ON THE NORM RESIDUE HOMOMORPHISM OF DEGREE TWO

ALEXANDER MERKURJEV

To S. Vostokov on his 60th birthday

1. Introduction

It was proven in [5] that for every field F of characteristic not 2, the norm residue homomorphism

$$h_F : K_2F/2K_2F \to {}_2\mathrm{Br}\, F,$$

taking the class of a symbol $\{a, b\}$ to the class of the quaternion algebra $(a, b)_F$ in the Brauer group is an isomorphism. The proof used a specialization argument reducing the problem to the study of the function field of a conic curve and a comparison theorem of A. Suslin on the K_2-group of the function field of a conic curve [10] that in its turn, was based on Quillen's computation of the higher K-theory of a conic curve. Other "elementary" proofs of the bijectivity of h_F, avoiding higher K-theory, but still using the specialization argument were given in [1] and [12].

In the present paper we give another self-contained proof of the bijectivity of h_F avoiding the specialization argument. The proof is based on the exactness of the sequence (cf., [10])

$$K_2F \to K_2F(C) \xrightarrow{\partial} \coprod_{x\in C} F(x)^\times \xrightarrow{N} F^\times,$$

where C is a projective conic curve over a field F. The "elementary" proof of exactness of the sequence, we give here, uses a careful treatment of the geometry of a conic curve. We explore a bijective correspondence between closed points of degree 2 on C and quadratic subfields of the corresponding quaternion algebra.

2. Milnor K-theory of fields

Let F be a field. The *graded Milnor ring $K_*(F)$ of F* is the factor ring of the tensor ring over $\mathbb{Z}$ of the multiplicative group $F^\times$ by the ideal generated by the tensors of the form $a \otimes b$ with $a + b = 1$ (see [7]). The class of a tensor $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ in $K_*(F)$ is denoted by $\{a_1, a_2, \ldots, a_n\}$ and is called a *symbol*. We have $K_0(F) = \mathbb{Z}$, $K_1(F) = F^\times$ and $K_2(F)$ is generated by the symbols $\{a, b\}$ with $a, b \in F^\times$ that are subject to the following relations:

(M1) $\{aa', b\} = \{a, b\} + \{a', b\}$, $\{a, bb'\} = \{a, b\} + \{a, b'\}$;

The author was supported in part by NSF Grant #0355166.

(M2) $\{a, b\} = 0$ if $a + b = 1$.

A field homomorphism $F \to E$ induces a ring homomorphism $K_*(F) \to K_*(E)$, $u \mapsto u_E$, making K_* a functor from the category of fields to the category of graded abelian groups.

Let L be a field with a discrete valuation v and residue field F. There is the *residue* homomorphism

$$\partial : K_*L \to K_{*-1}F$$

uniquely determined by the following condition. If $a_0, a_1, \dots, a_n \in L^\times$ such that $v(a_i) = 0$ for all $i = 1, 2, \dots n$ then

$$\partial(\{a_0, a_1, \dots, a_n\}) = v(a_0)\{\bar{a}_1, \dots, \bar{a}_n\},$$

where $\bar{a} \in F$ denotes the residue of a.

If $p \in L^\times$ is a prime element, i.e., $v(p) = 1$, we define the *specialization homomorphism*

$$s_p : K_*L \to K_*F$$

by the formula $s_p(u) = \partial(\{-p\} \cdot u)$. We have

$$s_p(\{a_1, a_2, \dots, a_n\}) = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\},$$

where $b_i = a_i / p^{v(a_i)}$.

Example 2.1. Consider the discrete valuation v of the field of rational functions $F(t)$ given by the irreducible polynomial t. For every $u \in K_*F$, we have $s_t(u_{F(t)}) = u$. In particular, the homomorphism $K_*F \to K_*F(t)$ is injective.

If E/F is a finite field extension, there is the $K_*(F)$-linear *norm homomorphism*

$$N_{E/F} : K_*(E) \to K_*(F)$$

that coincides with the usual norm map on $K_1(E) = E^\times$ [3, Ch.IX, §3].

Let F be a field of characteristic different from 2. For every $a, b \in F^\times$ the class of the quaternion algebra $(a, b)_F$ (see Example 3.3) in the Brauer group $\mathrm{Br}(F)$ has exponent 2. Moreover, the algebra $(a, b)_F$ is split if $a + b = 1$ (Example 3.1). The class of $(a, b)_F$ in $\mathrm{Br}(F)$ is bilinear with respect to a and b. Hence there is a well defined *norm residue homomorphism*

$$h_F : K_2F/2K_2F \to {}_2\mathrm{Br}\, F,$$

taking $\{a, b\} + 2K_2F$ to the class of the quaternion algebra $(a, b)_F$.

The rest of the paper is devoted to the proof of the following theorem.

Theorem 2.2. *For every field F of characteristic not 2, the norm residue homomorphism*

$$h_F : K_2F/2K_2F \to {}_2\mathrm{Br}\, F,$$

is an isomorphism.

3. Geometry of conic curves

In this section we establish interrelations between projective conic curves and corresponding quaternion algebras. The basic reference is the book [4].

3.1. **Quaternion algebras and conic curves.** Let F be a field (of arbitrary characteristic). A *quaternion F-algebra* is a dimension 4 central simple F-algebra. A quaternion algebra is either a division algebra or it is split, i.e., isomorphic to the matrix algebra $M_2(F)$.

Example 3.1. Let L/F be a Galois quadratic field extension and let $b \in F^\times$. We define the quaternion algebra $(L/F, b)$ as the vector space $L \oplus Lv$, where v is a symbol, with the multiplication rules $v^2 = b$ and $vx = \sigma(x)v$, where $x \in L$ and σ is the generator of the Galois group of L/F. The algebra $(L/F, b)$ is split if and only if b is a norm in the quadratic extension L/F. In fact any quaternion F-algebra is isomorphic to $(L/F, b)$ for some L/F and b.

If $\operatorname{char}(F) \neq 2$, we have $L = F(\sqrt{a})$ for some $a \in F^\times$. We write $(a, b)_F$ for $(L/F, b)$.

Every quaternion algebra Q carries a canonical involution $a \mapsto \bar{a}$. If $Q = (L/F, b)$ and $a = x + yv$ for $x, y \in L$, then $\bar{a} = \sigma(x) - yv$. There are the *reduced trace* linear map

$$\operatorname{Trd} : Q \to F, \quad a \mapsto a + \bar{a}$$

and the *reduced norm* quadratic map

$$\operatorname{Nrd} : Q \to F, \quad a \mapsto a\bar{a}.$$

Every element $a \in Q$ satisfies the equation

$$a^2 - \operatorname{Trd}(a)a + \operatorname{Nrd}(a) = 0.$$

Set

$$V = \operatorname{Ker}(\operatorname{Trd}) = \{a \in Q : \ \bar{a} = -a\},$$

so that V is a 3-dimensional subspace of Q. Note that $x^2 = -\operatorname{Nrd}(x) \in F$ for any $x \in V$, and the map $q : V \to F$ given by $q(x) = x^2$ is a quadratic form on V. The space V is the orthogonal complement to 1 in Q with respect to the non-degenerate bilinear form on Q:

$$(a, b) \mapsto \operatorname{Trd}(ab).$$

The equation $q(x) = 0$ defines a smooth projective *conic curve* C in the projective plane $\mathbb{P}(V)$.

The following proposition is well known.

Proposition 3.2. *The following conditions are equivalent:*

(1) *Q is split;*
(2) *C has a rational point;*
(3) *C is isomorphic to the projective line $\mathbb{P}^1$.*

If Q is a division algebra, the degree of any finite splitting field extension is even. Therefore, the degree of every closed point of C is even. Moreover, since Q is split over a quadratic subfield of Q, the conic C has a point of degree 2. Thus, the image of the degree homomorphism $\deg : \operatorname{Pic}(C) \to \mathbb{Z}$ is equal to $2\mathbb{Z}$. Note also that the degree homomorphism is injective since it is so over a splitting field. In other words, any divisor on C of degree zero is principal.

Example 3.3. If $\operatorname{char} F \neq 2$, there is a basis $1, i, j, k$ of Q such that $a = i^2 \in F^\times$, $b = j^2 \in F^\times$, $k = ij = -ji$. Then $V = Fi \oplus Fj \oplus Fk$ and C is given by the equation $aX^2 + bY^2 - abZ^2 = 0$.

Example 3.4. If $\operatorname{char} F = 2$, there is a basis $1, i, j, k$ of Q such that $a = i^2 \in F$, $b = j^2 \in F$, $k = ij = ji + 1$. Then $V = F1 \oplus Fi \oplus Fj$ and C is given by the equation $X^2 + aY^2 + bZ^2 + YZ = 0$.

For every $a \in Q$ define the linear form l_a on V by the formula

$$l_a(x) = \operatorname{Trd}(ax).$$

Since Trd is a non-degenerate bilinear form on Q, every linear form on V is equal to l_a for some $a \in Q$.

The proof of the following statement is straightforward.

Lemma 3.5. *Let $a, b \in Q$ and $\alpha, \beta \in F$. Then*
(1) $l_a = l_b$ *if and only if* $a - b \in F$;
(2) $l_{\alpha a + \beta b} = \alpha l_a + \beta l_b$;
(3) $l_{\bar{a}} = -l_a$;
(4) $l_{a^{-1}} = -(\operatorname{Nrd} a)^{-1} \cdot l_a$ *if a is invertible.*

Every element $a \in Q \setminus F$ generates a quadratic subalgebra $F[a] = F \oplus Fa$ of Q. Conversely, every quadratic subalgebra K of Q is of the form $F[a]$ for any $a \in K \setminus F$. By Lemma 3.5, the linear form l_a on V does not depend, up to a multiple, on the choice of $a \in K \setminus F$. Hence the line in $\mathbb{P}(V)$ given by the equation $l_a(x) = 0$ is determined by K. The intersection of this line with the conic C is a degree two effective divisor on C. Thus, we have got the maps

$$\boxed{\begin{array}{c}\text{Quadratic}\\ \text{subalgebras of } Q\end{array}} \to \boxed{\begin{array}{c}\text{Rational points}\\ \text{of } \mathbb{P}(V^*)\end{array}} = \boxed{\begin{array}{c}\text{Lines}\\ \text{in } \mathbb{P}(V)\end{array}} \to \boxed{\begin{array}{c}\text{Degree 2 effective}\\ \text{divisors on } C\end{array}}$$

Proposition 3.6. *These two maps are bijections.*

Proof. The first map is a bijection since every line in $\mathbb{P}(V)$ is given by the equation $l_a = 0$ for some $a \in Q \setminus F$ and a generates a quadratic subalgebra of Q. The second map is a bijection since the embedding of C as a closed subscheme of $\mathbb{P}(V)$ is given by a complete linear system. $\square$

Remark 3.7. Degree 2 effective divisors on C are rational points of the symmetric square S^2C. Proposition 3.6 essentially asserts that S^2C is isomorphic to the projective plane $\mathbb{P}(V^*)$.

Suppose Q is a division algebra. The conic curve C has no rational points. Quadratic subalgebras of Q are quadratic (maximal) subfields of Q. A degree 2 effective cycle on C is a closed point of degree 2. Thus, by Proposition 3.6, we have bijections

$$\boxed{\begin{array}{c}\text{Quadratic}\\ \text{subfields of } Q\end{array}} \xrightarrow{\sim} \boxed{\begin{array}{c}\text{Rational points}\\ \text{of } \mathbb{P}(V^*)\end{array}} = \boxed{\begin{array}{c}\text{Lines}\\ \text{in } \mathbb{P}(V)\end{array}} \xrightarrow{\sim} \boxed{\begin{array}{c}\text{Points of}\\ \text{degree 2 in } C\end{array}}$$

In what follows we will be frequently using the constructed bijection between the set of quadratic subfields of Q and the set of degree 2 closed points of C.

3.2. **Key identity.** In the following proposition we write a multiple of the quadratic form q on V as a degree two polynomial of linear forms.

Proposition 3.8. *For any $a, b, c \in Q$,*

$$l_{a\bar{b}} \cdot l_c + l_{b\bar{c}} \cdot l_a + l_{c\bar{a}} \cdot l_b = \big(\mathrm{Trd}(cba) - \mathrm{Trd}(abc)\big) \cdot q.$$

Proof. We write T for Trd in the proof. For every $x \in V$ we have:

$$\begin{aligned}
l_{a\bar{b}}(x) \cdot l_c(x) &= T(a\bar{b}x)T(cx) \\
&= T\big(a(T(b) - b)x\big)T(cx) \\
&= T(ax)T(b)T(cx) - T(abx)T(cx) \\
&= T(ax)T(b)T(cx) - T\big(abT(cx)x\big) \\
&= T(ax)T(b)T(cx) - T(abc)x^2 + T(abx\bar{c}x),
\end{aligned}$$

$$\begin{aligned}
l_{b\bar{c}}(x) \cdot l_a(x) &= T(b\bar{c}x)T(ax) \\
&= T\big((T(b) - \bar{b})\bar{c}x\big)T(ax) \\
&= T(\bar{c}x)T(b)T(ax) - T(\bar{b}\bar{c}x)T(ax) \\
&= -T(ax)T(b)T(cx) - T\big(\bar{b}\bar{c}xT(ax)\big) \\
&= -T(ax)T(b)T(cx) - T(\bar{b}\bar{c}xax) + T(\bar{b}\bar{c}\bar{a})x^2 \\
&= -T(ax)T(b)T(cx) - T(ax\bar{b}\bar{c}x) + T(cba)x^2
\end{aligned}$$

$$\begin{aligned}
l_{c\bar{a}}(x) \cdot l_b(x) &= T(c\bar{a}x)T(bx) \\
&= -T(a\bar{c}x)T(bx) \\
&= -T\big(aT(bx)\bar{c}x\big) \\
&= -T(abx\bar{c}x) + T(ax\bar{b}\bar{c}x).
\end{aligned}$$

It remains to add all three equalities. □

3.3. **Residue fields of points of C and quadratic subfields of Q.** Suppose Q is a division algebra. Recall that quadratic subfields of Q correspond bijectively to degree 2 points of C. We would like to identify a quadratic subfield of Q with the residue field of the corresponding point in C of degree 2.

Choose a quadratic subfield $K \subset Q$. For every $a \in Q \setminus K$ one has $Q = K \oplus aK$. We define the map

$$\varphi_a : V^* \to K$$

by the rule: if $c = u + av$ for $u, v \in K$, then $\varphi_a(l_c) = v$. Clearly,

$$\varphi_a(l_c) = 0 \Longleftrightarrow c \in K.$$

By Lemma 3.5, φ_a is a well defined F-linear map. For another element $b \in Q \setminus K$ we have

$$\varphi_b(l_c) = \varphi_b(l_a)\varphi_a(l_c), \tag{1}$$

hence the maps φ_a and φ_b differ by the multiple $\varphi_b(l_a) \in K^\times$. The map φ_a extends in a usual way to an F-algebra homomorphism

$$\varphi_a : S^\bullet(V^*) \to K$$

(here $S^\bullet$ denotes the symmetric algebra).

Let $x \in C \subset \mathbb{P}(V)$ be the point of degree 2 corresponding to the quadratic subfield K. The local ring $\mathcal{O}_{\mathbb{P}(V),x}$ is the subring of the quotient field of the symmetric algebra $S^\bullet(V^*)$ generated by the fractions $\frac{l_c}{l_d}$ for all $c \in Q$ and $d \in Q \setminus K$.

Fix an element $a \in Q \setminus F$. We define the F-algebra homomorphism

$$\varphi : \mathcal{O}_{\mathbb{P}(V),x} \to K$$

by the formula

$$\varphi\Big(\frac{l_c}{l_d}\Big) = \frac{\varphi_a(l_c)}{\varphi_a(l_d)}.$$

Note that $\varphi_a(l_d) \neq 0$ since $d \notin K$ and the map φ does not depend on the choice of $a \in Q \setminus K$ in view of (1).

We claim that the map φ vanishes on the quadratic form q defining C in $\mathbb{P}(V)$. Proposition 3.8 gives a formula for a multiple of the quadratic form q with the coefficient $\alpha = \operatorname{Trd}(cba) - \operatorname{Trd}(abc)$.

Lemma 3.9. *There exist $a \in Q \setminus K$, $b \in K$ and $c \in Q$ such that $\alpha \neq 0$.*

Proof. Pick any $b \in K \setminus F$ and any $a \in Q$ such that $ab \neq ba$. Clearly, $a \in Q \setminus K$. Then $\alpha = \operatorname{Trd}\big((ba - ab)c\big)$ is nonzero for some $c \in Q$ since the bilinear form Trd is non-degenerate on Q. □

Choose a, b and c as in Lemma 3.9. We have $\varphi_a(l_b) = 0$ since $b \in K$, $\varphi_a(l_a) = 1$ and $\varphi_a(l_{a\bar{b}}) = \bar{b}$. Write $c = u + av$ for $u, v \in K$, then $\varphi_a(l_c) = v$. Since $b\bar{c} = b\bar{u} + b\bar{v}\bar{a} = b\bar{u} + \operatorname{Trd}(b\bar{v}\bar{a}) - av\bar{b}$, we have $\varphi_a(l_{b\bar{c}}) = -v\bar{b}$ and by Proposition 3.8,

$$\alpha\varphi(q) = \varphi_a(l_{a\bar{b}})\varphi_a(l_c) + \varphi_a(l_{b\bar{c}})\varphi_a(l_a) + \varphi_a(l_{c\bar{a}})\varphi_a(l_b) = \bar{b}v - v\bar{b} = 0.$$

Since $\alpha \neq 0$, we have $\varphi(q) = 0$.

The local ring $\mathcal{O}_{C,x}$ coincides with the factor ring $\mathcal{O}_{\mathbb{P}(V),x}/q\mathcal{O}_{\mathbb{P}(V),x}$. Therefore, φ factors through an F-algebra homomorphism

$$\varphi : \mathcal{O}_{C,x} \to K.$$

Let $e \in K \setminus F$. The function $\frac{l_e}{l_a}$ is a local parameter of the local ring $\mathcal{O}_{C,x}$, i.e., it generates the maximal ideal of $\mathcal{O}_{C,x}$. Since $\varphi\left(\frac{l_e}{l_a}\right) = 0$, the map φ induces a field isomorphism

$$F(x) \xrightarrow{\sim} K \tag{2}$$

of degree 2 field extensions of F. We have proved

Proposition 3.10. *Let Q be a division quaternion algebra, let $K \subset Q$ be a quadratic subfield and let $x \in C$ be the corresponding point of degree 2. Then the residue field $F(x)$ is canonically isomorphic to K over F. Let $a \in Q$ and $b \in Q\backslash K$. Write $a = u+bv$ for unique $u, v \in K$. Then the value $(\frac{l_a}{l_b})(x) \in F(x)$ of the function $\frac{l_a}{l_b}$ at the point x corresponds to the element $v \in K$ under the isomorphism (2).*

4. Key exact sequence

Let C be a smooth curve over a field F. For every (closed) point $x \in C$ there is the residue homomorphism

$$\partial_x : K_2F(C) \to K_1F(x) = F(x)^\times$$

induced by the discrete valuation of the local ring $\mathcal{O}_{C,x}$.

In this section we prove the following

Theorem 4.1. *Let C be a conic curve over a field F. The sequence*

$$K_2F \to K_2F(C) \xrightarrow{\partial} \coprod_{x \in C} F(x)^\times \xrightarrow{N} F^\times,$$

where $\partial = \coprod \partial_x$ and N is given by the norm maps $N_{F(x)/F}$, is exact.

Remark 4.2. Theorem 4.1 was originally proven in [10] as a consequence of Quillen's computation of the higher K-theory of a conic curve [8, §8, Th. 4.1] and a theorem of Rehmann and Stuhler on the group K_2 of a quaternion algebra [9].

4.1. Filtration on $K_2F(C)$.

For a divisor $\mathfrak{a}$ on C set

$$L(\mathfrak{a}) = \{f \in F(C)^\times \text{ such that } div(f) + \mathfrak{a} \geq 0\} \cup \{0\}.$$

The set $L(\mathfrak{a})$ is a linear F-subspace of $F(C)$. The following lemma is a simple case of the Riemann-Roch theorem.

Lemma 4.3.

$$\dim L(\mathfrak{a}) = \begin{cases} \deg \mathfrak{a} + 1, & \textit{if } \deg \mathfrak{a} \geq 0; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Extending the base field we can assume that C splits, i.e., $C \simeq \mathbb{P}^1$ by Proposition 3.2. The result follows by a direct computation. □

If C splits, the statement of Theorem 4.1 is Milnor's computation of $K_2F(t)$ given in [7, Th. 2.3]. So we may (and will) assume that C is not split. We know that the degree of every closed point of C is even.

Fix a closed point $x_0 \in C$ of degree 2. For every $n \in \mathbb{Z}$, set $L_n = L(nx_0)$. Clearly,

$$L_n \cdot L_m \subset L_{n+m}.$$

By Lemma 4.3, $\dim L_n = 2n + 1$ if $n \geq 0$, $L_n = 0$ if $n < 0$ and $L_0 = F$. We also write $L_n^\times$ for $L_n \setminus \{0\}$. Note that the value $g(x)$ in $F(x)$ is defined for every $g \in L_n^\times$ and a point $x \neq x_0$.

Since any divisor on C of degree zero is principal, for every point $x \in C$ of degree $2n$ we can choose a function $p_x \in L_n^\times$ such that $div(p_x) = x - nx_0$. In particular, $p_{x_0} \in F^\times$. Note that p_x is uniquely determined up to a scalar multiple. Clearly, $p_x(x) = 0$ if $x \neq x_0$. Every function in $L_n^\times$ can be written as the product of a nonzero constant and finitely many p_x for some points x of degree at most $2n$.

Lemma 4.4. *Let $x \in C$ be a point of degree $2n$ different from x_0, and let $g \in L_m$ be such that $g(x) = 0$. Then $g = p_x q$ for some $q \in L_{m-n}$. In particular, $g = 0$ if $m < n$.*

Proof. Consider the F-linear map

$$e_x : L_m \to F(x), \quad e_x(g) = g(x).$$

If $m < n$, the map e_x is injective since x does not belong to the support of the divisor of a function in $L_m^\times$. Suppose that $m = n$ and $g \in \operatorname{Ker} e_x$. Then $div(g) = x - nx_0$ and hence g is a multiple of p_x. Thus, the kernel of e_x is 1-dimensional. By dimension count (Lemma 4.3), e_x is surjective.

Therefore, for arbitrary $m \geq n$, the map e_x is surjective and

$$\dim \operatorname{Ker} e_x = \dim L_m - \deg(x) = 2m + 1 - 2n.$$

The image of the injective linear map $L_{m-n} \to L_m$ of the multiplication by p_x is contained in $\operatorname{Ker} e_x$ and has dimension $\dim L_{m-n} = 2m + 1 - 2n$. Therefore, $\operatorname{Ker} e_x = p_x L_{m-n}$. $\square$

For every $n \in \mathbb{Z}$, let M_n be the subgroup of $K_2F(C)$ generated by the symbols $\{f, g\}$ with $f, g \in L_n^\times$, i.e., $M_n = \{L_n^\times, L_n^\times\}$. We have the following filtration:

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset K_2F(C). \tag{3}$$

Note that M_0 coincides with the image of the homomorphism $K_2F \to K_2F(C)$ and $K_2F(C)$ is the union of all M_n. Indeed, the group $F(C)^\times$ is the union of the subsets $L_n^\times$.

If $f \in L_n^\times$, the degree of every point of the support of $div(f)$ is at most $2n$. In particular, $\partial_x(M_{n-1}) = 0$ for every point x of degree $2n$. Therefore, for every $n \geq 0$ we have a well defined homomorphism

$$\partial_n : M_n/M_{n-1} \to \coprod_{\deg x = 2n} F(x)^\times$$

induced by ∂_x over all points $x \in C$ of degree $2n$.

We refine the filtration (3) by adding an extra term M' between M_0 and M_1. Set $M' = \{L_1^\times, L_0^\times\} = \{L_1^\times, F^\times\}$. In other words, the group M' is generated by M_0 and the symbols of the form $\{p_x, \alpha\}$ for all points $x \in C$ of degree 2 and all $\alpha \in F^\times$.

Denote by A' the subgroup of $\coprod_{\deg x=2} F(x)^\times$ consisting of all families (α_x) such that $\alpha_x \in F^\times$ for all x and $\prod_x \alpha_x = 1$. Clearly, $\partial_1(M'/M_0) \subset A'$.

Theorem 4.1 is a consequence of the following three propositions.

Proposition 4.5. *If $n \geq 2$, the map*

$$\partial_n : M_n/M_{n-1} \to \coprod_{\deg x=2n} F(x)^\times$$

is an isomorphism.

Proposition 4.6. *The restriction $\partial' : M'/M_0 \to A'$ of ∂_1 is an isomorphism.*

Proposition 4.7. *The sequence*

$$0 \to M_1/M' \xrightarrow{\partial_1} \left(\coprod_{\deg x=2} F(x)^\times \right) /A' \xrightarrow{N} F^\times$$

is exact.

Proof of Theorem 4.1. Since $K_2F(C)$ is the union of M_n, it is sufficient to prove that the sequence

$$0 \to M_n/M_0 \xrightarrow{\partial} \coprod_{\deg x\leq 2n} F(x)^\times \xrightarrow{N} F^\times$$

is exact for every $n \geq 1$. We proceed by induction on n. The case $n = 1$ follows from Propositions 4.6 and 4.7. The induction step is guaranteed by Proposition 4.5. □

4.2. **Proof of Proposition 4.5.** We will construct the inverse map of ∂_n.

Lemma 4.8. *Let $x \in C$ be a point of degree $2n > 2$. Then for every $u \in F(x)^\times$ there exist $f \in L_{n-1}^\times$ and $h \in L_1^\times$ such that $(\frac{f}{h})(x) = u$.*

Proof. The F-linear map

$$e_x : L_{n-1} \to F(x), \quad f \mapsto f(x)$$

is injective by Lemma 4.4. Hence,

$$\dim \operatorname{Coker} e_x = \deg(x) - \dim L_{n-1} = 2n - (2n-1) = 1.$$

Consider the F-linear map

$$g : L_1 \to \operatorname{Coker} e_x, \quad g(h) = u \cdot h(x) + \operatorname{Im} e_x.$$

Since $\dim L_1 = 3$, the kernel of g contains a nonzero function $h \in L_1^\times$. We have $u \cdot h(x) = f(x)$ for some $f \in L_{n-1}^\times$. Since $\deg x > 2$ the value $h(x)$ is nonzero. Hence $u = (\frac{f}{h})(x)$. □

Let $x \in C$ be a point of degree $2n > 2$. We define a map

$$\psi_x : F(x)^\times \to M_n/M_{n-1}$$

as follows. By Lemma 4.8, for an element $u \in F(x)^\times$ choose $f \in L_{n-1}^\times$ and $h \in L_1^\times$ such that $(\frac{f}{h})(x) = u$. We set

$$\psi_x(u) = \Big\{p_x, \frac{f}{h}\Big\} + M_{n-1}.$$

Lemma 4.9. *The map ψ_x is a well defined homomorphism.*

Proof. Let $f' \in L_{n-1}^\times$ and $h' \in L_1^\times$ be two functions with $(\frac{f'}{h'})(x) = u$. Then $f'h - fh' \in L_n$ and $(f'h - fh')(x) = 0$. By Lemma 4.4, $f'h - fh' = \lambda p_x$ for some $\lambda \in F$. If $\lambda = 0$, $\frac{f}{h} = \frac{f'}{h'}$.

Suppose $\lambda \neq 0$. Since $\frac{\lambda p_x}{f'h} + \frac{fh'}{f'h} = 1$ we have

$$0 = \Big\{\frac{\lambda p_x}{f'h}, \frac{fh'}{f'h}\Big\} \equiv \Big\{p_x, \frac{f}{h}\Big\} - \Big\{p_x, \frac{f'}{h'}\Big\} \quad mod\ M_{n-1}.$$

Hence, $\{p_x, \frac{f}{h}\} + M_{n-1} = \{p_x, \frac{f'}{h'}\} + M_{n-1}$, so that the map ψ is well defined.

Let $u_3 = u_1u_2 \in F(x)^\times$. Choose $f_i \in L_{n-1}^\times$ and $h_i \in L_1^\times$ such that $(\frac{f_i}{h_i})(x) = u_i$ $(i = 1, 2, 3)$. The function $f_1f_2h_3 - f_3h_1h_2$ belongs to L_{2n-1} and has zero value at x. By Lemma 4.4, $f_1f_2h_3 - f_3h_1h_2 = p_xq$ for some $q \in L_{n-1}$. Since $\frac{p_xq}{f_1f_2h_3} + \frac{f_3h_1h_2}{f_1f_2h_3} = 1$ we have

$$0 = \Big\{\frac{p_xq}{f_1f_2h_3}, \frac{f_3h_1h_2}{f_1f_2h_3}\Big\} \equiv \Big\{p_x, \frac{f_3}{h_3}\Big\} - \Big\{p_x, \frac{f_1}{h_1}\Big\} - \Big\{p_x, \frac{f_2}{h_2}\Big\} \quad mod\ M_{n-1}.$$

Thus, $\psi_x(u_3) = \psi_x(u_1) + \psi_x(u_2)$. □

By Lemma 4.9, we have the homomorphism

$$\psi_n = \sum \psi_x : \coprod_{\deg x = 2n} F(x)^\times \to M_n/M_{n-1}.$$

We claim that ∂_n and ψ_n are isomorphisms inverse to each other. If x is a point of degree $2n > 2$ and $u \in F(x)^\times$, choose $f \in L_{n-1}^\times$ and $h \in L_1^\times$ such that $(\frac{f}{h})(x) = u$. We have

$$\partial_x(\Big\{p_x, \frac{f}{h}\Big\}) = \Big(\frac{f}{h}\Big)(x) = u$$

and the symbol $\{p_x, \frac{f}{h}\}$ has no nontrivial residues at other points of degree $2n$. Therefore, $\partial_n \circ \psi_n$ is the identity.

To finish the proof of Proposition 4.5 it is sufficient to show that ψ_n is surjective. The group M_n/M_{n-1} is generated by classes of the form $\{p_x, g\} + M_{n-1}$ and $\{p_x, p_y\} + M_{n-1}$, where $g \in L_{n-1}^\times$ and x, y are distinct points of degree $2n$. Clearly

$$\{p_x, g\} + M_{n-1} = \psi_x\big(g(x)\big),$$

hence $\{p_x, g\} + M_{n-1} \in \operatorname{Im} \psi_n$.

By Lemma 4.8, there are $f \in L_{n-1}^{\times}$ and $h \in L_1^{\times}$ such that $p_x(y) = (\frac{f}{h})(y)$. The function $p_x h - f$ belongs to $L_{n+1}^{\times}$ and has zero value at y. By Lemma 4.4, $p_x h - f = p_y q$ for some $q \in L_1^{\times}$. Since $\frac{p_y q}{p_x h} + \frac{f}{p_x h} = 1$ we have

$$0 = \left\{ \frac{p_y q}{p_x h}, \frac{f}{p_x h} \right\} \equiv \{p_x, p_y\} \quad mod \ \mathrm{Im}(\psi_n). \quad \square$$

4.3. **Proof of Proposition 4.6.** We define a homomorphism

$$\rho : A' \to M'/M_0$$

by the rule

$$\rho\Big(\coprod \alpha_x\Big) = \sum_{\deg x = 2} \{p_x, \alpha_x\} + M_0.$$

Since $\partial_x\{p_x, \alpha\} = \alpha$, $\partial_{x_0}\{p_x, \alpha\} = \alpha^{-1}$ for every $x \neq x_0$ and the product of all α_x is equal to 1, the composition $\partial' \circ \rho$ is the identity. Clearly, ρ is surjective. $\square$

4.4. **Generators and relations of $A(Q)/A'$.** It remains to prove Proposition 4.7. Now the quaternion division algebra Q defining the conic curve C comes into play. By Proposition 3.10, the norm homomorphism

$$\coprod_{\deg x = 2} F(x)^{\times} \to F^{\times}$$

is canonically isomorphic to the norm homomorphism

$$\coprod K^{\times} \to F^{\times}, \tag{4}$$

where the coproduct is taken over all quadratic subfields $K \subset Q$. Note that the norm map $N_{K/F} : K^{\times} \to F^{\times}$ is the restriction of the reduced norm Nrd on K. Let $A(Q)$ be the kernel of the norm homomorphism (4). Under the identification the subgroup A' of $\coprod F(x)^{\times}$ corresponds to the subgroup of $A(Q)$ (we still denote it by A') consisting of all families (a_K) with $a_K \in F^{\times}$ and $\prod a_K = 1$. In other words, A' is the intersection of $A(Q)$ and $\coprod F^{\times}$. Now Proposition 4.7 asserts that the canonical homomorphism

$$\partial_1 : M_1/M' \to A(Q)/A' \tag{5}$$

is an isomorphism. In the proof of Proposition 4.7 we will construct the inverse isomorphism. In order to do that it is convenient to have a presentation of the group $A(Q)/A'$ by generators and relations.

We define a map (not a homomorphism!)

$$Q^{\times} \to \big(\coprod K^{\times}\big)/A', \quad a \mapsto \widetilde{a}$$

as follows. If $a \in Q^{\times}$ is not a scalar, it is contained in a unique quadratic subfield K of Q. Therefore, a defines an element of the coproduct $\coprod K^{\times}$. We denote by $\widetilde{a}$ the corresponding class in $\big(\coprod K^{\times}\big)/A'$. If $a \in F^{\times}$, of course, a

belongs to all quadratic subfields. Nevertheless a defines a unique element $\widetilde{a}$ of the factor group $\big(\coprod K^\times\big)/A'$ (we place a in any quadratic subfield). Clearly

$$\widetilde{(ab)} = \widetilde{a} \cdot \widetilde{b} \quad \text{if } a \text{ and } b \text{ commute.} \tag{6}$$

(Note that we use the multiplicative notation for the operation in the factor group.) Obviously, the group $\big(\coprod K^\times\big)/A'$ is an abelian group generated by $\widetilde{a}$ for all $a \in Q^\times$ with the set of defining relations given by (6).

The group $A(Q)/A'$ is generated (as an abelian group) by the products $\widetilde{a}_1\widetilde{a}_2 \dots \widetilde{a}_n$, where $a_i \in Q^\times$ with $\mathrm{Nrd}(a_1a_2 \dots a_n) = 1$, with the following set of defining relations:

(1) $(\widetilde{a}_1\widetilde{a}_2 \dots \widetilde{a}_n) \cdot (\widetilde{a}_{n+1}\widetilde{a}_{n+2} \dots \widetilde{a}_{n+m}) = (\widetilde{a}_1\widetilde{a}_2 \dots \widetilde{a}_{n+m})$;
(2) $\widetilde{a}\widetilde{b}\widetilde{(a^{-1})}\widetilde{(b^{-1})} = 1$;
(3) If a_{i-1} and a_i commute, then $\widetilde{a}_1 \dots \widetilde{a}_{i-1}\widetilde{a}_i \dots \widetilde{a}_n = \widetilde{a}_1 \dots \widetilde{a_{i-1}a_i} \dots \widetilde{a}_n$.

The set of generators is too large for our purposes. In the following subsection we will find another presentation of $A(Q)/A'$ (Corollary 4.22). More precisely, we will define an abstract group G by generators and relations (with the "better" set of generators) and prove that G is isomorphic to $A(Q)/A'$.

4.5. **The group G.** Let Q be a division quaternion algebra over a field F. Consider the abelian group G defined by generators and relations as follows. The sign $*$ will be used to denote the operation in G (and 1 for the identity element).

Generators: the symbols (a, b, c) for all ordered triples a, b, c of elements of $Q^\times$ such that $abc = 1$. Note that if (a, b, c) is a generator of G then so are the cyclic permutations (b, c, a) and (c, a, b).

Relations:
$(R1)$: $(a, b, cd) * (ab, c, d) = (b, c, da) * \big(bc, d, a\big)$ for all $a, b, c, d \in Q^\times$ such that $abcd = 1$;
$(R2)$: $(a, b, c) = 1$ if a and b commute.

For an (ordered) sequence $a_1, a_2, \dots, a_n$ $(n \geq 1)$ of elements of $Q^\times$ such that $a_1a_2 \dots a_n = 1$ we define a symbol

$$(a_1, a_2, \dots, a_n) \in G$$

by induction on n. The symbol is trivial if $n = 1$ or 2. If $n \geq 3$ we set

$$(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_{n-2}, a_{n-1}a_n) * (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n).$$

Note that if $a_1a_2 \dots a_n = 1$ then $a_2 \dots a_na_1 = 1$.

Lemma 4.10. *The symbols do not change under cyclic permutations, i.e.,* $(a_1, a_2, \dots, a_n) = (a_2, \dots, a_n, a_1)$ *if* $a_1a_2 \dots a_n = 1$.

Proof. Induction on n. The statement is clear if $n = 1$ or 2. If $n = 3$,

$$\begin{aligned}(a_1, a_2, a_3) &= (a_1, a_2, a_3) * (a_1a_2, a_3, 1) \text{ (relation } R2)\\ &= (a_2, a_3, a_1) * (a_2a_3, 1, a_1) \text{ (relation } R1)\\ &= (a_2, a_3, a_1) \text{ (relation } R2).\end{aligned}$$

Suppose that $n \geq 4$. We have

$$\begin{aligned}(a_1, a_2, \dots, a_n) =&(a_1, \dots, a_{n-2}, a_{n-1}a_n) * (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n) \text{ (definition)}\\ =&(a_2, \dots, a_{n-2}, a_{n-1}a_n, a_1) * (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n) \text{ (induction)}\\ =&(a_2, \dots, a_{n-2}, a_{n-1}a_na_1) * (a_2a_3 \dots a_{n-2}, a_{n-1}a_n, a_1)\\ &* (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n) \text{ (definition)}\\ =&(a_2, \dots, a_{n-2}, a_{n-1}a_na_1) * (a_1, a_2a_3 \dots a_{n-2}, a_{n-1}a_n)\\ &* (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n) \text{ (case } n = 3)\\ =&(a_2, \dots, a_{n-2}, a_{n-1}a_na_1) * (a_2a_3 \dots a_{n-2}, a_{n-1}, a_na_1)\\ &* (a_2a_3 \dots a_{n-1}, a_n, a_1) \text{ (relation } R1)\\ =&(a_2, \dots, a_{n-2}, a_{n-1}, a_na_1) * (a_2a_3 \dots a_{n-1}, a_n, a_1) \text{ (definition)}\\ =&(a_2, \dots a_n, a_1) \text{ (definition)}.\end{aligned}$$

□

Lemma 4.11. *If $a_1a_2 \dots a_n = 1$ and a_{i-1} commutes with a_i for some i, then $(a_1, \dots, a_{i-1}, a_i, \dots, a_n) = (a_1, \dots, a_{i-1}a_i, \dots, a_n)$.*

Proof. We may assume that $n \geq 3$ and $i = n$ by Lemma 4.10. We have

$$\begin{aligned}(a_1, \dots, a_{n-2}, a_{n-1}, a_n) &= (a_1, \dots, a_{n-2}, a_{n-1}a_n) * (a_1a_2 \dots a_{n-2}, a_{n-1}, a_n) \text{ (definition)}\\ &= (a_1, \dots, a_{n-2}, a_{n-1}a_n) \text{ (relation } R2).\end{aligned}$$

□

Lemma 4.12. $(a_1, \dots, a_n) * (b_1, \dots, b_m) = (a_1, \dots, a_n, b_1, \dots, b_m)$.

Proof. Induction on m. By Lemma 4.11, we may assume that $m \geq 3$. We have

$$\begin{aligned}L.H.S. &= (a_1, \dots, a_n) * (b_1, \dots, b_{m-1}b_m) * (b_1b_2 \dots b_{m-2}, b_{m-1}, b_m) \text{ (definition)}\\ &= (a_1, \dots, a_n, b_1, \dots, b_{m-1}b_m) * (b_1b_2 \dots b_{m-2}, b_{m-1}, b_m) \text{ (induction)}\\ &= (a_1, \dots, a_n, b_1, \dots, b_m) \text{ (definition)}.\end{aligned}$$

□

As usual, we write $[a, b]$ for the commutator $aba^{-1}b^{-1}$.

Lemma 4.13. *Let $a, b \in Q^\times$.*

1. For every nonzero $b' \in Fb + Fba$ one has $[a, b] = [a, b']$. Similarly, $[a, b] = [a', b]$ for every nonzero $a' \in Fa + Fab$.

2. For every nonzero $b' \in Fb + Fba + Fbab$ there exists $a' \in Q^\times$ such that $[a, b] = [a', b] = [a', b']$.

Proof. 1. We have $b' = bx$, where $x \in F + Fa$. Hence x commutes with a and therefore $[a, b] = [a, b']$. The proof of the second statement is similar.

2. There is nonzero $a' \in Fa + Fab$ such that $b' \in Fb + Fba'$. By the first part, $[a, b] = [a', b] = [a', b']$. □

Corollary 4.14. *1. Let $[a, b] = [c, d]$. Then there are $a', b' \in Q^\times$ such that $[a, b] = [a', b] = [a', b'] = [c, b'] = [c, d]$.*

2. Every two commutators in $Q^\times$ can be written in the form $[a, b]$ and $[c, d]$ with $b = c$.

Proof. 1. If $[a, b] = 1 = [c, d]$, we can take $a' = b' = 1$. Otherwise, the sets $\{b, ba, bab\}$ and $\{d, dc\}$ are linearly independent. Let b' be a nonzero element in the intersection of the subspaces $Fb + Fba + Fbab$ and $Fd + Fdc$. The statement follows from Lemma 4.13.

2. Let $[a, b]$ and $[c, d]$ be two commutators. We may clearly assume that $[a, b] \neq 1 \neq [c, d]$, so that the sets $\{b, ba, bab\}$ and $\{c, cd\}$ are linearly independent. Choose a nonzero element b' in the intersection of $Fb + Fba + Fbab$ and $Fc + Fcd$. By Lemma 4.13, $[a, b] = [a', b']$ for some $a' \in Q^\times$ and $[c, d] = [b', d]$. □

Lemma 4.15. *Let $h \in Q^\times$. The following conditions are equivalent:*

1. *$h = [a, b]$ for some $a, b \in Q^\times$;*
2. *$h \in [Q^\times, Q^\times]$;*
3. $\mathrm{Nrd}(h) = 1$.

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.
(3) ⇒ (1); Let K be a separable quadratic subfield containing h. (If h is purely inseparable, then $h^2 \in F$ and therefore $h = 1$.) Since $N_{K/F}(h) = \mathrm{Nrd}(h) = 1$, by the classical Hilbert theorem 90, $h = \bar{b}b^{-1}$ for some $b \in K^\times$. By the Noether-Skolem theorem, $\bar{b} = aba^{-1}$ for some $a \in Q^\times$. □

Let $h \in Q^\times$ be such that $\mathrm{Nrd}(h) = 1$. Then by Lemma 4.15, $h = [a, b] = aba^{-1}b^{-1}$ for some $a, b \in Q^\times$. Consider the following element

$$\widehat{h} = (b, a, b^{-1}, a^{-1}, h) \in G.$$

Lemma 4.16. *The element $\widehat{h}$ does not depend on the choice of a and b.*

Proof. Let $h = [a, b] = [c, d]$. By Corollary 4.14(1) we may assume that either $a = c$ or $b = d$. Consider the first case (the latter case is similar). We write

$d = bx$, where x commutes with a. We have

$$
\begin{aligned}
(d, a, d^{-1}, a^{-1}, h) &= (bx, a, x^{-1}b^{-1}, a^{-1}, h) \\
&= (bx, x^{-1}, b^{-1}) * (b, x, a, x^{-1}b^{-1}, a^{-1}, h) \text{ (Lemmas 4.11, 4.12)} \\
&= (bx, x^{-1}, b^{-1}) * (a^{-1}, h, b, x, a, x^{-1}b^{-1}) \text{ (Lemma 4.10)} \\
&= (a^{-1}, h, b, x, a, x^{-1}b^{-1}, bx, x^{-1}, b^{-1}) \text{ (Lemma 4.12)} \\
&= (a^{-1}, h, b, a, b^{-1}) \text{ (Lemma 4.11)} \\
&= (b, a, b^{-1}, a^{-1}, h) \text{ (Lemma 4.10).}
\end{aligned}
$$

□

Lemma 4.17. *For every $h_1, h_2 \in [Q^\times, Q^\times]$ we have*

$$\widehat{h_1 h_2} = \widehat{h_1} * \widehat{h_2} * (h_1 h_2, h_2^{-1}, h_1^{-1}).$$

Proof. By Corollary 4.14(2), $h_1 = [a_1, c]$ and $h_2 = [c, b_2]$ for some $a_1, b_2, c \in Q^\times$. Then $h_1 h_2 = [a_1 b_2^{-1}, b_2 c b_2^{-1}]$ and

$$
\begin{aligned}
\widehat{h_1} * \widehat{h_2} * (h_1 h_2, h_2^{-1}, h_1^{-1}) &= (c, a_1, c^{-1}, a_1^{-1}, h_1, h_2, b_2, c, b_2^{-1}, c^{-1}) * (h_1 h_2, h_2^{-1}, h_1^{-1}) \\
&= (b_2, c, b_2^{-1}, c^{-1}, c, a_1, c^{-1}, a_1^{-1}, h_1, h_2) * (h_2^{-1}, h_1^{-1} h_1 h_2) \\
&= (b_2, c, b_2^{-1}, a_1, c^{-1}, a_1^{-1}, h_1 h_2) \\
&= (b_2, c, b_2^{-1}, b_2 c^{-1} b_2^{-1}) * (b_2 c b_2^{-1}, a_1, c^{-1}, a_1^{-1}, h_1 h_2) \\
&= (b_2^{-1}, b_2 c^{-1} b_2^{-1}, b_2, c) * (c^{-1}, a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1) \\
&= (b_2^{-1}, b_2 c^{-1} b_2^{-1}, b_2, c, c^{-1}, a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1) \\
&= (b_2^{-1}, b_2 c^{-1} b_2^{-1}, b_2, a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1 b_2^{-1}, b_2 a_1^{-1}, a_1) \\
&= (b_2, a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1 b_2^{-1}, b_2 c^{-1} b_2^{-1}) * (b_2 a_1^{-1}, a_1, b_2^{-1}) \\
&= (b_2 a_1^{-1}, a_1, b_2^{-1}, b_2, a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1 b_2^{-1}, b_2 c^{-1} b_2^{-1}) \\
&= (b_2 a_1^{-1}, h_1 h_2, b_2 c b_2^{-1}, a_1 b_2^{-1}, b_2 c^{-1} b_2^{-1}) \\
&= (b_2 c b_2^{-1}, a_1 b_2^{-1}, b_2 c^{-1} b_2^{-1}, b_2 a_1^{-1}, h_1 h_2) \\
&= \widehat{h_1 h_2}.
\end{aligned}
$$

□

Let $a_1, a_2, \ldots, a_n \in Q^\times$ such that $\mathrm{Nrd}(h) = 1$ where $h = a_1 a_2 \ldots a_n$. We set

$$((a_1, a_2, \ldots, a_n)) = (a_1, a_2, \ldots, a_n, h^{-1}) * \widehat{h} \in G.$$

Lemma 4.18. $((a_1, a_2, \ldots, a_n)) * ((b_1, b_2, \ldots, b_m)) = ((a_1, \ldots, a_n, b_1, \ldots, b_m))$.

Proof. Set $h = a_1 \dots a_n$, $h' = b_1 \dots b_m$. We have

$$\begin{aligned} L.H.S. &= (a_1, a_2, \dots, a_n, h^{-1}) * (b_1, b_2, \dots, b_m, (h')^{-1}) * \widehat{h} * \widehat{h'} \\ &= (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m, (h')^{-1}, h^{-1}) * \widehat{h} * \widehat{h'} \\ &= (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m, (hh')^{-1}) * (hh', (h')^{-1}, h) * \widehat{h} * \widehat{h'} \\ &= (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m, (hh')^{-1}) * \widehat{hh'} \text{ (Lemma 4.17)} \\ &= R.H.S. \end{aligned}$$

□

The following Lemma is a consequence of the definition and Lemma 4.11.

Lemma 4.19. *If a_{i-1} commutes with a_i for some i, then* $((a_1, \dots, a_{i-1}, a_i, \dots, a_n)) = ((a_1, \dots, a_{i-1}a_i, \dots, a_n))$.

Lemma 4.20. $((a, b, a^{-1}, b^{-1})) = 1$.

Proof. Set $h = [a, b]$. We have

$$L.H.S. = (a, b, a^{-1}, b^{-1}, h^{-1}) * \widehat{h} = (a, b, a^{-1}, b^{-1}, h^{-1}) * (b, a, b^{-1}, a^{-1}, h) = 1.$$

□

We would like to establish an isomorphism between G and $A(Q)/A'$. We define a homomorphism $\pi : G \to A(Q)/A'$ by the formula

$$\pi(a, b, c) = \tilde{a}\tilde{b}\tilde{c} \in A(Q)/A',$$

where $a, b, c \in Q^\times$ such that $abc = 1$. Clearly, π is well defined.

Let $a_1, a_2, \dots, a_n \in Q^\times$ such that $a_1 a_2 \dots a_n = 1$. By induction on n we get

$$\pi(a_1, a_2, \dots, a_n) = \widetilde{a}_1 \widetilde{a}_2 \dots \widetilde{a}_n \in A(Q)/A'.$$

Let $h \in [Q^\times, Q^\times]$. Write $h = [a, b]$ for $a, b \in Q^\times$. We have

$$\pi(\widehat{h}) = \pi(b, a, b^{-1}, a^{-1}, h) = \widetilde{h}.$$

If $a_1, a_2, \dots, a_n \in Q^\times$ such that $\mathrm{Nrd}(h) = 1$ where $h = a_1 a_2 \dots a_n$, then

$$\pi((a_1, a_2, \dots, a_n)) = \pi(a_1, a_2, \dots, a_n, h^{-1}) * \pi(\widehat{h}) = \widetilde{a}_1 \widetilde{a}_2 \dots \widetilde{a}_n. \tag{7}$$

Define a homomorphism $\theta : A(Q)/A' \to G$ as follows. Let $a_1, a_2, \dots, a_n \in Q^\times$ be such that $\mathrm{Nrd}(a_1 a_2 \dots a_n) = 1$. We set

$$\theta(\widetilde{a}_1 \widetilde{a}_2 \dots \widetilde{a}_n) = ((a_1, a_2, \dots, a_n)). \tag{8}$$

The relation at the end of subsection 4.4 and Lemmas 4.18, 4.19 and 4.20 show that θ is a well defined homomorphism. Formulas (7) and (8) give

Proposition 4.21. *The maps π and θ are isomorphisms inverse to each other.*

Corollary 4.22. *The group $A(Q)/A'$ is generated by the products $\tilde{a}\tilde{b}\tilde{c}$ for all ordered triples a, b, c of elements of $Q^\times$ such that $abc = 1$ satisfying the following set of defining relations:*

$(R1')$ $\big(\tilde{a}\tilde{b}(\widetilde{cd})\big) \cdot \big((\widetilde{ab})\tilde{c}\tilde{d}\big) = \big(\tilde{b}\tilde{c}(\widetilde{da})\big) \cdot \big((\widetilde{da})\tilde{b}\tilde{c}\big)$ for all $a, b, c, d \in Q^\times$ such that $abcd = 1$;

$(R2')$ $\tilde{a}\tilde{b}\tilde{c} = 1$ if a and b commute.

4.6. **Proof of Proposition 4.7.** We need to prove that the homomorphism ∂_1 in (5) is an isomorphism.

The fraction $\frac{l_a}{l_b}$ for $a, b \in Q \setminus F$ can be considered as a nonzero rational function on C, $\frac{l_a}{l_b} \in F(C)^\times$.

Lemma 4.23. *Let K_0 be the quadratic subfield of Q corresponding to the fixed point x_0 and let $b \in K_0 \setminus F$. Then the space L_1 consists of fractions $\frac{l_a}{l_b}$ for all $a \in Q$.*

Proof. Obviously $\frac{l_a}{l_b} \in L_1$. It follows from Lemma 3.5, that the space of all fractions $\frac{l_a}{l_b}$ is 3-dimensional and by Lemma 4.3, $\dim L_1 = 3$. $\square$

By Lemma 4.23, the group M' is generated by symbols of the form $\{\frac{l_a}{l_b}, \alpha\}$ for all $a, b \in Q \setminus F$ and $\alpha \in F^\times$ and the group M_1 is generated by symbols $\{\frac{l_a}{l_b}, \frac{l_c}{l_d}\}$ for all $a, b, c, d \in Q \setminus F$.

Let $a, b, c \in Q$ be such that $abc = 1$. We define an element

$$[a, b, c] \in M_1/M'$$

as follows. If at least one of a, b and c belongs to $F^\times$ we set $[a, b, c] = 0$. Otherwise the linear forms l_a, l_b and l_c are nonzero and we set

$$[a, b, c] = \Big\{\frac{l_a}{l_c}, \frac{l_b}{l_c}\Big\} + M'.$$

Lemma 3.5 and the equality $\{u, -u\} = 0$ in $K_2F(C)$ yield:

Lemma 4.24. *Let $a, b, c \in Q^\times$ be such that $abc = 1$ and let $\alpha \in F^\times$. Then*

1. *$[a, b, c] = [b, c, a]$;*
2. *$[\alpha a, \alpha^{-1} b, c] = [a, b, c]$;*
3. *$[a, b, c] + [c^{-1}, b^{-1}, a^{-1}] = 0$;*
4. *If a and b commute, then $[a, b, c] = 0$.*

Lemma 4.25. *$\partial_1[a, b, c] = \widetilde{\tilde{a}\tilde{b}\tilde{c}}$.*

Proof. We may assume that none of a, b and c is a constant. Let x, y and z be the points of C of degree 2 corresponding to quadratic subfields $F[a]$, $F[b]$ and $F[c]$ that we identify with $F(x)$, $F(y)$ and $F(z)$ respectively.

Consider the following element in the class $[a, b, c]$:

$$w = \Big\{\frac{l_a}{l_c}, \frac{l_b}{l_c}\Big\} + \Big\{\frac{l_b}{l_c}, \mathrm{Nrd}(a)\Big\} + \Big\{\frac{l_b}{l_a}, -\mathrm{Nrd}(b)\Big\}.$$

By Proposition 3.10 (we identify residue fields with the corresponding quadratic extensions) and Lemma 3.5,

$$\begin{aligned}
\partial_x(w) &= \frac{l_b}{l_c}(x)\big(-\operatorname{Nrd}(b)\big)^{-1} = -\operatorname{Nrd}(b)\frac{l_{b^{-1}}}{l_{b^{-1}a^{-1}}}(x)\big(-\operatorname{Nrd}(b)\big)^{-1} = a,\\
\partial_y(w) &= \frac{l_c}{l_a}(y)\big(-\operatorname{Nrd}(ab)\big) = -\operatorname{Nrd}(a)^{-1}\frac{l_{b^{-1}a^{-1}}}{l_{a^{-1}}}(x)\big(-\operatorname{Nrd}(ab)\big)\\
&= -\operatorname{Nrd}(a)^{-1}\bar{b}^{-1}\big(-\operatorname{Nrd}(ab)\big) = b,\\
\partial_z(w) &= -\frac{l_a}{l_b}(z)\operatorname{Nrd}(a)^{-1} = \operatorname{Nrd}(a)\frac{l_{bc}}{l_b}(x)\operatorname{Nrd}(a)^{-1} = c.
\end{aligned}$$

□

Lemma 4.26. *Let $a, b, c, d \in Q \setminus F$ be such that $cd, da \notin F$ and $abcd = 1$. Then*

$$\Big\{\frac{l_a l_c}{l_{cd} l_{da}}, \frac{l_b l_d}{l_{cd} l_{da}}\Big\} \in M'.$$

Proof. Plugging in Proposition 3.8 the elements c^{-1}, ab and b for a, b and c respectively and using Lemma 3.5 we get elements $\alpha, \beta, \gamma \in F^\times$ such that on the conic C,

$$\alpha l_a l_c + \beta l_b l_d + \gamma l_{cd} l_{da} = 0.$$

Then

$$-\frac{\alpha l_a l_c}{\gamma l_{cd} l_{da}} - \frac{\beta l_b l_d}{\gamma l_{cd} l_{da}} = 1$$

and

$$0 = \Big\{-\frac{\alpha l_a l_c}{\gamma l_{cd} l_{da}}, -\frac{\beta l_b l_d}{\gamma l_{cd} l_{da}}\Big\} \equiv \Big\{\frac{l_a l_c}{l_{cd} l_{da}}, \frac{l_b l_d}{l_{cd} l_{da}}\Big\} \quad mod\ M'.$$

□

Proposition 4.27. *Let $a, b, c, d \in Q^\times$ be such that $abcd = 1$. Then*

$$[a, b, cd] + [ab, c, d] = [b, c, da] + [bc, d, a].$$

Proof. We first notice that if one of the elements a, b, ab, c, d, cd belongs to $F^\times$, the equality holds. Indeed, if $a \in F^\times$, then the equality reads $[ab, c, d] = [b, c, da]$ and follows from Lemma 4.24. If $\alpha = ab \in F^\times$, then again by Lemma 4.24,

$$L.H.S. = 0 = [b, c, da] + [(da)^{-1}, \alpha^{-1}c^{-1}, \alpha b^{-1}] = R.H.S.$$

Now assume that none of the elements belong to $F^\times$. By Lemma 4.26, we have in M_1/M':

$$
\begin{aligned}
0 &= \left\{\frac{l_a l_c}{l_{cd} l_{da}}, \frac{l_b l_d}{l_{cd} l_{da}}\right\} + M' \\
&= \left\{\frac{l_a}{l_{cd}}, \frac{l_b}{l_{cd}}\right\} + \left\{\frac{l_c}{l_{da}}, \frac{l_b}{l_{da}}\right\} + \left\{\frac{l_a}{l_{cd}}, \frac{l_d}{l_{da}}\right\} + \left\{\frac{l_c}{l_{da}}, \frac{l_d}{l_{cd}}\right\} + M' \\
&= [a, b, cd] - [b, c, da] + \left(\left\{\frac{l_a}{l_{da}}, \frac{l_d}{l_{da}}\right\} + \left\{\frac{l_{da}}{l_{cd}}, \frac{l_d}{l_{da}}\right\}\right) + \left\{\frac{l_c}{l_{da}}, \frac{l_d}{l_{cd}}\right\} + M' \\
&= [a, b, cd] - [b, c, da] - [bc, d, a] + \left(\left\{\frac{l_{da}}{l_{cd}}, \frac{l_d}{l_{cd}}\right\} + \left\{\frac{l_c}{l_{da}}, \frac{l_d}{l_{cd}}\right\}\right) + M' \\
&= [a, b, cd] - [b, c, da] - [bc, d, a] + [ab, c, d].
\end{aligned}
$$

□

We are going to use the presentation of the group $A(Q)/A'$ by generators and relations given in Corollary 4.22. We define a homomorphism

$$\mu : A(Q)/A' \to M_1/M'$$

by the formula

$$\mu(\widetilde{\widetilde{a}\widetilde{b}\widetilde{c}}) = [a, b, c]$$

for all $a, b, c \in Q$ such that $abc = 1$. It follows from Lemma 4.24(4) and Proposition 4.27 that μ is well defined. Lemma 4.25 implies that $\partial_1 \circ \mu$ is the identity.

To show that μ is the inverse of ∂_1 it is sufficient to prove that μ is surjective.

The group M_1/M' is generated by elements of the form $w = \{\frac{l_{a'}}{l_{c'}}, \frac{l_{b'}}{l_{c'}}\} + M'$ for $a', b', c' \in Q \setminus F$. We may assume that $1, a', b'$ and c' are linearly independent (otherwise, $w = 0$). In particular, $1, a', b'$ and $a'b'$ form a basis of Q, hence

$$c' = \alpha + \beta a' + \gamma b' + \delta a'b'$$

for some $\alpha, \beta, \gamma, \delta \in F$ with $\delta \neq 0$. We have

$$(\gamma\delta^{-1} + a')(\beta + \delta b') = \varepsilon + c'$$

for $\varepsilon = \beta\gamma\delta^{-1} - \alpha$. Set

$$a = \gamma\delta^{-1} + a', b = \beta + \delta b', c = (\varepsilon + c')^{-1}.$$

We have $abc = 1$. It follows from Lemma 3.5 that

$$w = \left\{\frac{l_{a'}}{l_{c'}}, \frac{l_{b'}}{l_{c'}}\right\} + M' = \left\{\frac{l_a}{l_c}, \frac{l_b}{l_c}\right\} + M' = [a, b, c].$$

By definition of μ, we have $\mu(\widetilde{\widetilde{a}\widetilde{b}\widetilde{c}}) = [a, b, c] = w$, hence μ is surjective. The proof of Proposition 4.7 is complete. □

5. Hilbert theorem 90 for K_2

Let L/F be a Galois quadratic field extension with the Galois group $G = \{1, \sigma\}$. For every field extension E/F linearly disjoint with L/F, the field $LE = L \otimes_F E$ is a quadratic Galois extension of E with the Galois group isomorphic to G. The group G acts naturally on $K_2(LE)$. We write $(1-\sigma)u$ for $\sigma(u) - u$, $u \in K_2(LE)$. Set

$$V(E) = K_2(LE)/(1-\sigma)K_2(LE).$$

If $E \to E'$ is a homomorphism of field extensions of F linearly disjoint with L/F, there is a natural homomorphism

$$V(E) \to V(E').$$

Proposition 5.1. [11, 11.1.2] *Let C be a conic curve over F and let L/F be a Galois quadratic field extension such that C is split over L. Then the natural homomorphism $V(F) \to V\big(F(C)\big)$ is injective.*

Proof. Let $u \in K_2L$ be such that $u_{L(C)} = (1-\sigma)v$ for some $v \in K_2L(C)$. For a closed point $x \in C$ the L-algebra $L(x) = L \otimes_F F(x)$ is isomorphic to the product of residue fields $L(y)$ for all closed points $y \in C_L$ over $x \in C$. By $\partial_x(v) \in L(x)^\times$ we denote the product of $\partial_y(v) \in L(y)^\times$ for all y over x.

Set $a_x = \partial_x(v) \in L(x)^\times$. We have

$$a_x/\sigma(a_x) = \partial_x(v)/\sigma(\partial_x(v)) = \partial_x\big((1-\sigma)v\big) = \partial_x\big(u_{L(C)}\big) = 1,$$

i.e., $a_x \in F(x)^\times$. By Theorem 4.1, applied to C_L,

$$\prod_{x\in C} N_{F(x)/F}(a_x) = N_{L/F}\big(\prod_{y\in C_L} N_{L(y)/L}(a_y)\big) = N_{L/F}\big(\prod_{y\in C_L} N_{L(y)/L}(\partial_y(v))\big) = 1.$$

It follows from Theorem 4.1, applied to C, that there is $w \in K_2F(C)$ such that $\partial_x(w) = a_x$ for all $x \in C$. Set $v' = v - w_{L(X)} \in K_L(C)$. Since

$$\partial_x(v') = \partial_x(v)\partial_x(w)^{-1} = a_x a_x^{-1} = 1,$$

again, by Theorem 4.1, applied to C_L, there exists $s \in K_2L$ with $s_{L(C)} = v'$. We have

$$(1-\sigma)s_{L(C)} = (1-\sigma)v' = (1-\sigma)v = u_{L(C)},$$

i.e., $(1-\sigma)s - u$ splits over $L(C)$. Since $L(C)/L$ is a purely transcendental extension, we have $(1-\sigma)s - u = 0$ (see Example 2.1) and hence $u = (1-\sigma)s \in \operatorname{Im}(1-\sigma)$. □

Corollary 5.2. *For any finitely generated subgroup $H \subset F^\times$ there is a field extension F'/F linearly disjoint with L/F such that the natural homomorphism $V(F) \to V(F')$ is injective and $H \subset N_{L'/F'}(L'^\times)$ where $L' = LF'$.*

Proof. By induction it is sufficient to assume that H is generated by one element b. Set $F' = F(C)$, where C is the conic curve associated with the quaternion algebra $Q = (L/F, b)$. Since Q is split over F' we have $b \in N_{L'/F'}(L'^\times)$ by Example 3.1. The conic C is split over L, therefore, the homomorphism $V(F) \to V(F')$ is injective by Proposition 5.1. □

For any two elements $x, y \in L^\times$ we write $\langle x, y\rangle$ for the class of the symbol $\{x, y\}$ in $V(F)$. Consider the group homomorphism

$$f = f_F : N_{L/F}(L^\times) \otimes F^\times \to V(F); \quad f\big(N_{L/F}(x) \otimes a\big) = \langle x, a\rangle.$$

The map f if well defined. Indeed, if $N_{L/F}(x) = N_{L/F}(y)$ for $x, y \in L^\times$, we have $y = xz\sigma(z)^{-1}$ for some $z \in L^\times$ by the classical Hilbert theorem 90. Then $\{y, a\} = \{x, a\} + (1 - \sigma)\{z, a\}$ and hence $\langle y, a\rangle = \langle x, a\rangle$.

Lemma 5.3. *Let $b \in N_{L/F}(L^\times)$. Then $f\big(b \otimes (1 - b)\big) = 0$.*

Proof. If $b = c^2$ for some $c \in F^\times$ then

$$f\big(b \otimes (1 - b)\big) = \langle c, 1 - c^2\rangle = \langle c, 1 - c\rangle + \langle c, 1 + c\rangle = \langle -1, 1 + c\rangle = 0$$

since $-1 = z\sigma(z)^{-1}$ for some $z \in L^\times$.

Now assume that b is not a square in F. Set

$$F' = F[t]/(t^2 - b), \quad L' = L[t]/(t^2 - b).$$

Note that L' is either a field or product of two copies of the field F'. Let $u \in F'$ be the class of t, so that $u^2 = b$. Choose $x \in L^\times$ with $N_{L/F}(x) = b$. Note that $N_{L'/F'}(\frac{x}{u}) = \frac{b}{u^2} = 1$ and $N_{L'/L}(1 - u) = 1 - b$.

The automorphism σ extends to an automorphism of L' over F'. By the classical Hilbert theorem 90 applied to the extension L'/F', there is $v \in L'^\times$ such that $v\sigma(v)^{-1} = \frac{x}{u}$. We have

$$f(b, 1-b) = \langle x, 1-b\rangle = \langle x, N_{L'/L}(1-u)\rangle = N_{L'/L}\langle x, 1-u\rangle = N_{L'/L}\langle \frac{x}{u}, 1-u\rangle =$$

$$N_{L'/L}\langle v\sigma(v)^{-1}, 1 - u\rangle = (1 - \sigma)N_{L'/L}\langle v, 1 - u\rangle = 0.$$

□

Theorem 5.4. (Hilbert theorem 90 for K_2, [6, Th. 14.1]) *Let L/F be a Galois quadratic extension and let σ be the generator of* $\mathrm{Gal}(L/F)$*. Then the sequence*

$$K_2L \xrightarrow{1-\sigma} K_2L \xrightarrow{N_{L/F}} K_2F$$

is exact.

Proof. Let $u \in K_2L$ be an element such that $N_{L/F}(u) = 0$. Since the group K_2L is generated by symbols of the form $\{x, a\}$ with $x \in L^\times$ and $a \in F^\times$ [3, Ch.IX, 2.5] we can write

$$u = \sum_{j=1}^{m} \{x_j, a_j\}$$

for some $x_j \in L^\times$ and $a_j \in F^\times$, and

$$N_{L/F}(u) = \sum_{j=1}^{m} \{N_{L/F}(x_j), a_j\} = 0.$$

Hence by definition of K_2F, we have in $F^\times \otimes F^\times$:

$$\sum_{j=1}^{m} N_{L/F}(x_j)\otimes a_j = \sum_{i=1}^{n} \pm \big(b_i \otimes (1-b_i)\big) \tag{9}$$

for some $b_i \in F^\times$. Clearly, the equality (9) holds in $H\otimes F^\times$ for some finitely generated subgroup $H\subset F^\times$ containing all $N_{L/F}(x_j)$ and b_i.

By Corollary 5.2, there is a field extension F'/F such that the natural homomorphism $V(F)\to V(F')$ is injective and $H\subset N_{L'/F'}(L'^\times)$ where $L'=LF'$. The equality (9) then holds in $N_{L'/F'}(L'^\times)\otimes F'^\times$. Now we apply the map $f_{F'}$ to both sides of (9). By Lemma 5.3, the class of $u_{L'}$ in $V(F')$ is equal to

$$\sum_{j=1}^{m}\langle x_j, a_j\rangle = f_{F'}\big(\sum_{j=1}^{m} N_{L/F}(x_j)\otimes a_j\big) = \sum_{i=1}^{n} \pm f_{F'}\big(b_i\otimes(1-b_i)\big) = 0,$$

i.e., $u_{L'}\in(1-\sigma)K_2L'$. Since the map $V(F)\to V(F')$ is injective, we get $u\in(1-\sigma)K_2L$. □

Theorem 5.5. [6, Th. 14.2] *Let $u\in K_2F$ be an element such that $2u=0$. Then $u=\{-1,a\}$ for some $a\in F^\times$. In particular, $u=0$ if $\operatorname{char}(F)=2$.*

Proof. Let $G=\{1,\sigma\}$. Consider a G-action on the field $L=F((t))$ of Laurent power series defined by

$$\sigma(t)=\begin{cases} -t, & \text{if } \operatorname{char} F\neq 2;\\ \frac{t}{1+t}, & \text{if } \operatorname{char} F = 2.\end{cases}$$

We get a quadratic Galois extension L/E, where $E=L^G$.

Consider the diagram

$$\begin{array}{ccc} K_2L & \xrightarrow{1-\sigma} & K_2L \\ \partial\downarrow & & \downarrow s \\ F^\times & \xrightarrow{\{-1\}} & K_2F, \end{array}$$

where ∂ is the residue homomorphism of the canonical descrete valuation of L, $s=s_t$ is the specialization homomorphism of the parameter t and the bottom homomorphism is the multiplication by $\{-1\}$. We claim that the diagram is commutative. The group K_2L is generated by elements of the form $\{f,g\}$ and $\{t,g\}$ for all power series f and g in $F[[t]]$ with nonzero constant term. If $\operatorname{char} F\neq 2$, we have

$$\begin{aligned} s\circ(1-\sigma)\{f,g\} &= s(\{f,g\}-\{\sigma f,\sigma g\})\\ &= \{f(0),g(0)\}-\{(\sigma f)(0),(\sigma g)(0)\}\\ &= 0 = \{-1\}\cdot\partial\{f,g\}, \end{aligned}$$

$$\begin{aligned} s \circ (1-\sigma)\{t,g\} &= s(\{-t,g\} - \{t,\sigma g\}) \\ &= \{-1, g(0)\} \\ &= \{-1\} \cdot \partial\{t,g\}. \end{aligned}$$

In the case $\operatorname{char} F = 2$ we obviously have $s(u) = s(\sigma u)$ for every $u \in K_2L$, hence $s \circ (1-\sigma) = 0$.

Since $N_{L/F}(u_L) = 2u_E = 0$, by Theorem 5.4, $u = (1-\sigma)v$ for some $v \in K_2(L)$. The commutativity of the diagram yields

$$u = s(u_L) = s\big((1-\sigma)v\big) = \{-1, \partial(v)\}.$$

□

6. Proof of the main theorem

In this section we give a proof of Theorem 2.2.

6.1. **Injectivity of h_F.** From now on we assume that F is a field of characteristic different from 2. Let $h_F(u + 2K_2F) = 0$ for an element $u \in K_2F$. Let u be a sum of n symbols. By induction on n we prove that $u \in 2K_2F$. The cases $n = 1$ and $n = 2$ were considered in [2].

Write u in the form $u = \{a, b\} + v$ for $a, b \in F^\times$ and an element $v \in K_2F$ that is a sum of $n-1$ symbols. Let C be the conic curve over F corresponding to the quaternion algebra $Q = (a, b)_F$ and set $L = F(C)$. The conic C is given by the equation

$$aX^2 + bY^2 = abZ^2$$

in the projective coordinates. Set $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. Since $\frac{x^2}{b} + \frac{y^2}{a} = 1$, we have

$$0 = \Big\{\frac{x^2}{b}, \frac{y^2}{a}\Big\} = 2\Big\{x, \frac{y^2}{a}\Big\} - 2\{b, y\} - \{a, b\}$$

and therefore $\{a, b\} = 2r$ in K_2L for $r = \{x, \frac{y^2}{a}\} - \{b, y\}$. Let $p \in C$ be the degree 2 point given by $Z = 0$. The element r has only one nontrivial residue at the point p, $\partial_p(r) = -1$.

Since the quaternion algebra $(a, b)_F$ is split over L, we have $h_L(v_L + 2K_2L) = 0$. By induction, $v_L = 2w$ for some element $w \in K_2L$.

Set $c_x = \partial_x(w)$ for every point $x \in C$. Since

$$c_x^2 = \partial_x(2w) = \partial_x(v_L) = 1,$$

we have $c_x = (-1)^{n_x}$ for $n_x = 0$ or 1. The degree of every point of C is even, hence

$$\sum_{x \in C} n_x \deg(x) = 2m$$

for some $m \in \mathbb{Z}$. Since every degree zero divisor on C is principal, there is a function $f \in L^\times$ with the degree zero divisor $\sum n_x x - mp$. Set

$$w' = w + \{-1, f\} + kr \in K_2L$$

where $k = m + n_p$. If $x \in C$ is a point different from p, we have

$$\partial_x(w') = \partial_x(w) \cdot (-1)^{n_x} = 1.$$

Since also

$$\partial_p(w') = \partial_p(w) \cdot (-1)^m \cdot (-1)^k = (-1)^{n_p+m+k} = 1,$$

we have $\partial_x(w') = 1$ for all $x \in C$. By Theorem 4.1, $w' = s_L$ for some $s \in K_2F$. Hence

$$v_L = 2w = 2w' - 2kr = 2s_L - \{a^k, b\}_L.$$

Set $v' = v - 2s + \{a^k, b\} \in K_2F$; we have $v'_L = 0$. The conic C splits over the quadratic extension $E = F(\sqrt{a})$. The field extension $E(C)/E$ is purely transcendental and $v'_{E(C)} = 0$. Hence $v'_E = 0$ (see Example 2.1) and therefore $2v' = N_{E/F}(v'_E) = 0$. By Theorem 5.5, $v' = \{-1, d\}$ for some $d \in F^\times$. Hence modulo $2K_2F$ the element v is the sum of two symbols $\{a^k, b\}$ and $\{-1, d\}$. Thus we are reduced to the case $n = 2$. $\square$

6.2. **Surjectivity of h_F.** We write k_2F for $K_2F/2K_2F$.

Proposition 6.1. *Let L/F be a quadratic extension. Then the sequence*

$$k_2F \to k_2L \xrightarrow{N_{L/F}} k_2F$$

is exact.

Proof. Let $u \in K_2L$ such that $N_{L/F}(u) = 2v$ for some $v \in K_2F$. Then $N_{L/F}(u - v_L) = 2v - 2v = 0$ and by Theorem 5.4, $u - v_L = (1 - \sigma)w$ for some $w \in K_2L$. Hence

$$u = v_L + (1 - \sigma)w = (v + N_{L/F}(w))_L - 2\sigma w.$$

$\square$

Let $s \in {}_2\operatorname{Br} F$. Suppose first that F has no odd degree extensions. By induction on the index of s we prove that $s \in \operatorname{Im}(h_F)$. Let L/F be a quadratic extension such that $\operatorname{ind}(u_L) < \operatorname{ind}(u)$. By induction, $s_L = h_L(u)$ for some $u \in k_2L$. We have

$$h_F(N_{L/F}(u)) = N_{L/F}(h_L(u)) = N_{L/F}(s_L) = 0.$$

It follows from the injectivity of h_F that $N_{L/F}(u) = 0$ and by Proposition 6.1, $u = v_L$ for some $v \in k_2F$. Then

$$h_F(v)_L = h_L(v_L) = h_L(u) = s_L$$

hence $s - h_F(v)$ is split over L and therefore it is the class of a quaternion algebra. Thus $s - h_F(v) = h_F(w)$, where $w \in k_2F$ is a symbol and $s = h_F(v + w) \in \operatorname{Im}(h_F)$.

In the general case, by the first part of the proof, there exists an odd degree extension E/F such that $s_E = h_E(v)$ for some $v \in k_2E$. Then

$$s = N_{E/F}(s_E) = N_{E/F}\big(h_E(v)\big) = h_F\big(N_{E/F}(v)\big). \quad \square$$

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Department of Mathematics, University of California, Los Angeles, CA 90095-1555

E-mail address: merkurev@math.ucla.edu