torsor up to canonical isomorphism. We denote this group by $H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$.

6. Invariants of algebraic groups

Let G be an algebraic group over a field F. Consider the functor

$$\operatorname{Tors}_G : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Sets}},$$

where $\textsf{\textit{Fields}}_F$ is the category of field extensions of F, taking a field K to the set $\operatorname{Tors}_G(K)$ of isomorphism classes of G-torsors over $\operatorname{Spec} K$. Let

$$H : \textsf{\textit{Fields}}_F \longrightarrow \textsf{\textit{Abelian Groups}}$$

be another functor. As defined in [11], an *H-invariant* of G is a morphism of functors

$$I : G\text{- torsors} \longrightarrow H,$$

viewed as functors to *Sets*. In other words, an invariant is a natural in K collection of maps of sets $\mathrm{Tors}_G(K) \longrightarrow H(K)$ for every field extension K/F.

We write $\mathrm{Inv}(G, H)$ for the group of H-invariants of G.

An invariant $I \in \mathrm{Inv}(G, H)$ is called *normalized* if $I(E) = 0$ for every trivial G-torsor E. The normalized invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and

$$\mathrm{Inv}(G, H) \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}}.$$

Let $h : E \longrightarrow X$ be a weakly versal torsor defined in §5. The generic fiber E^{gen} of h is the *generic torsor* over $\mathrm{Spec}\, F(X)$. The evaluation at the generic torsor yields a homomorphism

$$\theta_G : \mathrm{Inv}(G, H) \longrightarrow H^n(F(X), H), \quad I \mapsto I(E^{\mathrm{gen}}).$$

Consider the following "injectivity condition" on the functor H:

(IC): *The homomorphism $H(K) \longrightarrow H(K(Y))$ is injective for every smooth variety Y over a field extension K/F with a K-rational point.*

The following statement was previously known in the case G is smooth (see [11]).

Proposition 6.1. *Suppose that a functor H satisfies the condition (IC). Then the map θ_G is injective.*

Proof. Let $I \in \mathrm{Inv}(G, H)$ be an invariant such that $I(E^{\mathrm{gen}}) = 0$. We will show that for every G-torsor $T \longrightarrow \mathrm{Spec}(K)$ for a field extension K/F, the value $I(T)$ is trivial. Replacing K by $K(t)$ if necessary, we may assume that K is infinite.

Consider the following commutative diagram

$$\begin{tikzcd} T \arrow[d] & T \times E \arrow[l, "g"'] \arrow[r] \arrow[d, "f"'] & E \arrow[d] \\ \mathrm{Spec}(K) & Y \arrow[l] \arrow[r] & X \end{tikzcd}$$

with $Y = (T \times E)/G$, the two fiber product squares and the vertical morphisms the G-torsors. Let $K' = K(Y)$ be the function field of Y and T' the generic fiber of f. Then the G-torsors

$$T_{K'} := T \times_K \mathrm{Spec}(K') \quad \text{and} \quad (E^{\mathrm{gen}})_{K'} := (E^{\mathrm{gen}}) \times_{F(X)} \mathrm{Spec}(K')$$

are both isomorphic to T'. It follows that

$$I(T)_{K'} = I(T_{K'}) = I(T') = I\big((E^{\mathrm{gen}})_{K'}\big) = I(E^{\mathrm{gen}})_{K'} = 0.$$

The torsor E is weakly versal, hence, there is a G-equivariant morphism $T \longrightarrow E$. Therefore, the morphism g in the diagram has a G-equivariant section. It follows that Y has a K-rational point. By assumption, the map $H(K) \longrightarrow H(K')$ is injective and hence $I(T) = 0$. $\square$

We will be considering the cohomology functors H taking a field K/F to the Galois cohomology $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ and write $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *cohomological invariants of G of degree n with coefficients in* $\mathbb{Q}/\mathbb{Z}(j)$.

Lemma 6.2. *The functors $H(K) = H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ satisfy the condition (IC) for all n and j.*

Proof. Let Y be a variety over a field K with a K-rational point y. The completion of the local ring $O_{Y,y}$ at y is the of power series ring $K[[t_1, t_2, \dots, t_d]]$, where $d = \dim(Y)$. Therefore, $K(Y)$ is a subfield of the iterated power series field $K((t_1)) \dots ((t_d))$. Thus, it suffices to show that for every field L, the map $H(L) \longrightarrow H\big(L((t))\big)$ is injective. This was proved in [11, Part 2, Proposition A.9]. $\square$

Consider a standard G-torsor $U \longrightarrow U/G =: X$ and let $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ be its generic fiber. Since the pull-back of $U \longrightarrow U/G$ with respect to any of the two projections $U^2/G \longrightarrow X$ coincides with the G-torsor $U^2 \longrightarrow U^2/G$, the pull-backs of the generic G-torsor $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ with respect to the two morphisms $\operatorname{Spec} F(E^2/G) \longrightarrow \operatorname{Spec} F(X)$ induced by the projections are isomorphic. Hence for every invariant $I \in \mathrm{Inv}\big(G, H^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ we have

$$p_1^*\big(I(E_{\mathrm{gen}})\big) = I\big(p_1^*(E_{\mathrm{gen}})\big) = I\big(p_2^*(E_{\mathrm{gen}})\big) = p_2^*\big(I(E_{\mathrm{gen}})\big)$$

in $H^n\big(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j)\big)$. It follows that the image of θ_G is contained in the subgroup

$$H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \subset H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \subset H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big).$$

Theorem 6.3. *Let G be an algebraic group over a field F. Then θ_G yields an isomorphism*

$$\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\sim} H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big).$$

Proof. The inverse isomorphism was constructed in [2, Theorem 3.4] in the case F is an infinite field as follows. Let $u \in H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$. We define an invariant in $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ by taking a G-torsor $J \longrightarrow \operatorname{Spec}(K)$ to the image of u under the pull-back homomorphism

$$H^0_{\mathrm{Zar}}\big(\mathrm{B}G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^0_{\mathrm{Zar}}\big(\operatorname{Spec}(K), \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big) = H^n(K, \mathbb{Q}/\mathbb{Z}(j)),$$

induced by a G-equivariant morphism $f : J \longrightarrow U$. By Lemma 5.2, the result is independent of the choice of f.

Now let F be a finite field. It suffices to prove that for a given prime integer p, the p-primary components $A(F)$ and $B(F)$ of the groups in the statement are isomorphic. Let l be a prime integer different from p and F'/F an infinite Galois field extension with the Galois group Δ a pro-l-group. Since the field F' is infinite, the map $A(F') \longrightarrow B(F')$ is an isomorphism. By a restriction-corestriction argument, the map $A(F) \longrightarrow B(F)$ is identified with the homomorphism $A(F')^\Delta \longrightarrow B(F')^\Delta$ of the groups of Δ-invariant elements and hence is an isomorphism. $\square$

6.1. **Unramified invariants.** Let G be an algebraic group over F. An invariant $I \in \operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ is called *unramified* if for every field extension K/F and every $T \in \operatorname{Tors}_G(K)$, we have $I(T) \in H^n_{\mathrm{nr}}(K/F, \mathbb{Q}/\mathbb{Z}(j))$. We will write $\operatorname{Inv}^n_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}(j))$ for the subgroup of all unramified invariants in $\operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$.

Proposition 6.4. *Let G be an algebraic group over a field F. Then an invariant $I \in \operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ is unramified if and only if*

$$I(E_{\mathrm{gen}}) \in H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big).$$

Proof. By an argument similar to the one in the proof of Theorem 6.3 we may assume that F is infinite. Suppose $I(E_{\mathrm{gen}}) \in H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)$ and let $T \in \operatorname{Tors}_G(K)$. Consider the variety $Y = (T \times U)/G$ and the natural morphisms $Y \longrightarrow \operatorname{Spec}(K)$ and $Y \longrightarrow U/G$. As in the proof of Proposition 6.1, the torsors E_{gen} and T are isomorphic over the field $K(Y)$. It follows that

$$I(T)_{F(Y)} = I(E_{\mathrm{gen}})_{F(Y)} \in H^n_{\mathrm{nr}}\big(K(Y), \mathbb{Q}/\mathbb{Z}(j)\big).$$

Since $K(Y)/K$ is a purely transcendental field extension, we have $I(T) \in H^n_{\mathrm{nr}}\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ by Lemma 2.3. $\square$

We write $H^n_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}/\mathbb{Z}(j)\big)$ for $H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)$.

Corollary 6.5. *Let G be an algebraic group over a field F. Then θ_G yields an isomorphism $\operatorname{Inv}^n_{\mathrm{nr}}\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\sim} H^n_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}/\mathbb{Z}(j)\big)$.*

Proposition 3.4 then implies the following corollary.

Corollary 6.6. *If an algebraic group over F admits a non-constant unramified cohomological invariant with values in $\mathbb{Q}/\mathbb{Z}(j)$ for some j, then the classifying space $\mathrm{B}G$ is not retract rational over F.*

7. Degree 1 invariants with coefficients in Galois module

Let G be an algebraic group over a field F. Write $\pi_0(G)$ for the factor group of G modulo the connected component of the identity G° of G. It is an étale algebraic group over F.

Let M be a discrete Γ_F-module, where $\Gamma_F = \operatorname{Gal}(F_{\mathrm{sep}}/F)$ and let

$$\alpha : \pi_0(G_{\mathrm{sep}}) \longrightarrow M$$

be a Γ_F-homomorphism. For every field extension K of F we then have the composition

$$I^\alpha_K : H^1(K, G) \longrightarrow H^1\big(K, \pi_0(G)\big) \xrightarrow{\alpha^*} H^1(K, M),$$

where the first map is induced by the canonical surjection $G \longrightarrow \pi_0(G)$ and the second one by α. We can view I^α as a normalized H-invariant of the group G

with the functor H defined by $H(K) = H^1(K, M)$. We write $\operatorname{Inv}^1(G, M)$ for the group of H-invariants.

Note that the functor H satisfied the condition (IC). Indeed, if Y is a smooth variety over a field extension K/F with a K-rational point, then K is algebraically closed in $K(Y)$ and hence the restriction homomorphism $\Gamma_{K(Y)} \longrightarrow \Gamma_K$ is surjective. Therefore, the inflation homomorphism $H^1(K, M) \longrightarrow H^1(K(Y), M)$ is injective.

The following proposition was mentioned in [16, §31.15] without proof.

Proposition 7.1. *The map*

$$\varphi : \operatorname{Hom}_\Gamma\big(\pi_0(G_{\mathrm{sep}}), M\big) \longrightarrow \operatorname{Inv}^1(G, M)_{\mathrm{norm}} \quad \textit{given by } \alpha \mapsto I^\alpha$$

is an isomorphism.

Proof. Consider the case when G is connected. We need to show that every invariant $I \in \operatorname{Inv}^1(G, M)_{\mathrm{norm}}$ is trivial. Let E be a G-torsor over a field extension K/F. Since G is connected, the variety of G and hence the one of E are geometrically irreducible. Therefore, the separable closure of K in the function field $K(E')$ of the associated reduced scheme $E' = E_{\mathrm{red}}$ coincides with K. It follows that the restriction homomorphism $\Gamma_{K(E')} \longrightarrow \Gamma_K$ is surjective and hence the inflation homomorphism

$$H^1(K, M) \longrightarrow H^1(K(E'), M)$$

is injective. As E is trivial over $K(E')$, we have $I(E)_{K(E')} = 0$ and hence $I(E) = 0$ by the injectivity.

In the general case, we construct a map ψ inverse to φ. Let $U \longrightarrow U/G$ be a standard versal G-torsor. Its generic fiber E^{gen} is the *generic* torsor over the field $F(U/G)$.

Let $I(E^{\mathrm{gen}}) \in H^1(F(U/G), M)$ be the value of a normalized invariant I at the generic torsor E^{gen}. Since $U/G^\circ \longrightarrow U/G$ is a surjective morphism, we can view $F(U/G)$ as a subfield of $F(U/G^\circ)$. The image of $I(E^{\mathrm{gen}})$ in $H^1(F(U/G^\circ), M)$ is the value of the restriction of I on G°. By the first part of the proof, the latter value is trivial.

For a field extension K/F, write

$$N(K) := \operatorname{Ker}\big(H^1(K(U/G), M) \longrightarrow H^1(K(U/G^\circ), M)\big).$$

We have proved that $I(E^{\mathrm{gen}}) \in N(F)$.

The field extension $F_{\mathrm{sep}}(U/G^\circ)$ of $F_{\mathrm{sep}}(U/G)$ is Galois with Galois group $\pi_0(G_{\mathrm{sep}})$. By exactness of the inflation-restriction sequence, we have

$$N(F_{\mathrm{sep}}) = H^1(\pi_0(G_{\mathrm{sep}}), M) = \operatorname{Hom}(\pi_0(G_{\mathrm{sep}}), M)$$

since $\pi_0(G_{\mathrm{sep}})$ acts trivially on M. We define the map

$$\psi : \operatorname{Inv}^1(G, M) \longrightarrow \operatorname{Hom}_\Gamma\big(\pi_0(G_{\mathrm{sep}}), M\big)$$

by taking I to the image of $I(E^{\mathrm{gen}})$ under the homomorphism

$$N(F) \longrightarrow N(F_{\mathrm{sep}})^\Gamma = \operatorname{Hom}_\Gamma(\pi_0(G_{\mathrm{sep}}), M).$$

To prove that the composition $\psi\circ\varphi$ is the identity, we may assume that F is separably closed. Let $I=\varphi(\alpha)$ for some $\alpha\in\operatorname{Hom}(\pi_0(G_{\mathrm{sep}}),M)$. By construction, the element $I(E^{\mathrm{gen}})$ in $H^1(K,M)=\operatorname{Hom}(\Gamma_K,M)$, where $K=F(U/G)$, is equal to the composition $\Gamma_K\longrightarrow\pi_0(G)\xrightarrow{\alpha}M$. It follows that $\psi(I)=\alpha$.

Now we prove that the composition $\varphi\circ\psi$ is the identity. For a field extension F'/F consider a homomorphism

$$\beta_{F'}:\operatorname{Hom}_\Gamma(\pi_0(G),M)\longrightarrow H^1(F'(U/G),M)$$

as follows. The element $\beta_{F'}(\alpha)$ is defined as the image of the class of the generic G-torsor under the composition

$$H^1(F'(U/G),G)\longrightarrow H^1(F'(U/G),\pi_0(G))\xrightarrow{\alpha^*}H^1(F'(U/G),M).$$

Taking $F'=F_{\mathrm{sep}}$ we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_\Gamma(\pi_0(G),M) \arrow[r, "\beta_F"] \arrow[d] & H^1(F(U/G),M) \arrow[d, "j"] \\
\operatorname{Hom}(\pi_0(G),M) \arrow[r, "\beta_{F'}"] & H^1(F'(U/G),M).
\end{tikzcd}$$

Note that the bottom map $\beta_{F'}$ is the inflation.

Let $I\in\operatorname{Inv}^1(G,M)$, $\alpha=\psi(I)\in\operatorname{Hom}_\Gamma(\pi_0(G),M)$ and $I'=\varphi(\alpha)$. We need to show that $I=I'$. By construction, $\beta_F(\alpha)=I'(E)$, where E is the generic G-torsor over $F(U/G)$. By the definition of ψ, $j(I(E))$ is the image of α under the diagonal map in the diagram. It follows that $I'(E)-I(E)\in\operatorname{Ker}(j)$. By the inflation-restriction sequence, $\operatorname{Ker}(j)=H^1(F,M)$, hence $I'(E)-I(E)=b_{F(U/G)}$ for an element $b\in H^1(F,M)$. The torsor E is trivial over $F(U)$, hence b vanishes over $F(U)$. It follows that $b=0$ as the map $H^1(F,M)\longrightarrow H^1(F(U),M)$ is injective by (IC).

We proved that I and I' are equal at the generic torsor and hence $I=I'$ by Proposition 6.1. $\square$

8. Brauer Invariants

The invariants with values in the Brauer group are the degree 2 cohomological invariants: $\operatorname{Inv}\bigl(G,\mathrm{Br}\bigr)=\operatorname{Inv}^2\bigl(G,\mathbb{Q}/\mathbb{Z}(1)\bigr)$. Let G be a (connected) semisimple group over F and

$$1\longrightarrow C\longrightarrow\widetilde{G}\longrightarrow G\longrightarrow 1$$

an exact sequence with $\widetilde{G}$ a simply connected semisimple group and C finite central subgroup of $\widetilde{G}$ of multiplicative type. For every character $\chi\in C^*:=$

$\mathrm{Hom}(C, \mathbf{G}_{\mathrm{m}})$, consider the push-out diagram

$$\begin{tikzcd} 1 \arrow[r] & C \arrow[r] \arrow[d] & \widetilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & G' \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

Let K/F be a field extension. By Hilbert Theorem 90, the sequence

$$1 \longrightarrow \mathbf{G}_{\mathrm{m}}(K_{\mathrm{sep}}) \longrightarrow G'(K_{\mathrm{sep}}) \longrightarrow G(K_{\mathrm{sep}}) \longrightarrow 1$$

is exact. Therefore, we have the connecting map

$$\delta_K(\chi) : \mathrm{Tors}_G(K) = H^1(K, G) \longrightarrow H^2(K, \mathbf{G}_{\mathrm{m}}) = \mathrm{Br}(K).$$

This collection of maps $\delta_K(\chi)$ over all K/F is an invariant of G (depending on χ) with values in the Brauer group. Thus, we have a homomorphism

$$\delta : C^* \longrightarrow \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}.$$

Theorem 8.1. ([2, Theorem 2.4]) The map $\delta : C^* \longrightarrow \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$ is an isomorphism.

The following theorem was proved by F. Bogomolov [3, Lemma 5.7] in characteristic zero and in [2, Theorem 5.10] in general.

Theorem 8.2. *Let G be a semisimple group over a field F. Then $\mathrm{Inv}_{\mathrm{nr}}(G, \mathrm{Br})_{\mathrm{norm}} = 0$ and $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big) = \mathrm{Br}(F)$.*

Example 8.3. (see [22] and [11]) Let G be a cyclic group of order 8 and $F = \mathbb{Q}$. A G-torsor over a field extension K/F is a G-Galois cyclic algebra L over K. The class of the central cyclic K-algebra $(L/K, 16)$ of degree 8 in $\mathrm{Br}(K)$ is a non-constant unramified degree 2 invariant ([11, Proposition 33.15]). It follows that $\mathrm{B}G$ is not retract rational. In other words, if V is a faithful representation of G over $\mathbb{Q}$, then the variety V/G is not retract rational.

Example 8.4. Let F be an algebraically closed field. D. Saltman in [21] constructed a finite group G such that $\mathrm{B}G$ is not retract rational over F (a counter-example to Noether's problem) by exhibiting a nontrivial unramified invariant of G with values in the Brauer group. It was proved in [3] that $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big)$ for a finite group G is isomorphic to the subgroup of $H^3(G, \mathbb{Z})$ consisting of all classes having trivial restrictions to all bicyclic subgroups of G. In [13] examples of finite groups of order p^5, p odd and 2^6 with nontrivial group $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big)$ were given.

Example 8.5. Let G be a cyclic group of prime order p. By [22], $\mathrm{B}G$ is retract rational. But it was shown in [24] that $\mathrm{B}G$ is not stably rational if $p = 47$ and $F = \mathbb{Q}$.

9. Invariants of degree 3 with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$

Let G be a split semisimple group over F and C the kernel of the universal cover of G. The cohomological cup-product

$$H^2(K,\mathbb{Q}/\mathbb{Z}(1))\otimes K^\times \longrightarrow H^3(K,\mathbb{Q}/\mathbb{Z}(2))$$

for any field extension K/F yields a pairing

$$\operatorname{Inv}^2\big(G,\mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}}\otimes F^\times \longrightarrow \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}.$$

By Theorem 8.1, the group $\operatorname{Inv}^2\big(G,\mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}}$ is isomorphic to C^*, so we get a pairing

$$\tau : C^*\otimes F^\times \longrightarrow \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}.$$

The image of τ is the group of *decomposable* invariants. We write $\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}$ for the cokernel of τ.

Let $T\subset G$ be a split maximal torus. The Weyl group W of G acts on the character group T^* of T and therefore, on the symmetric powers $S^n(T^*)$ of T^*. Let $x\in T^*$ and $\{x_1,\dots,x_m\}$ the W-orbit of x in T^*. Then the element $c_2(x) := \sum_{i<j} x_ix_j$ belongs to $S^2(T^*)^W$. Write $\operatorname{Dec}(G)$ for the subgroup of $S^2(T^*)^W$ generated by the elements $c_2(x)$ over all $x\in T^*$, so $\operatorname{Dec}(G)$ is the subgroup of the "obvious" elements in $S^2(T^*)^W$.

Theorem 9.1. [17, Theorem 3.9] *Let G be a split semisimple group over F, $T\subset G$ a split maximal torus and C the kernel of the universal cover of G. Then there is an exact sequence*

$$0\longrightarrow C^*\otimes F^\times \xrightarrow{\tau} \operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}} \longrightarrow S^2(T^*)^W/\operatorname{Dec}(G)\longrightarrow 0.$$

It follows from this theorem that

$$\operatorname{Inv}^3\big(G,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}\simeq S^2(T^*)^W/\operatorname{Dec}(G).$$

This group was computed when G is semisimple (M. Rost, see [11]), when G is adjoint (see [17]) and when G is (almost) simple in [1].

The unramified invariants of semisimple groups over an algebraically closed fields were considered in [18].

Theorem 9.2. [18, Proposition 8.1. and Theorems 8.4 and 11.3] *Let G be a semisimple group over an algebraically closed field F and p a prime integer different from* $\operatorname{char}(F)$. *Then the p-primary component of* $\operatorname{Inv}^3_{\mathrm{nr}}(G,p)$ *is trivial in the following cases:*

- *G is a simply connected or adjoint group.*
- *G is a simple group.*
- *p is odd.*

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