ESSENTIAL p-DIMENSION OF $\mathbf{PGL}(p^2)$

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Abstract. Let p be a prime integer and F a field of characteristic different from p. We prove that the essential p-dimension of the group $\mathbf{PGL}_F(p^2)$ is equal to p^2+1. This integer measures complexity of the class of central simple algebras of degree p^2 over field extensions of F.

1. Introduction

Informally, the essential dimension of an "algebraic structure" over a field F is the smallest number of parameters required to define this structure over a field extension of F (see [1] or [10]). Thus, the essential dimension measures complexity of the structure.

Let p be a prime integer. Essential p-dimension of an "algebraic structure" measures complexity of the structure modulo the "effects of degree prime to p" (see [11]). In practice, the essential p-dimension is easier to compute than the essential dimension.

The formal definition of the essential (p-)dimension is as follows. Let p denote either a prime integer or 0. An integer k is said to be *prime to* p if k is prime to p when $p>0$ and $k=1$ when $p=0$. Let F be a field. Consider the category Fields/F of field extensions of F and field homomorphisms over F. Let $\mathcal{F}:\mathsf{Fields}/F\to\mathsf{Sets}$ be a functor (an "algebraic structure") and $K,E\in\mathsf{Fields}/F$. An element $\alpha\in\mathcal{F}(E)$ is said to be *p-defined over K* (and K is called a *field of p-definition of α*) if there exist a finite field extension E'/E of degree prime to p (so $E'=E$ if $p=0$), a field homomorphism $K\to E'$ over F and an element $\beta\in\mathcal{F}(K)$ such that the image of α under the map $\mathcal{F}(E)\to\mathcal{F}(E')$ coincides with the image of β under the map $\mathcal{F}(K)\to\mathcal{F}(E')$. The *essential p-dimension of α*, denoted $\operatorname{ed}_p^{\mathcal{F}}(\alpha)$, is the least transcendence degree $\operatorname{tr.deg}_F(K)$ over all fields of p-definition K of α. The *essential p-dimension of the functor $\mathcal{F}$* is

$$\operatorname{ed}_p(\mathcal{F})=\sup\{\operatorname{ed}_p^{\mathcal{F}}(\alpha)\},$$

where the supremum is taken over fields $E\in\mathsf{Fields}/F$ and all $\alpha\in\mathcal{F}(E)$.

We write $\operatorname{ed}(\mathcal{F})$ for $\operatorname{ed}_0(\mathcal{F})$ and simply call $\operatorname{ed}(\mathcal{F})$ the *essential dimension of $\mathcal{F}$*. Clearly, $\operatorname{ed}(\mathcal{F})\ge\operatorname{ed}_p(\mathcal{F})$ for all p.

Let G be an algebraic group over F. The *essential p-dimension of G* is the essential p-dimension of the functor $\mathcal{F}_G:\mathsf{Fields}/F\to\mathsf{Sets}$ taking a field E to the set of isomorphism classes of all G-torsors (principal homogeneous G-spaces) over $\operatorname{Spec}(E)$.

If $G = \mathbf{PGL}_n$ over F, the functor $\mathcal{F}_G$ is isomorphic to the functor taking a field E to the set of isomorphism classes of central simple E-algebras of degree n. Let p be a prime integer and let p^r be the highest power of p dividing n. Then $\mathrm{ed}_p\big(\mathbf{PGL}_F(n)\big) = \mathrm{ed}_p\big(\mathbf{PGL}_F(p^r)\big)$ [11, Lemma 8.5.5]. Every central simple E-algebra of degree p is cyclic over a finite field extension of degree prime to p, hence $\mathrm{ed}_p\big(\mathbf{PGL}_F(p)\big) = 2$ [11, Lemma 8.5.7] as we just need two parameters to define a cyclic algebra. It is shown in [8, Cor. 3.10] and [11, Th. 8.6] that $4 \le \mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) \le p^2 + 1$.

We prove the following:

Theorem 1.1. *Let p be a prime integer and F a field of characteristic different from p. Then*

$$\mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = p^2 + 1.$$

Corollary 1.2. (Rost) *If F is a field of characteristic different from 2, then* $\mathrm{ed}\big(\mathbf{PGL}_F(4)\big) = \mathrm{ed}_2\big(\mathbf{PGL}_F(4)\big) = 5$.

Proof. By Theorem 1.1, we have $\mathrm{ed}\big(\mathbf{PGL}_F(4)\big) \ge \mathrm{ed}_2\big(\mathbf{PGL}_F(4)\big) = 5$. On the other hand, $\mathrm{ed}\big(\mathbf{PGL}_F(4)\big) \le 5$ by [8]. □

We use the following notation:

$\mathsf{X}(F)$ is the character group of the absolute Galois group $\mathrm{Gal}(F_{sep}/F)$ of a field F.

$\mathrm{Br}(F)$ is the Brauer group of F. For a field extension L/F, we write $\mathrm{Br}(L/F)$ for the relative Brauer group $\mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}(L)\big)$.

$\mathbb{G}_m$ denotes the multiplicative group $\mathrm{Spec}\, F[t, t^{-1}]$ over F.

For a finite separable field extension L/F, we write $R_{L/F}$ for the corestriction operation (see [7, §20.5]). In particular, $R_{L/F}(\mathbb{G}_{m,L})$ is the multiplicative group of L considered as an algebraic group (torus) over F. We write $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ for the torus of norm 1 elements in L.

If A is a central simple algebra over F, then $\mathrm{SB}(A)$ denotes the Severi-Brauer variety of A of reduced rank 1 right ideals in A [7, §1.C].

If p is a prime integer and B is a torsion abelian group, we write $B\{p\}$ for the p-primary component of B.

In the present paper, the word "scheme" over a field F means a separated scheme of finite type over F and a "variety" over F is an integral scheme over F. If X is a scheme over F and E/F is a field extension, then $X(E) = \mathrm{Mor}_F(\mathrm{Spec}(E), X)$ is the set of points of X over E. We write X_E for the scheme $X \times_F \mathrm{Spec}(E)$ over E.

2. Algebraic tori

2.1. **R-trivial homomorphisms of algebraic tori.** Let T be an algebraic torus over a field F. As usual, we write T^* for the character group of T over a separable closure F_{sep} of F. The group T^* is a Γ-lattice, where $\Gamma = \mathrm{Gal}(F_{sep}/F)$ is the absolute Galois group of F.

A torus P is *quasi-trivial* if P^* is a permutation lattice, i.e., there is a Γ-invariant $\mathbb{Z}$-basis of P^*.

Let E/F be a field extension. Recall that the group of R-equivalence classes $T(E)/R$ is the factor group of $T(E)$ modulo the subgroup $RT(E)$ of all elements that are R-equivalent to 1 (see [2, §5] and [14, Ch. 6]). If P is a quasi-trivial torus, then $P(E)/R = 1$.

Example 2.1. [2, Prop. 15] Let L/F be a finite Galois field extension and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ the torus of norm 1 elements in L. Then the subgroup $RT(F)$ is generated by elements of the form $\sigma(u)/u$ over all $\sigma \in \mathrm{Gal}(L/F)$ and $u \in L^\times$.

Proposition 2.2. *Let $f : T_0 \to T_1$ be a homomorphism of algebraic tori over F. Let $1 \to S_1 \to P_1 \to T_1 \to 1$ be an exact sequence with P_1 a quasi-trivial torus such that for any field extension E/F, the image of $P_1(E) \to T_1(E)$ coincides with $RT_1(E)$ (for example, a flasque resolution of T_1 satisfies this property by* [2, Th. 2]*). Then the following conditions are equivalent:*

(1) *For any field extension E/F, the homomorphism $T_0(E)/R \to T_1(E)/R$ induced by f is trivial.*
(2) *The image of the generic point of T_0 in $T_1\big(F(T_0)\big)/R$ is trivial.*
(3) *There exists a commutative diagram of homomorphisms of algebraic tori*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_0 & \longrightarrow & M_0 & \longrightarrow & T_0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & {\scriptstyle f}\downarrow & & \\ 1 & \longrightarrow & S_1 & \longrightarrow & P_1 & \longrightarrow & T_1 & \longrightarrow & 1. \end{array}$$

with exact rows and a quasi-trivial torus P_0.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): By assumption, there is a point $x \in P_1\big(F(T_0)\big)$ such that the image of x in $T_1\big(F(T_0)\big)$ coincides with the f-image of the generic point of T_0. The point x yields a rational morphism $g : T_0 \dashrightarrow P_1$ such that the composition $T_0 \dashrightarrow P_1 \to T_1$ coincides with f. Let $U \subset T_0$ be the domain of definition of g. The Γ-lattice of characters T_0^* is identified with the factor group $F_{\mathrm{sep}}[T_0]^\times/F_{\mathrm{sep}}^\times$, moreover, T_0^* is a sublattice in $\Lambda := F_{\mathrm{sep}}[U]^\times/F_{\mathrm{sep}}^\times$ and the factor lattice Λ/T_0^* is a permutation lattice (see the proof of [2, Prop. 5]). Let M_0 and P_0 be the tori with the character lattices Λ and Λ/T_0^* respectively. The morphism $U \to P_1$ yields a homomorphism of Γ-lattices $P_1^* \to \Lambda$ and therefore, a homomorphism $M_0 \to P_1$ of tori. By construction, the compositions $M_0 \to P_1 \to T_1$ coincides with the composition $M_0 \to T_0 \to T_1$.

(3) $\Rightarrow$ (1): Let E/F be a field extension. The composition $M_0(E)/R \to T_0(E)/R \to T_1(E)/R$ is trivial as it factors through the trivial group $P_1(E)/R$. The first homomorphism in the composition is surjective since P_0 is a quasi-trivial torus. Hence the homomorphism $T_0(E)/R \to T_1(E)/R$ is trivial. □

We say that a homomorphism of algebraic tori $f : T_0 \to T_1$ is *R-trivial* if f satisfies the equivalent conditions of Proposition 2.2.

Example 2.3. Let p be a prime integer and let $(K_1/F, \sigma)$ and $(K'/F, \tau)$ be cyclic field extensions of degree p^k $(k \geq 1)$ and p respectively, $L_1 = K_1 \otimes_F K'$ and $G_1 = \mathrm{Gal}(L_1/F)$. We assume that L_1 is a field.

Let $K_0 \subset K_1$ be the subfield of degree p over F and set $L_0 = K_0 \otimes_F K'$. We write G_0 for the Galois group of L_0/F.

Let $T_0 = R^{(1)}_{L_0/F}(\mathbb{G}_{m,L_0})$ and $T_1 = R^{(1)}_{L_1/F}(\mathbb{G}_{m,L_1})$ be the norm 1 tori for the extensions L_0/F and L_1/F respectively. Then T_0 is a subtorus in T_1. We claim that the inclusion homomorphism $f : T_0 \to T_1$ is not R-trivial.

For $i = 0, 1$, there is an exact sequence of G_i-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G_i] \to T_i^* \to 0. \tag{1}$$

If M is a G_1-lattice and X a G_1-module, we write $\widehat{\mathrm{Ext}}^i_{G_1}(M, X)$ for the Tate cohomology group $\widehat{H}^i(G_1, M^* \otimes_{\mathbb{Z}} X)$. It follows from (1) that

$$\widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T_0^*) \simeq \widehat{H}^{-1}(G_1, T_0^*) = \widehat{H}^{-1}(G_0, T_0^*) \simeq \widehat{H}^0(G_0, \mathbb{Z}) \simeq \mathbb{Z}/p^2\mathbb{Z}.$$

Moreover, the class of the map $f^* : T_1^* \to T_0^*$ corresponds to $1 + p^2\mathbb{Z}$, so f^* has order p^2 in $\widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T_0^*)$.

Let P_1 be the product of two copies of the torus $R_{L_1/F}(\mathbb{G}_{m,L_1})$ and $\alpha : P_1 \to T_1$ the homomorphism taking (u, v) to $\sigma(u)\tau(v)/uv$. For a field extension E/F, the image of $P_1(E)$ in $T_1(E)$ coincides with $RT_1(E)$ (see Example 2.1). Set $S_1 := \mathrm{Ker}(\alpha)$, so we have an exact sequence of tori

$$0 \to S_1 \to P_1 \to T_1 \to 0.$$

Suppose the homomorphism $f : T_0 \to T_1$ is R-trivial. By Proposition 2.2, there is a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_0 & \longrightarrow & M_0 & \longrightarrow & T_0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 1 & \longrightarrow & S_1 & \longrightarrow & P_1 & \longrightarrow & T_1 & \longrightarrow & 1. \end{array}$$

with exact rows and a quasi-trivial torus P_0. As P_1^* is a free G_1-module, the right vertical map in the commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{Ext}}^0_{G_1}(T_0^*, T_0^*) & \longrightarrow & \widehat{\mathrm{Ext}}^0_{G_1}(T_1^*, T_0^*) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Ext}}^1_{G_1}(P_0^*, T_0^*) & \longrightarrow & \widehat{\mathrm{Ext}}^1_{G_1}(S_1^*, T_0^*) \end{array}$$

is an isomorphism. It follows that the image of the identity of T_0^* in $\widehat{\mathrm{Ext}}^1_{G_1}(S_1^*, T_0^*)$ has order p^2. We get a contradiction by showing that the group $\widehat{\mathrm{Ext}}^1_{G_1}(P_0^*, T_0^*)$ has exponent p. The G_1-lattice P_0^* is a direct sum of lattices of the form $\mathbb{Z}[G_1/H_1]$, where H_1 is a subgroup of G_1. Let H_0 be the image of H_1 under the surjection $G_1 \to G_0$. We have

$$\widehat{\mathrm{Ext}}^1_{G_1}\big(\mathbb{Z}[G_1/H_1], T_0^*\big) = \widehat{\mathrm{Ext}}^1_{H_1}(\mathbb{Z}, T_0^*) = H^1(H_1, T_0^*) = H^1(H_0, T_0^*) = H^2(H_0, \mathbb{Z}).$$

The latter group is isomorphic to the character group of H_0 that is of exponent p as so is H_0.

2.2. **Characters, cyclic algebras and tori.** Let F be a field and $\Gamma = \mathrm{Gal}(F_{sep}/F)$ the absolute Galois group of F. The character group $\mathsf{X}(F)$ of Γ is equal to

$$\mathrm{Hom}_{cont}(\Gamma, \mathbb{Q}/\mathbb{Z}) = H^1(F, \mathbb{Q}/\mathbb{Z}) \simeq H^2(F, \mathbb{Z}).$$

For a character $\chi \in \mathsf{X}(F)$ set $F(\chi) = (F_{sep})^{\mathrm{Ker}(\chi)}$. Then $F(\chi)/F$ is a cyclic field extension of degree $\mathrm{ord}(\chi)$. The Galois group $\mathrm{Gal}\big(F(\chi)/F\big)$ has a canonical generator σ such that $\chi(\tilde{\sigma}) = \mathrm{ord}(\chi)^{-1} + \mathbb{Z}$ for any lifting $\tilde{\sigma}$ of σ to Γ.

Let K/F be cyclic field extension. Choose a character $\chi \in \mathsf{X}(F)$ such that $K = F(\chi)$. The cup-product

$$\mathsf{X}(F) \otimes F^\times = H^2(F, \mathbb{Z}) \otimes H^0(F, F_{sep}^\times) \to H^2(F, F_{sep}^\times) = \mathrm{Br}(F)$$

takes $\chi \otimes a$ to the class $\chi \cup (a)$ of a cyclic algebra split by K. In fact, every element of $\mathrm{Br}(K/F)$ is of the form $\chi \otimes a$ for some $a \in F^\times$.

Let L be an étale F-algebra of dimension n and $S = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$. The exact sequence

$$1 \to \mathbb{G}_m \to R_{L/F}(\mathbb{G}_{m,L}) \to S \to 1$$

and Hilbert Theorem 90 yield an isomorphism $\theta : H^1(F, S) \xrightarrow{\sim} \mathrm{Br}(L/F)$. Let $\alpha \in H^1(F, S)$ and let S_α be the corresponding principal homogeneous space of S. As S is an open subscheme of the projective space $\mathbb{P}_F(L)$, the variety S_α is an open subset of the Severi-Brauer variety $SB(A)$ of a central simple F-algebra A_α of degree n such that $[A_\alpha] = \theta(\alpha)$ in $\mathrm{Br}(L/F)$. Moreover, S_α is trivial if and only if A_α is split.

Let $\chi \in \mathsf{X}(F)$ and $L = F(\chi)$. Then $S \simeq R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ by Hilbert Theorem 90 and $[A_\alpha] = \chi \cup a$ for some $a \in F^\times$. Moreover, the principal homogeneous space S_α coincides with the fiber S_a of the norm homomorphism $R_{L/F}(\mathbb{G}_{m,L}) \to \mathbb{G}_m$ over a.

2.3. **Bicyclic algebras and tori.** Let χ and η be two characters in $\mathsf{X}(F)$ of order n and m respectively. Then the fields $K = F(\chi)$ and $K' = F(\eta)$ are cyclic extensions of F of degree n and m respectively. Set $L = K \otimes_F K'$, so L is a bicyclic extension of F of degree nm. The group $G = \mathrm{Gal}(K/F) \times \mathrm{Gal}(K'/F)$ acts naturally on L by automorphisms and G is generated by elements σ and τ such that $L^\sigma = K'$ and $L^\tau = K$.

Let I_G be the augmentation ideal in the group ring $\Lambda := \mathbb{Z}[G]$, i.e., $I_G = \mathrm{Ker}(\varepsilon)$, where $\varepsilon : \Lambda \to \mathbb{Z}$ is defined by $\varepsilon(\rho) = 1$ for all $\rho \in G$. We have:

$$\mathrm{Br}(L/F) = H^2(G, L^\times) = \mathrm{Ext}^2_G(\mathbb{Z}, L^\times) \simeq \mathrm{Ext}^1_G(I_G, L^\times). \tag{2}$$

Consider the exact sequences of G-modules

$$0 \to M \to \Lambda^2 \xrightarrow{f} I_G \to 0, \tag{3}$$

where $f(x,y)=(\sigma-1)x+(\tau-1)y$ and $M=\mathrm{Ker}(f)$ and

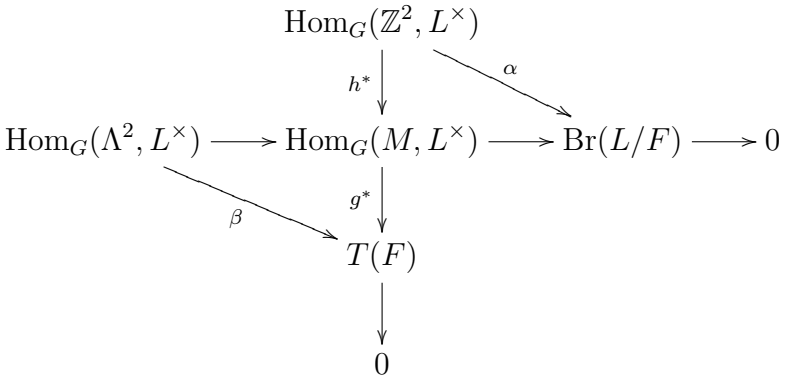
$$0\to\Lambda/\mathbb{Z}N_G\xrightarrow{g}M\xrightarrow{h}\mathbb{Z}^2\to 0, \tag{4}$$

where $N_G=\sum_{\rho\in G}\rho\in\Lambda$, $g(x+\mathbb{Z}N_G)=\big((\tau-1)x,(1-\sigma)x\big)$ and $h(x,y)=\big(\varepsilon(x)/n,\varepsilon(y)/m\big)$.

Let T be the torus of norm 1 elements for the extension L/F. We have

$$T(F)=\mathrm{Hom}_G\big(\Lambda/\mathbb{Z}N_G,L^\times\big). \tag{5}$$

The exact sequences (3) (4), the isomorphisms (2) and (5) and Hilbert Theorem 90 yield a commutative diagram:



It follows that the cokernels of α and β are naturally isomorphic. The image of $\alpha: F^{\times 2}\to\mathrm{Br}(L/F)$ is the subgroup of *decomposable elements* $\mathrm{Br}_{dec}(L/F)$ of $\mathrm{Br}(L/F)$ generated by $\chi\cup(a)$ and $\eta\cup(b)$ with $a,b\in F^\times$.

The cokernel of $\beta: L^{\times 2}\to T(F)$ is the group of R-equivalence classes $T(F)/R$ (see Example 2.1). We have proved:

Proposition 2.4. *Let L/F is a bicyclic extension and $T=R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Then there is a natural isomorphism*

$$T(F)/R\simeq\mathrm{Br}(L/F)/\mathrm{Br}_{dec}(L/F).$$

Example 2.5. The torus $T=R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ is not rational if L/F is a bicyclic field extension of degree p^2 by [14, §4.8]. Moreover, T is not R-trivial *generically*, i.e., there is a field extension E/F such that $T(E)/R\ne 1$. It fact, the image of the generic point of T in $T\big(F(T)\big)/R$ is not trivial, i.e., the identity map of T is not R-tivial (see Example 2.3).

3. Degree of points of the norm 1 torus for a bicyclic field extension

3.1. **Chow groups and push-forward homomorphism.** Let X be a scheme over a field F. We write $\mathrm{Z}(X)$ for the group of algebraic cycles on X, i.e., the free abelian group generated by points of X. We write $\mathrm{CH}(X)$ for the factor group of $\mathrm{Z}(X)$ by the subgroup of cycles rationally equivalent to 0 (see [3,

§1.3]). The groups $\mathrm{Z}(X)$ and $\mathrm{CH}(X)$ are graded by the dimension of points. If $x \in X$ is a point of dimension i, $[x]$ denotes the class of x in $\mathrm{CH}_i(X)$.

If X is a variety of dimension d, then the group $\mathrm{CH}_d(X)$ is infinite cyclic generated by the class of the generic point of X.

Let $f : X \to Y$ be a morphism of schemes over F. The push-forward homomorphism $f_* : \mathrm{Z}(X) \to \mathrm{Z}(Y)$ is a graded homomorphism defined by

$$f_*(x) = \begin{cases} [F(x) : F(y)] \cdot y, & \text{if } [F(x) : F(y)] \text{ is finite;} \\ 0, & \text{otherwise,} \end{cases}$$

where $x \in X$ and $y = f(x)$. If f is a proper morphism, then f_* factors through the rational equivalence, defining the push-forward homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ still denoted by f_* (see [3, §1.4]).

3.2. **Degree of a point.** Let X be a scheme over a field F, $a \in X(E)$ a point over a field extension E/F and $\{x\}$ the image of $a : \mathrm{Spec}(E) \to X$. The *dimension of* a is the integer $\dim(a) := \dim(x)$. If $f : X \to Y$ is a morphism of varieties over F and $a \in X(E)$ for a field extension E/F, we have $\dim(a) \geq \dim\big(f(a)\big)$. If $d = \dim(a)$, we define the *class* $[a]$ *of* a in $\mathrm{CH}_d(X)$ as follows:

$$[a] := \begin{cases} [E : F(x)] \cdot [x], & \text{if } [E : F(x)] \text{ is finite;} \\ 0, & \text{otherwise.} \end{cases}$$

In addition, if X is a variety, the *degree of* a is the integer $\deg(a)$ satisfying $[a] = \deg(a) \cdot [x]$ if $\dim(a) = \dim(X)$ and x is the generic point of X, and $\deg(a) = 0$ otherwise.

If E'/E is a field extension and $a \in X(E)$, we write $a_{E'}$ for the image of a in $X(E')$. If E'/E is finite, we have $\deg(a_{E'}) = [E' : E] \cdot \deg(a)$.

If $E = F(X)$ is the function field of X and $a \in X(E)$ is the generic point, then $\deg(a) = 1$.

Proposition 3.1. *Let $f : X \to Y$ be a proper morphism of varieties over F and let $a \in X(E)$ be a point over a field extension E/F. Then $[f(a)] = f_*([a])$ in $\mathrm{CH}(Y)$.*

Proof. Let $\{x\}$ be the image of a in X and $y = f(x)$. If one of the field extensions $E/F(x)$ and $F(x)/F(y)$ is infinite, then $[f(a)] = 0$ and $f_*([a]) = 0$. We may assume that E is a finite extension of $F(y)$. Then

$$\begin{aligned} [f(a)] &= [E : F(y)] \cdot [y] \\ &= [E : F(x)]\big([F(x) : F(y)] \cdot [y]\big) \\ &= [E : F(x)] \cdot f_*([x]) \\ &= f_*([a]). \end{aligned}$$

□

If Z is a scheme over F, we write $n(Z)$ for the $\gcd[F(z) : F]$ over all closed points $z \in Z$.

Example 3.2. Let T be an algebraic torus over F. We write $i(T)$ for the greatest common divisor of the integers $[E : F]$ over all finite field extensions E/F such that T is isotropic over E. If X is a smooth complete geometrically irreducible variety containing T as an open set, then $n(X \setminus T) = i(T)$ by [2, Lemme 12] (see also [9, Lemma 5.1]).

We shall need a variant of a push-forward homomorphism for morphisms that are not proper.

Proposition 3.3. *Let X be a complete variety over F, $U \subset X$ an open subvariety, $Z = X \setminus U$ and $f : U \to Y$ a morphism over F, where Y is a variety of dimension d over F. If $n = n(Z_{F(Y)})$, then the push-forward homomorphism on cycles $f_* : \mathrm{Z}(U) \to \mathrm{Z}(Y)$, composed with the projection $\mathrm{Z}(Y) \to \mathrm{Z}_d(Y) = \mathbb{Z}$, gives rise to a well defined homomorphism*

$$f_\star : \mathrm{CH}(U) \to \mathbb{Z}/n\mathbb{Z}.$$

Moreover, for any point $a \in U(E)$ over a field extension E/F, one has $f_\star([a]) = \deg\big(f(a)\big)$ modulo n.

Proof. We define the map $f_\star$ to be trivial on all homogeneous components $\mathrm{CH}_i(U)$ except $i = d$, so we just need to define $f_\star$ on $\mathrm{CH}_d(U)$.

We claim that the image of the push-forward homomorphism

$$s_* : \mathrm{CH}_d(Z \times Y) \to \mathrm{CH}_d(Y) = \mathbb{Z}$$

for the projection $s : Z \times Y \to Y$ is contained in $n\mathbb{Z}$. Let $u \in Z \times Y$ be a point of dimension d. If $s(u)$ is not the generic point of Y, then $s_*([u]) = 0$. Otherwise, u is a closed point in $Z_{F(Y)} \subset Z \times Y$ and $s_*([u])$ coincides with the degree of this closed point and hence is divisible by n. The claim is proven.

The map s_* factors as $s_* = q_* \circ i_*$, where $i : Z \times Y \to X \times Y$ is the closed embedding and $q : X \times Y \to Y$ is the projection. By localization [3, §1.8], $\mathrm{CH}_d(U \times Y)$ is canonically isomorphic to the cokernel of i_*. By the claim, q_* gives rise to a homomorphism $\mathrm{CH}_d(U \times Y) \to \mathbb{Z}/n\mathbb{Z}$. Composing it with the push-forward homomorphism for the closed embedding $(1_U, f) : U \to U \times Y$, we get the required homomorphism $f_\star : \mathrm{CH}_d(U) \to \mathbb{Z}/n\mathbb{Z}$. The last equality in the statement follows from Proposition 3.1 applied to q. □

Example 3.4. Let T be an algebraic torus over F and $n = i(T)$ (see Example 3.2). Then the structure morphism $T \to \mathrm{Spec}(F)$ gives rise to a homomorphism $\mathrm{CH}_0(T) \to \mathbb{Z}/n\mathbb{Z}$ that takes the class of a closed point $t \in T$ to $[F(t) : F]$ modulo n.

3.3. **Chow groups of tori and Severi-Brauer varieties.** Let p be a prime integer and let Z be the product of r copies of the projective space $\mathbb{P}_F(W)$, where W is a vector space of dimension $n > 0$ over F. Then

$$\mathrm{CH}(Z) = \mathbb{Z}[\mathbf{h}] := \mathbb{Z}[h_1, h_2, \dots, h_r],$$

with $h_i^n = 0$ for all i, where h_i is pull-back on Z of the class of a hyperplane on the ith factor of Z. Moreover, $\mathbb{Z}[\mathbf{h}]$ is the factor ring of the polynomial ring

on the variables $t_1, t_2, \dots, t_r$ by the ideal generated by $t_1^n, t_2^n, \dots, t_r^n$. Note that the homogeneous ith component $\mathbb{Z}[\mathbf{h}]_i$ is trivial if $i > r(n-1)$ and $\mathbb{Z}[\mathbf{h}]_{r(n-1)} = \mathbb{Z}h^{n-1}$, where $h := h_1 h_2 \cdots h_p$.

Let K/F be a Galois field extension with a cyclic Galois group H of prime order p and let σ be a generator of H. Let V be a vector space of dimension $n > 0$ over K. Consider the variety $X = R_{K/F}\big(\mathbb{P}_K(V)\big)$ over F. Then X_K is the product of p copies of $\mathbb{P}_K(V)$. The group H acts on the product by cyclic permutation of the factors. We have the graded ring homomorphism

$$\mathrm{CH}(X) \to \mathrm{CH}(X_K) = \mathbb{Z}[\mathbf{h}],$$

where $\mathbf{h} = (h_1, h_2, \dots, h_p)$.

The group H acts on $\mathbb{Z}[\mathbf{h}]$ permuting cyclically the h_i's. Hence the image of the map $\mathrm{CH}(X) \to \mathbb{Z}[\mathbf{h}]$ is contained in the subring $\mathbb{Z}[\mathbf{h}]^H$ of H-invariant elements, so we have the graded ring homomorphism

$$\mathrm{CH}(X) \to \mathbb{Z}[\mathbf{h}]^H$$

(which is in fact an isomorphism). The image of an element $\alpha \in \mathrm{CH}(X)$ in $\mathbb{Z}[\mathbf{h}]^H$ is denoted by $\bar{\alpha}$. For example, if α is the class of the subscheme $R_{K/F}\big(\mathbb{P}_K(W)\big)$ of X, where W is a K-subspace of V of codimension $i = 0, 1, \dots, n-1$, then $\bar{\alpha} = h^i$.

Consider the trace homomorphism

$$\mathrm{tr} : \mathbb{Z}[\mathbf{h}] \to \mathbb{Z}[\mathbf{h}]^H$$

defined by $\mathrm{tr}(x) = \sum_{i=0}^{p-1} \sigma^i(x)$. We write I for the image of tr. Clearly, I is a graded ideal in $\mathbb{Z}[\mathbf{h}]^H$. Note that

$$\big(\mathbb{Z}[\mathbf{h}]^H\big)_j = \begin{cases} I_j, & \text{if } p \text{ does not divide } j; \\ \mathbb{Z}h^i + I_j, & \text{if } j = pi. \end{cases} \tag{6}$$

It follows that $\mathbb{Z}[\mathbf{h}]^H$ is generated by I and h^i, $i = 0, 1, \dots, n-1$ as an abelian group. Moreover, $ph^j \in I$ for all j and $I_{p(n-1)} = p\mathbb{Z}h^{n-1}$.

Let A be a central simple algebra over K of degree n and let $Y = R_{K/F}\big(\mathrm{SB}(A)\big)$, where $\mathrm{SB}(A)$ is the Severi-Brauer variety of A over K. The function field E of Y splits A and is linearly disjoint with K/F. Therefore, $Y_E \simeq X_E$ and we have the ring homomorphism

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_E) \simeq \mathrm{CH}(X_E) \to \mathbb{Z}[\mathbf{h}]^H.$$

The image of an element $\alpha \in \mathrm{CH}(Y)$ in $\mathbb{Z}[\mathbf{h}]^H$ is denoted by $\bar{\alpha}$.

Proposition 3.5. *Let K/F be a cyclic field extension of a prime degree p, let A be a nonsplit central simple K-algebra of degree p and $Y = R_{K/F}\big(\mathrm{SB}(A)\big)$. Then the image of the map $\mathrm{CH}(Y) \to \mathbb{Z}[\mathbf{h}]^H$ is contained in $\mathbb{Z} + I$.*

Proof. Consider a more general situation: A is a central simple K-algebra of index p and degree n. Let $\alpha \in \mathrm{CH}(Y)$. We shall prove in the cases 1 and 2 below that $\bar{\alpha} \in \mathbb{Z} + I$. By (6), we may assume that $\alpha \in \mathrm{CH}^{pi}(Y)$ for $i = 1, 2, \dots, n-1$. Let $a \in \mathbb{Z}$ be such that $\bar{\alpha} \equiv ah^i$ modulo I. It suffices to prove that a is divisible by p.

Case 1: $i = n-1$. We have $\bar\alpha = bh^{n-1}$ for some $b \equiv a$ modulo p as $I_{p(n-1)} = p\mathbb{Z}h^{n-1}$. Since h^{n-1} is the class of a rational point of Y over a splitting field and the degree of every closed point of Y is divisible by p, we have $b \in p\mathbb{Z}$. Therefore, $a \in p\mathbb{Z}$.

Case 2: i divides $n-1$. Write $n-1 = ij$. We have $\alpha^j \in \mathrm{CH}^{p(n-1)}(Y)$ and $\alpha^j \equiv a^j h^{n-1}$ modulo I. By Case 1, a^j and hence a is divisible by p.

Now assume that A is a central division K-algebra of degree p and $\alpha \in \mathrm{CH}^{pi}(Y)$ with $i = 1, 2, \dots, p-1$. We shall prove that $\bar\alpha \in I$. Write $ik + pm = 1$ for some integers k and $m > 0$. The Severi-Brauer variety $\mathrm{SB}(M_m(A))$ can be identified with the variety of the reduced rank 1 right A-submodules in the free right A-module A^m. The projection to last component A of A^m gives rise to a rational morphism $\mathrm{SB}(M_m(A)) \to \mathrm{SB}(A)$ that is defined on the complement U of the variety $\mathrm{SB}(M_{m-1}(A))$ embedded into $\mathrm{SB}(M_m(A))$ as a closed subvariety via the inclusion $A^{m-1} \to A^m$, $(a_1, \dots, a_{m-1}) \mapsto (a_1, \dots, a_{m-1}, 0)$. Moreover, the projection $U \to \mathrm{SB}(A)$ is a vector bundle.

Let $Y' = R_{K/F}\big(\mathrm{SB}(M_m(A))\big)$ and $U' = R_{K/F}(U)$. Then U' is an open subscheme of Y' and the natural morphism $U' \to Y$ is a vector bundle. Hence we have a surjective homomorphism

$$\mathrm{CH}(Y') \to \mathrm{CH}(U') \simeq \mathrm{CH}(Y).$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{CH}(Y') & \longrightarrow & \mathrm{CH}(Y) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}[\mathbf{h}']^H & \longrightarrow & \mathbb{Z}[\mathbf{h}]^H \end{array},$$

where the bottom map takes a monomial $\mathbf{h}'^{\alpha}$ to $\mathbf{h}^{\alpha}$ if $\alpha_i < p$ for all i and to 0 otherwise, is commutative. Lift α to an element $\alpha' \in \mathrm{CH}^{pi}(Y')$. As i divides $pm - 1$, by Case 2 applied to the algebra $M_m(A)$, we have $\bar\alpha' \in I'$. Since the bottom map in the diagram takes I' to I, we have $\bar\alpha \in I$. □

Let K'/F be a cyclic field extension of degree p and

$$S = \big(R^{(1)}_{K'/F}(\mathbb{G}_{m,K'})\big)^r \simeq \big(R_{K'/F}(\mathbb{G}_{m,K'})/\,\mathbb{G}_m\big)^r$$

for some $r > 0$. We view the variety of the group S as an open subset of $Z := \mathbb{P}_F(K')^r$. Hence the restriction gives a surjective ring homomorphism

$$(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}] = \mathrm{Ch}(Z) \to \mathrm{Ch}(S),$$

where $\mathbf{h} = (h_1, h_2, \dots, h_r)$, $h_i^p = 0$ for all i, and we write Ch for the Chow groups modulo p. We shall also write $\tilde h_i$ for the image of h_i in $\mathrm{Ch}^1(S)$. The class in $\mathrm{Ch}^{r(p-1)}(S)$ of a rational point of S is equal to $\tilde h^{p-1}$, where $\tilde h = \tilde h_1 \tilde h_2 \cdots \tilde h_r \in \mathrm{Ch}^p(S)$. As $i(S) = p$, we have $\tilde h^{p-1} \ne 0$ by Example 3.4.

Proposition 3.6. *The map $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}] \to \mathrm{Ch}(S)$ is a ring isomorphism.*

Proof. Suppose that $f(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_r) = 0$ for a nonzero homogeneous polynomial f over $\mathbb{Z}/p\mathbb{Z}$. Suppose that a monomial $h_1^{\alpha_1} \cdots h_r^{\alpha_r}$ enters f with a nonzero coefficient. Multiplying the equality by $\tilde{h}_1^{\beta_1} \cdots \tilde{h}_r^{\beta_r}$ with $\beta_i = p-1-\alpha_i$, we get $\tilde{h}^{p-1} = 0$, a contradiction. □

For an element α in $\mathrm{Ch}(S)$ we shall write $\bar{\alpha}$ for the corresponding element in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$.

Consider the homomorphism $f : S \times S \to S$ defined by $f(x, y) = xy^{-1}$. Recall that as $i(S) = p$, by Example 3.2 and Proposition 3.3, we have the homomorphism

$$f_\star : \mathrm{CH}_{r(p-1)}(S \times S) \to \mathbb{Z}/p\mathbb{Z}. \tag{7}$$

Lemma 3.7. *For any $\alpha \in \mathrm{Ch}^i(S)$ and $\beta \in \mathrm{Ch}^j(S)$ with $i + j = r(p-1)$, we have*

$$\bar{\alpha} \cdot \bar{\beta} = f_\star(\alpha \times \beta) h^{p-1}$$

in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$.

Proof. It suffices to consider the case when α and β are monomials in $\tilde{h}_i$. As both sides of the equality commute with products, we may assume that $r = 1$, i.e., $S = R_{K'/F}(\mathbb{G}_{m,K'})/\mathbb{G}_m$, and $\alpha = \tilde{h}^i$, $\beta = \tilde{h}^j$. The cycles α and β are represented by $\mathbb{P}(U) \cap S$ and $\mathbb{P}(W) \cap S$, where U and W are F-subspaces of K' of codimension i and j respectively. The fiber of the restriction

$$f' : \big(\mathbb{P}(U) \cap S\big) \times \big(\mathbb{P}(W) \cap S\big) \to S$$

of f over a point s of S is isomorphic to $\mathbb{P}(U \cap sW) \cap S$. The vector space $U \cap sW$ is one-dimensional for a generic s, hence f' is a birational isomorphism and $f_\star(\alpha \times \beta) = 1 + p\mathbb{Z}$. On the other hand, $\bar{\alpha} \cdot \bar{\beta} = h^i \cdot h^j = h^{p-1}$. □

Let L/F be a bicyclic field extension of degree p^2 and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Choose a subfield K of L of degree p over F and let $t \in K^\times$ be an element with $N_{K/F}(t) = 1$, i.e., t is an F-point of the torus $R^{(1)}_{K/F}(\mathbb{G}_{m,K})$. Write S_t for the fiber of the norm homomorphism $T \to R^{(1)}_{K/F}(\mathbb{G}_m)$ over t. The variety S_t is a principal homogeneous space of the torus $S = R_{K/F}\big(R^{(1)}_{L/K}(\mathbb{G}_{m,L})\big) \simeq R_{K/F}\big(R_{L/K}(\mathbb{G}_{m,L})/\mathbb{G}_{m,K}\big)$.

The variety S_t is canonically isomorphic to an open subscheme of the variety $Y := R_{K/F}\big(\mathrm{SB}(A_t)\big)$ for a central simple K-algebra A_t of degree p (see Section 2.2). Over the function field E of $\mathrm{SB}(A_t)$ over K, the varieties S_t and S become isomorphic to the torus $\big(R^{(1)}_{LE/E}(\mathbb{G}_{m,LE})\big)^p$, where $LE = L \otimes_K E$, so we can apply the constructions considered above to the torus S_E over E. In particular, we have the element $\bar{\alpha} \in (\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]$ well defined for any cycle α on S_t and S.

Consider the morphism

$$f : S_t \times S \to S_t, \qquad f(x, y) = xy^{-1}.$$

We have defined the homomorphism (see (7)):

$$f_\star : \mathrm{CH}_{p(p-1)}(S_t \times S) \to \mathrm{CH}_{p(p-1)}\big((S_t)_E \times S_E\big) \to \mathbb{Z}/p\mathbb{Z}.$$

Proposition 3.8. *Suppose that the principal homogeneous space S_t is not trivial. Then $f_\star(\alpha\times\tilde{h}^j)=0$ for any $\alpha\in \mathrm{Ch}^{p(p-j-1)}(S_t)$ and $j=0,1,\dots,p-2$.*

Proof. As S_t is not trivial, the algebra A_t is not split. We can lift α to a cycle β in $\mathrm{Ch}(Y)$. By Proposition 3.5, $\bar{\beta}$ belongs to the image $\tilde{I}$ of the ideal I in $(\mathbb{Z}/p\mathbb{Z})[\mathbf{h}]^H$. It follows that $\bar{\alpha}\cdot h^j=\bar{\beta}\cdot h^j\in\tilde{I}_{p(p-1)}=0$. Lemma 3.7 (applied to the field extension E of F and $r=p$) shows that $f_\star(\alpha\times\tilde{h}^j)=0$. □

3.4. **A key proposition.** Let p be a prime integer, L/F a bicyclic field extension of degree p^2, $G=\mathrm{Gal}(L/F)$, σ and τ generators of G. Consider the tori $T=R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ of norm 1 elements in L/F and $P=R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$, both of dimension $d:=p^2-1$. The torus T (respectively, P) becomes isotropic over a field extension E/F if and only if $E\otimes_F L$ is not a field. It follows that $i(T)=i(P)=i(T\times P)=p$.

Consider the morphisms f and g from $T\times P$ to T defined by $f(t,v)=t$ and $g(t,v)=t\sigma(v)/v$. By Proposition 3.3 and Example 3.2, f and g give rise to a well defined homomorphisms $f_\star$ and $g_\star$ from $\mathrm{CH}_d(T\times P)$ to $\mathbb{Z}/p\mathbb{Z}$.

Proposition 3.9. *The maps $f_\star$ and $g_\star$ coincide.*

Proof. The torus P is an open subscheme in the projective space $\mathbb{P}_F(L)$, hence the ring $\mathrm{CH}(P)$ is generated by the restriction to P of the class e of a hyperplane in $\mathbb{P}_F(L)$. Moreover, by the Projective Bundle Theorem [3, Th. 3.3], $\mathrm{CH}_d(T\times P)$ coincides with the sum of subgroups $\mathrm{CH}_i(T)\times e^i$ over all $i=0,1,\dots,d$.

Let $\beta\in\mathrm{CH}_i(T)$. It suffices to show that $f_\star(\beta\times e^i)=g_\star(\beta\times e^i)$ for any $i=0,1,\dots,d$. If $i=d$, the class e^i is represented by the identity point 1 of P. The equality follows from the fact that f and g coincide on $T\times\{1\}$.

Now assume that $i<d$. In this case $f_\star(\beta\times e^i)=0$ and we need to show that $g_\star(\beta\times e^i)=0$.

Let K be the subfield of σ-invariant elements in L of degree p over F. We have $pk+1\le p^2-i\le p(k+1)$ for some integer $k=0,\dots,p-1$. Consider a K-linear subspace W of L of K-dimension k such that $K\cap W=0$. Let V be an F-subspace of L of dimension p^2-i over F such that

$$F\oplus W\subset V\subset K\oplus W.$$

The class of $P\cap\mathbb{P}(V)$ in $\mathrm{CH}^i(P)$ is equal to e^i.

The torus $S:=R_{K/F}\big(R^{(1)}_{L/K}(\mathbb{G}_{m,L})\big)$ is the kernel of the norm homomorphism $T\to T_1:=R^{(1)}_{K/F}(\mathbb{G}_{m,K})$, so we have an exact sequence

$$1\to S\to T\to T_1\to 1. \tag{8}$$

By Hilbert Theorem 90, $S\simeq R_{K/F}\big(R_{L/K}(\mathbb{G}_{m,L})/\mathbb{G}_{m,K}\big)$. We view S as an open subscheme of $R_{K/F}\big(\mathbb{P}_K(L)\big)$. The map g factors as follows:

$$T\times P\xrightarrow{1_T\times l}T\times S\xrightarrow{r}T,$$

where $l : P \to S$ is defined by $l(v) = v/\sigma(v)$ and $r(t,s) = ts^{-1}$. The image of $P \cap \mathbb{P}_F(K \oplus W)$ under l is the variety $S \cap R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ of dimension pk in $S \simeq R_{K/F}\big(R_{L/K}(\mathbb{G}_{m,L})/\,\mathbb{G}_{m,K}\big)$. Hence, if $p^2 - i > pk+1$, then $\dim\big(P \cap \mathbb{P}(V)\big) > pk$, but dimension of the image of $P \cap \mathbb{P}(V)$ under l is at most pk, so $P \cap \mathbb{P}(V)$ loses dimension under l, therefore, $g_\star(\beta \times e^i) = 0$.

It remains to consider the case $p^2 - i = pk + 1$, $k = 1, \dots, p-1$, i.e., $V = F \oplus W$. Since the map $P \cap \mathbb{P}(V) \to R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ given by l is a birational isomorphism, and the class of $R_{K/F}\big(\mathbb{P}_K(K \oplus W)\big)$ in $\mathrm{CH}(S)$ is equal to h^{p-k-1}, where $h \in \mathrm{CH}^p(S)$ is the class given by a K-hyperplane in L, it suffices to show that $r_\star(\beta \times h^{p-k-1}) = 0$.

Let S_t be the fiber of the norm homomorphism $T \to T_1$ over the generic point t of T_1, so S_t is a principal homogeneous space of S over the function field $F(T_1)$. Denote by

$$r' : S_t \times S \to S_t$$

the morphism given by $r'(x,s) = xs^{-1}$. Thus we have a commutative diagram

$$\begin{array}{ccc} S_t \times S & \xrightarrow{r'} & S_t \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ T \times S & \xrightarrow{r} & T \end{array}$$

where m is the canonical morphism and $q = m \times 1_S$. It follows that $r_\star$ factors as the composition

$$\mathrm{CH}_d(T \times S) \xrightarrow{q^*} \mathrm{CH}_{p(p-1)}(S_t \times S) \xrightarrow{r'_\star} \mathbb{Z}/p\mathbb{Z}.$$

Thus, it suffices to show that $r'_\star(\alpha \times h^{p-k-1}) = 0$ for any $\alpha \in \mathrm{CH}^{pk}(S_t)$. This follows from Proposition 3.8 applied to the torus S over the field $F(T_1)$ (with $j = p-k-1$) if we show that S_t is a nontrivial principal homogeneous space of S. Suppose that S_t has a point over $F(T_1)$. It follows that the exact sequence (8) splits rationally, i.e., the torus T is birationally isomorphic to the product $S \times T_1$ and hence is a rational variety. But T is not rational (see Example 2.5), a contradiction. □

3.5. **Invariance of the degree under R-equivalence.**

Theorem 3.10. *Let p be a prime integer, L/F a bicyclic field extension of degree p^2 and $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. Let M/F be a field extension and let t and t' be R-equivalent points in $T(M)$. Then $\deg(t) \equiv \deg(t')$ modulo p.*

Proof. We have $t' = t \cdot \sigma(u)u^{-1} \cdot \tau(v)v^{-1}$ for some $u, v \in (LM)^\times$ (see Example 2.1). Let $t'' = t \cdot \sigma(u)u^{-1}$. It suffices to prove that $\deg(t) = \deg(t'')$ and $\deg(t') = \deg(t'')$ in $\mathbb{Z}/p\mathbb{Z}$. We shall prove the first equality (the second being similar). So replacing t' by t'' we may assume that $t' = t \cdot \sigma(u)u^{-1}$.

Consider the point $w = (t,u)$ in $(T \times P)(M)$ and two morphisms f and g from $T \times P$ to T as in Section 3.4. We have $f(w) = t$ and $g(w) = t'$. By

Propositions 3.3 and 3.9, we have in $\mathbb{Z}/p\mathbb{Z}$:

$$\deg(t) = \deg f(w) = f_\star([w]) = g_\star([w]) = \deg g(w) = \deg(t'). \qquad \square$$

4. Essential p-dimension of $\mathbf{PGL}(p^2)$

Let F be a field and p a prime integer different from $\mathrm{char}(F)$.

4.1. **Central simple algebras and discrete valuations.** Let v be a discrete valuation on a field extension E over F, N the residue field, $\widehat{E}$ the completion of E. Then N is a field extension of F. Let $\chi \in \mathsf{X}(F)$. Then $F(\chi)/F$ is a cyclic field extension of degree $\mathrm{ord}(\chi)$ with the choice of a generator of $\mathrm{Gal}\big(F(\chi)/F\big)$. The group $\mathsf{X}(N)$ is identified with the character group of the maximal unramified field extension of $\widehat{E}$. For a character $\chi \in \mathsf{X}(N)$, we write $\widehat{\chi}$ for the corresponding character in $\mathsf{X}(\widehat{E})$.

There is an exact sequence of p-groups [4, Prop. 7.7]:

$$0 \to \mathrm{Br}(N)\{p\} \xrightarrow{i} \mathrm{Br}(\widehat{E})\{p\} \xrightarrow{\partial} \mathsf{X}(N)\{p\} \to 0. \tag{9}$$

The first map preserves indices of algebras. For a central simple algebra C over N with $C \in \mathrm{Br}(N)\{p\}$ let $\widehat{C}$ be a central simple algebra over $\widehat{E}$ of the same degree representing the image of $[C]$ under i. For example, if $[C] = \chi \cup (\bar{u})$ for some $\chi \in \mathsf{X}(N)\{p\}$ and a unit $u \in \widehat{E}$, then $[\widehat{C}] = \widehat{\chi} \cup (u)$.

The choice of a prime element π in $\widehat{E}$ provides with a splitting of the sequence (9) by sending a character χ to the class of the cyclic algebra $\widehat{\chi} \cup (\pi)$. Thus for every central simple algebra A over $\widehat{E}$ we can write

$$[A] = [\widehat{C}] + \big(\widehat{\chi} \cup (\pi)\big)$$

in $\mathrm{Br}(\widehat{E})$ for unique $[C] \in \mathrm{Br}(N)\{p\}$ and $\chi = \partial\big([A]\big)$. Moreover (see [5, Th. 5.15(a)] or [13, Prop. 2.4]),

$$\mathrm{ind}(A) = \mathrm{ord}(\chi) \cdot \mathrm{ind}(C_{N(\chi)}). \tag{10}$$

Let E'/E be a finite field extension and v' a discrete valuation on E' extending v with residue field N'. Then for any $[A] \in \mathrm{Br}(E)\{p\}$ one has

$$\partial_{v'}\big([A]_{E'}\big) = e \cdot \partial_v([A])_{N'}, \tag{11}$$

where e is the ramification index of E'/E [4, Prop. 8.2].

4.2. **The functors $\mathcal{F}_1$ and $\mathcal{F}_2$.** We define the functors $\mathcal{F}_1$ and $\mathcal{F}_2$ from the category Fields/F of field extensions of F to the category Sets as follows. Let E/F be a field extension. Then $\mathcal{F}_1(E)$ is the set of isomorphism classes of central simple E-algebras of degree p^2. Thus, $\mathrm{ed}_p(\mathcal{F}_1) = \mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big)$.

Let $\mathcal{S}_2(E)$ be the class of pairs (B, K), where B is a central simple algebra of degree p^2 over E and K is a cyclic étale E-algebra of degree p such that $\mathrm{ind}(B_K) \le p$. We say that the pairs (B_1, K_1) and (B_2, K_2) are equivalent if $K_1 \simeq K_2$ over E and $[B_1] - [B_2] \in \mathrm{Br}(K_1/E) = \mathrm{Br}(K_2/E)$. Let $\mathcal{F}_2(E)$ be the set of equivalence classes in $\mathcal{S}_2(E)$. We write $[B, K]$ for the class in $\mathcal{F}_2(E)$ of a pair (B, K).

We say that the class $[B,K]$ is *decomposable* if $[B,K]=[B',K]$ with B' a split algebra.

Let $(B,K)\in\mathcal{S}_2(E)$ with K a field and let $\chi\in\mathsf{X}(E)$ be a character (of order p) such that $K=E(\chi)$ (see Section 2.2). As $\operatorname{ind}(B_K)\le p$, there is a central simple algebra C over the function field $E(y)$ (y is a variable) of degree p^2 such that

$$[C]=[B_{E(y)}]+\big(\chi_{E(y)}\cup(y)\big) \tag{12}$$

in $\operatorname{Br}\big(E(y)\big)$. We have $[C]\in\mathcal{F}_1\big(E(y)\big)$ and $\partial\big([C]\big)=\chi$, where ∂ is taken with respect to the discrete valuation v on $E(y)$ associated to y (see Section 4.1).

Consider the following condition $(*)$ on the pair (B,K) in $\mathcal{S}_2(E)$ and the character χ:

For any finite field extension N/E of degree prime to p, the class $[B,K]_N$ is not decomposable and the class of the algebra B_N in $\operatorname{Br}(N)$ cannot be written in the form $[B_N]=\rho\cup(s)$ for some $s\in N^\times$ and a character $\rho\in\mathsf{X}(N)$ of order p^2 such that $p\cdot\rho$ is a multiple of χ_N.

Proposition 4.1. *Let $\chi\in\mathsf{X}(E)$ be a character of prime order p, $K=E(\chi)$, B a central simple algebra of degree p^2 over E such that $(B,K)\in\mathcal{S}_2(E)$ and (B,K) together with χ satisfy the condition $(*)$. Then*

$$\operatorname{ed}_p^{\mathcal{F}_1}([C])\ge\operatorname{ed}_p^{\mathcal{F}_2}([B,K])+1$$

for the algebra C defined by (12).

Proof. Let $M/E(y)$ be a finite field extension of degree prime to p, $M_0\subset M$ a subfield over F and $[C_0]\in\mathcal{F}_1(M_0)$ such that

$$[C_0]_M=[C]_M \tag{13}$$

in $\mathcal{F}_1(M)$ and $\operatorname{ed}_p^{\mathcal{F}_1}([C])=\operatorname{tr.deg}_F(M_0)$. We extend the discrete valuation v on $E(y)$ to a discrete valuation v' on M with ramification index e' and inertia degree prime to p (see [6, Lemma 1.1]). Thus, the residue field N of v' is a finite extension of E of degree prime to p. Let v_0 be the restriction of v' to M_0 and N_0 its residue field. As $[N:E]$ is not divisible by p, it follows from (11) that $\partial([C]_M)=e'\cdot\chi_N\ne0$. Hence the algebra C_M is ramified, i.e., the class of C_M does not belong to the image of the map $\operatorname{Br}(O)\to\operatorname{Br}(M)$, where O is the valuation ring of v'. It follows that C_0 is also ramified, therefore v_0 is nontrivial and hence v_0 is a discrete valuation.

Let $\chi_0=\partial\big([C_0]\big)\in\mathsf{X}(N_0)\{p\}$ and $K_0=N_0(\chi_0)$. Choose a prime element π_0 in M_0 and write

$$[C_0]_{\widehat{M_0}}=[\widehat{B}_0]+\big(\widehat{\chi}_0\cup(\pi_0)\big) \tag{14}$$

in $\operatorname{Br}(\widehat{M_0})$, where B_0 is a central simple algebra over N_0 (see Section 4.1). By (10),

$$\operatorname{ind}(C_0)=\operatorname{ord}(\chi_0)\cdot\operatorname{ind}(B_0)_{K_0}. \tag{15}$$

Let e be the ramification index of M/M_0 and let π be a prime element in M. Write $\pi_0 = u\pi^e$ and $y = v\pi^{e'}$ with u and v units in M.

It follows from (13) and (11) that

$$e' \cdot \chi_N = \partial\big([C]_M\big) = \partial\big([C_0]_M\big) = e \cdot \partial\big([C_0]\big)_N = e \cdot (\chi_0)_N. \tag{16}$$

Recall that e' is relatively prime to p. It follows that χ_N is a multiple of $(\chi_0)_N$. In particular, $\operatorname{ord}(\chi_0)$ is divisible by p.

It follows from (13),(14) and (16) that

$$[\widehat{(B_0)_N}] + \big(\widehat{(\chi_0)_N} \cup (u)\big) = [\widehat{B}_N] + \big(\widehat{\chi}_N \cup (v)\big) \tag{17}$$

in $\operatorname{Br}(\widehat{M})$, hence

$$[(B_0)_N] + \big((\chi_0)_N \cup (\bar{u})\big) = [B_N] + \big(\chi_N \cup (\bar{v})\big) \tag{18}$$

in $\operatorname{Br}(N)$.

Since $\operatorname{ind}(C_0) \le p^2$, it follows from (10) and (15) that $\operatorname{ord}(\chi_0)$ divides p^2.

Case 1: $\operatorname{ord}(\chi_0) = p^2$. By (15), $\operatorname{ind}(B_0)_{K_0} = 1$, i.e., B_0 is split over K_0, hence $[B_0] = \chi_0 \cup (s_0)$ for some $s_0 \in N_0^\times$. It follows from (18) that $[B_N] = (\chi_0)_N \cup (s)$ for some $s \in N^\times$. If $\operatorname{ord}(\chi_0)_N = p$, then $(\chi_0)_N$ is a multiple of χ_N and hence $[B, K]_N$ is decomposable. If $\operatorname{ord}(\chi_0)_N = p^2$, the character $p \cdot (\chi_0)_N$ is a multiple of χ_N. In both cases, (B, K) and χ do not satisfy the condition $(*)$, a contradiction.

Case 2: $\operatorname{ord}(\chi_0) = p$. Then the characters χ_N and $(\chi_0)_N$ generate the same subgroup in $\mathsf{X}(N)$. It follows that

$$K_0 \otimes_{N_0} N \simeq N\big((\chi_0)_N\big) = N(\chi_N) \simeq K \otimes_E N. \tag{19}$$

By (15), we have $\operatorname{ind}(B_0)_{K_0} \le p$. Therefore, we may assume that $\deg(B_0) = p^2$ and hence $(B_0, K_0) \in \mathcal{S}_2(N_0)$. It follows from (18) that

$$[B]_N - [B_0]_N \in \operatorname{Br}(K \otimes_E N/N).$$

By (19), the pairs $(B_N, K \otimes_E N)$ and $((B_0)_N, K_0 \otimes_{N_0} N) = (B_0, K_0)_N$ are equivalent in $\mathcal{S}_2(N)$. It follows that the class of $[B, K]$ in $\mathcal{F}_2(E)$ is p-defined over N_0, therefore,

$$\operatorname{ed}_p^{\mathcal{F}_1}\big([C]\big) = \operatorname{tr.deg}_F(M_0) \ge \operatorname{tr.deg}_F(N_0) + 1 \ge \operatorname{ed}_p^{\mathcal{F}_2}\big([B, K]\big) + 1. \qquad \square$$

4.3. **The functor $\mathcal{F}_3$.** Let E/F be a field extension and let $\mathcal{S}_3(E)$ be the class of pairs (A, L), where A is a csa of degree p^2 over E and L is a bicyclic étale E-algebra of dimension p^2 such that L splits A, i.e., $[A] \in \operatorname{Br}(L/E)$. We say that the pairs (A_1, L_1) and (A_2, L_2) in $\mathcal{S}_3(E)$ are equivalent if $L_1 \simeq L_2$ and $[A_1] - [A_2] \in \operatorname{Br}_{dec}(L_1/E) = \operatorname{Br}_{dec}(L_2/E)$ (see Section 2.3). Let $\mathcal{F}_3(E)$ be the set of equivalence classes in $\mathcal{S}_3(E)$. We write $[A, L]$ for the equivalence class of (A, L) in $\mathcal{F}_3(E)$.

Let L be a bicyclic étale E-algebra of dimension p^2. We view the factor group $\operatorname{Br}(L/E)/\operatorname{Br}_{dec}(L/E)$ as a subset of $\mathcal{F}_3(E)$ identifying the class of an algebra A with $[A, L]$.

We say that a class $[A, L]$ is *decomposable* if $[A, L] = [A', L]$ with A' a split algebra.

Let $(A, L) \in S_3(E)$. Choose characters χ and η in $\mathsf{X}(E)$ such that $L = E(\chi, \eta) := E(\chi)E(\eta)$. Let $K = E(\chi)$ and $K' = E(\eta)$. As $\operatorname{ind}(A_K) \le p$, there is a central simple algebra B over the function field $E(x)$ (x is a variable) of degree p^2 such that

$$[B] = [A_{E(x)}] + \big(\eta_{E(x)} \cup (x)\big) \tag{20}$$

in $\operatorname{Br}\big(E(x)\big)$. We have $\big(B, K(x)\big) \in \mathcal{S}_2\big(E(x)\big)$ and $\partial\big([B]\big) = \eta$, where ∂ is taken with respect to the discrete valuation v on $E(x)$ associated to x.

Consider the following condition $(**)$ on the pair (A, L) in $\mathcal{S}_3(E)$ and the characters χ and η:

For any finite field extension N/E of degree prime to p, the class $[A, L]_N$ is not decomposable and the class of the algebra A_N in $\operatorname{Br}(N)$ cannot be written in the form $[A_N] = \big(\rho \cup (s)\big) + \big(\varepsilon \cup (t)\big)$ for some $s, t \in N^\times$ and characters $\varepsilon \in \mathsf{X}(N)$ of order p and $\rho \in \mathsf{X}(N)$ of order p^2 such that $\langle p \cdot \rho, \varepsilon\rangle = \langle \chi_N, \eta_N\rangle$.

Proposition 4.2. *Let $\chi, \eta \in \mathsf{X}(E)$ be linearly independent characters of prime order p, $K = E(\chi)$, $L = E(\chi, \eta)$, A a central simple algebra of degree p^2 over E such that $(A, L) \in \mathcal{S}_3(E)$ and (A, L) with with the characters χ and η satisfy the condition $(**)$. Then*

$$\operatorname{ed}_p^{\mathcal{F}_2}\big([B, K(x)]\big) \ge \operatorname{ed}_p^{\mathcal{F}_3}\big([A, L]\big) + 1$$

for the algebra B defined by (20).

Proof. Let $M/E(x)$ be a finite field extension of degree prime to p, $M_0 \subset M$ a subfield over F and $[B_0, R_0] \in \mathcal{F}_2(M_0)$ such that

$$[B_0, R_0]_M = [B, K(x)]_M$$

in $\mathcal{F}_2(M)$ and $\operatorname{ed}_p^{\mathcal{F}_2}\big([B, K(x)]\big) = \operatorname{tr.deg}_F(M_0)$. This equality means that

$$R := K(x) \otimes_{E(x)} M \simeq R_0 \otimes_{M_0} M \quad \text{and} \tag{21}$$

$$[B]_M = [B_0]_M + \big(\chi_M \cup (f)\big) \tag{22}$$

for some $f \in M^\times$.

We extend the discrete valuation v on $E(x)$ to a discrete valuation v' on M with ramification index e' and inertia degree prime to p (see [6, Lemma 1.1]). Thus, the residue field N of v' is a finite extension of E of degree prime to p. Let v_0 be the restriction of v' to M_0 and N_0 its residue field. As $[N : E]$ is not divisible by p, it follows from (11) that $\partial([B]_M) = e' \cdot \chi_N \ne 0$. Hence the algebra B_M is ramified. It follows that B_0 is also ramified, therefore v_0 is nontrivial and hence v_0 is a discrete valuation.

As $R = KM$, the valuation v' on M extends uniquely to a discrete valuation on R and R/M is unramified.

Let $\eta_0 = \partial([B_0]) \in \mathsf{X}(N_0)\{p\}$ and $K_0' = N_0(\eta_0)$. Choose a prime element π_0 in M_0 and write

$$[B_0]_{\widehat{M_0}} = [\widehat{A}_0] + \big(\widehat{\eta}_0 \cup (\pi_0)\big) \tag{23}$$

in $\mathrm{Br}(\widehat{M_0})$, where A_0 is a central simple algebra over N_0. By (10),

$$\mathrm{ind}(B_0) = \mathrm{ord}(\eta_0) \cdot \mathrm{ind}(A_0)_{K_0'}. \tag{24}$$

Let e be the ramification index of M/M_0 and let π be a prime in M. Write $\pi_0 = u\pi^e$, $x = v\pi^{e'}$ and $f = w\pi^k$ with u, v and w units in M.

It follows from (22) and (11) that

$$e' \cdot \eta_N = \partial([B]_M) = e \cdot \partial([B_0])_N + \partial\big(\chi_M \cup (f)\big) = e \cdot (\eta_0)_N + k \cdot \chi_N. \tag{25}$$

Note that the characters χ_N and η_N are linearly independent in $\mathsf{X}(N)$ since $[N : E]$ is not divisible by p.

As e' is relatively prime to p, η_N belongs to the subgroup of $\mathsf{X}(N)$ generated by $(\eta_0)_N$ and χ_N, and $\eta_0 \neq 0$ since χ_N and η_N are linearly independent. In particular, p divides $\mathrm{ord}(\eta_0)$.

It follows from (22), (23) and (25) that

$$[\widehat{(A_0)_N}] + \big(\widehat{(\eta_0)_N} \cup (u)\big) + \big(\widehat{\chi}_M \cup (w)\big) = [\widehat{A}_N] + \big(\widehat{\eta}_N \cup (v)\big) \tag{26}$$

in $\mathrm{Br}(\widehat{M})$, hence

$$[(A_0)_N] + \big((\eta_0)_N \cup (\bar{u})\big) + \big(\chi_N \cup (\bar{w})\big) = [A_N] + \big(\eta_N \cup (\bar{v})\big) \tag{27}$$

in $\mathrm{Br}(N)$.

Since $\mathrm{ind}(B_0) \le p^2$, it follows from (24) that $\mathrm{ord}(\eta_0) \le p^2$.

Case 1: $\mathrm{ord}(\eta_0) = p^2$. By (24), A_0 is split over $N_0(\eta_0)$, hence $[A_0] = \eta_0 \cup (s_0)$ for some $s_0 \in N_0^\times$. It follows from (27) that $[A_N] = \big((\eta_0)_N \cup (s)\big) + \big(\chi_N \cup (t)\big)$ for some $s, t \in N^\times$. If $\mathrm{ord}(\eta_0)_N = p$, by (25), $(\eta_0)_N$ is contained in $\langle \chi_N, \eta_N \rangle$ and hence $[A, L]_N$ is decomposable. If $\mathrm{ord}(\eta_0)_N = p^2$, then again by (25), $\langle p \cdot (\eta_0)_N, \chi_N \rangle = \langle \chi_N, \eta_N \rangle$. In both cases, (A, L) with the characters χ and η do not satisfy the condition $(**)$, a contradiction.

Case 2: $\mathrm{ord}(\eta_0) = p$. It follows from (25) that $(e, p) = 1$ and η_0 belongs to the subgroup generated by χ and η. Hence, by (21), the cyclic extension R_0/M_0 is unramified. Thus, there exists a character $\chi_0 \in \mathsf{X}(N_0)$ with $\widehat{R}_0 = \widehat{M_0}(\widehat{\chi}_0)$ and $(\chi_0)_N = \chi_N$.

It follows from (25) that

$$\langle (\chi_0)_N, (\eta_0)_N \rangle = \langle \chi_N, \eta_N \rangle$$

in $\mathsf{X}(N)$. Let $L_0 = N_0(\chi_0, \eta_0)$. Then

$$L_0 \otimes_{N_0} N = N\big((\chi_0)_N, (\eta_0)_N\big) = N(\chi_N, \eta_N) = L \otimes_E N \tag{28}$$

is a bicyclic field extension of degree p^2, hence so is the extension L_0/N_0. In particular, χ_0 and η_0 generate a subgroup of order p^2 in $\mathsf{X}(N_0)$.

Let $K_0 = N_0(\chi_0)$. It follows from (23) that

$$[B_0]_{\widehat{R}_0} = [\widehat{(A_0)}_{K_0}] + \big(\widehat{(\eta_0)}_{K_0} \cup (\pi_0)\big).$$

As $(B_0, R_0) \in \mathcal{S}_2(M_0)$, we have $\operatorname{ind}(B_0)_{R_0} \leq p$. Since the character $(\eta_0)_{K_0}$ is nontrivial, it follows from (10) that A_0 is split by $K_0((\eta_0)_{K_0}) = L_0$. We may then assume that $\deg(A_0) = p^2$ and hence $(A_0, L_0) \in \mathcal{S}_3(N_0)$.

It follows from (27) that $[A_N] - [(A_0)_N] \in \operatorname{Br}_{dec}(L \otimes_E N/N)$. By (28), the pairs $(A_N, L \otimes_E N)$ and $\big((A_0)_N, L_0 \otimes_{N_0} N\big) = (A_0, L_0)_N$ are equivalent in $\mathcal{S}_3(N)$. Then the class $[A, L]$ in $\mathcal{F}_3(E)$ is p-defined over N_0, therefore,

$$\operatorname{ed}_p^{\mathcal{F}_2}([B, K(x)]) = \operatorname{tr.deg}_F(M_0) \geq \operatorname{tr.deg}_F(N_0) + 1 \geq \operatorname{ed}_p^{\mathcal{F}_3}([A, L]) + 1. \quad \square$$

Let E be a field extension of F and L/E a bicyclic field extension of degree p^2. Write T for the torus over E of norm 1 elements for the field extension L/E. Let $t \in T\big(E(T)\big)$ be the generic point and let $[A, L(T)]$ be the corresponding element in $\mathcal{F}_3\big(E(T)\big)$ via the isomorphism between $T\big(E(T)\big)/R$ and $\operatorname{Br}(L(T)/E(T))/\operatorname{Br}_{dec}(L(T)/E(T))$ in Proposition 2.4.

Proposition 4.3. $\operatorname{ed}_p^{\mathcal{F}_3}\big([A, L(T)]\big) \geq p^2 - 1$.

Proof. Let $M/E(T)$ be a field extension of degree prime to p, $M_0 \subset M$ a subfield over F and $[A_0, L_0] \in \mathcal{F}_3(M_0)$ such that $[A_0, L_0]_M = [A, L(T)]_M$. We need to prove that $\operatorname{tr.deg}_F(M_0) \geq p^2 - 1$. Set $LM = L \otimes_E M$. As $L_0 \otimes_{M_0} M \simeq LM$, we may assume that $L_0 \subset LM$.

Let T_0 be the torus over M_0 of norm 1 elements for the extension L_0/M_0. We have $(T_0)_M \simeq T_M$. Consider the commutative diagram

$$\begin{array}{ccc} T_0(M_0)/R & \longrightarrow & T(M)/R \\ \downarrow & & \downarrow \\ \mathcal{F}_3(M_0) & \longrightarrow & \mathcal{F}_3(M) \end{array}$$

where the vertical injective maps are given by the isomorphisms in Proposition 2.4. The pair $[A_0, L_0]$ belongs to the image of the left vertical map in the diagram. Hence there exists an element $t_0 \in T_0(M_0)$ such that $(t_0)_M$ in $T_0(M) = T(M)$ is R-equivalent to t_M. We have $\deg(t) = 1$, therefore, $\deg(t_M)$ is not divisible by p as $[M : E(T)]$ is prime to p. By Theorem 3.10, $\deg((t_0)_M) \equiv \deg(t_M)$ modulo p, hence $\deg((t_0)_M) \neq 0$. It follows that $(t_0)_M$, viewed as a morphism $\operatorname{Spec}(M) \to T$ is dominant. Therefore, there is a field homomorphism $E(T) \to M$ over E taking t to $(t_0)_M$. The elements $\rho(t)$ over all $\rho \in G := \operatorname{Gal}(L/E)$ generate the field $L(T)$ over L. Hence the elements $\rho(t_0)_M$ generate a subfield in LM over L of the transcendence degree $\dim(T) = p^2 - 1$. As $t_0 \in L_0$ and L_0 is normal over M_0 and hence is G-invariant, the elements $\rho(t_0)$ generate a subfield in L_0 over F of the transcendence degree $p^2 - 1$. It follows that $\operatorname{tr.deg}_F(L_0) \geq p^2 - 1$, hence $\operatorname{tr.deg}_F(M_0) \geq p^2 - 1$. $\square$

Remark 4.4. Let L be a bicyclic field extension of degree p^2 of a field F of arbitrary characteristic and let $T = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. A similar argument as the

one in the proof of Proposition 4.3 shows that $\mathrm{ed}_p(T/R) = p^2 - 1$, where T/R is the functor taking a field E to $T(E)/R$.

4.4. **The main theorem.**

Theorem 4.5. *Let p be a prime integer and F a field of characteristic different from p. Then*

$$\mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = p^2 + 1.$$

Proof. Recall that $\mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = \mathrm{ed}_p(\mathcal{F}_1)$. First we prove the inequality $\mathrm{ed}_p(\mathcal{F}_1) \ge p^2 + 1$. We may replace F by any field extension. In particular, we may assume that there are linearly independent characters $\chi, \eta \in \mathsf{X}(F)$ of order p, hence $L := F(\chi, \eta)/F$ is a bicyclic field extension of degree p^2. Set $K = F(\chi)$ and $K' = F(\eta)$. Let T be the norm 1 torus for the extension L/F and set $E := F(T)$. Let $[A, LE]$ be the element of $\mathcal{F}_3(E)$ corresponding to the generic point $t \in T(E)$ via the isomorphism in Proposition 2.4. Consider the pair $\big(B, KE(x)\big) \in \mathcal{S}_2(E(x))$ with $[B] = [A_{E(x)}] + \big(\eta_{E(x)} \cup (x)\big)$ in $\mathrm{Br}(E(x))$ and the algebra C of degree p^2 over $E(x, y)$ with $[C] = [B_{E(x,y)}] + \big(\chi_{E(x,y)} \cup (y)\big)$ in $\mathrm{Br}\big(E(x, y)\big)$.

We claim that the pair (A, LE) in $\mathcal{S}_3(E)$ and the characters χ_E and η_E satisfy the condition $(**)$. Indeed, as $t \ne 1$ in $T(E)/R$ (see Section 2.3) we have $t_N \ne 1$ since $[N : E]$ is prime to p and hence $[A, LE]_N$ is not decomposable. Now suppose that $[A_N] = \big(\rho \cup (s)\big) + \big(\varepsilon \cup (t)\big)$ for a field extension N/E of degree prime to p, elements $s, t \in N^\times$ and characters $\varepsilon \in \mathsf{X}(N)$ of order p and $\rho \in \mathsf{X}(N)$ of order p^2 such that $\langle p \cdot \rho, \varepsilon\rangle = \langle \chi_N, \eta_N\rangle$. Let T_1 be the norm 1 torus for the field extension $L_1 = N(\rho, \varepsilon)$ over N. By Example 2.3, the inclusion homomorphism $T \to T_1$ is not R-trivial, i.e, the image of t in $T_1(E)/R$ is not trivial. It follows that the image of t in $T_1(N)/R$ is also non-trivial. By Proposition 2.4, $[A_N]$ does not belong to the kernel of the homomorphism $\mathrm{Br}(LN/N)/\mathrm{Br}_{dec}(LN/N) \to \mathrm{Br}(L_1/N)/\mathrm{Br}_{dec}(L_1/N)$, a contradiction. The claim is proved.

We claim that the pair $\big(B, KE(x)\big)$ in $\mathcal{S}_2\big(E(x)\big)$ and the character $\chi_{E(x)}$ satisfy the condition $(*)$. The same argument as in the previous claim applied to the field $E(x)$ shows that $\big(A_{E(x)}, LE(x)\big)$ in $\mathcal{S}_3(E(x))$ and the characters $\chi_{E(x)}$ and $\eta_{E(x)}$ satisfy the condition $(**)$. Let $N/E(x)$ be a finite field extension of degree prime to p. As $[A_{E(x)}] = [B] - \big(\eta_{E(x)} \cup (x)\big)$, the class $[B, KE(x)]_N$ is not decomposable. Suppose that $[B_N] = \rho \cup (s)$ for some $s \in N^\times$ and a character $\rho \in \mathsf{X}(N)$ of order p^2 such that $p \cdot \rho$ is a multiple of χ_N. Then $[A_N] = \big(\rho \cup (s)\big) - \big(\eta_N \cup (x)\big)$ and we have $\langle p \cdot \rho, \eta_N\rangle = \langle \chi_N, \eta_N\rangle$, a contradiction proving the claim.

By Propositions 4.1, 4.2 and 4.3,

$$\mathrm{ed}_p\big(\mathbf{PGL}_F(p^2)\big) = \mathrm{ed}_p(\mathcal{F}_1) \ge \mathrm{ed}_p^{\mathcal{F}_1}([C]) \ge \mathrm{ed}_p^{\mathcal{F}_2}([B, KE(x)]) + 1 \ge$$
$$\mathrm{ed}_p^{\mathcal{F}_3}\big([A, LE]\big) + 2 \ge (p^2 - 1) + 2 = p^2 + 1.$$

We shall show that $\mathrm{ed}_p(\mathcal{F}) \le p^2 + 1$. As mentioned in the introduction, this was shown in [8, Cor. 3.10(a)]. For completeness, we give the argument here.

Let $\mathcal{F}_1'(E)$ be the set of isomorphism classes of central simple E-algebras of degree p^2 that are crossed products with the group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. So $\mathcal{F}_1'$ is a subfunctor of $\mathcal{F}_1$. By [12, Th. 1.2], for every $[A] \in \mathcal{F}_1(E)$ there is a finite field extension E'/E of degree prime to p such that $[A_{E'}] \in \mathcal{F}_1'(E')$. Hence the inclusion of $\mathcal{F}_1'$ into $\mathcal{F}_1$ is p-surjective. It follows that $\mathrm{ed}_p(\mathcal{F}_1) \leq \mathrm{ed}_p(\mathcal{F}_1')$ [10, Prop. 1.3]. So it suffices to show that $\mathrm{ed}(\mathcal{F}_1') \leq p^2 + 1$.

Let E/F be a field extension and $[A] \in \mathcal{F}_1'(E)$. Then $[A] \in \mathrm{Br}(L/E)$ for a bicyclic field extension L/F of degree p^2 with Galois group G generated by σ and τ. The exact sequence (3) yields an epimorphism

$$\mathrm{Hom}_G(M, L^\times) \to \mathrm{Br}(L/E).$$

Choose a G-homomorphism $\varphi : M \to L^\times$ corresponding to $[A]$ in $\mathrm{Br}(L/E)$. Since $\mathrm{rank}(M) = p^2 + 1$, the image of φ is contained in $L_0^\times$, where L_0 is a G-invariant subfield of L with $\mathrm{tr.deg}_F(L_0) \leq p^2 + 1$. Note that G acts faithfully on M. Modifying φ by an element in the image of the map $\mathrm{Hom}_G(\Lambda^2, L^\times) \to \mathrm{Hom}_G(M, L^\times)$, we may assume that G acts faithfully on the image of φ and hence on L_0. Thus L_0 is a Galois extension of $E_0 := (L_0)^G$ with Galois group G and φ defines a central simple E_0-algebra A_0 with $[A_0] \in \mathrm{Br}(L_0/E_0)$ such that $A_0 \otimes_{F_0} E \simeq A$. Thus, A is defined over E_0, hence

$$\mathrm{ed}^{\mathcal{F}_1'}([A]) \leq \mathrm{tr.deg}_F(E_0) = \mathrm{tr.deg}_F(L_0) \leq p^2 + 1. \qquad \square$$

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