PERIODS OF PRINCIPAL HOMOGENEOUS SPACES OF ALGEBRAIC TORI

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Abstract. A generic torsor of an algebraic torus S over a field F is the generic fiber of a S-torsor $P \to T$, where P is a quasi-trivial torus containing S as a subgroup and $T = P/S$. The period of a generic S-torsor over a field extension K/F, i.e., the order of the class of the torsor in the group $H^1(K,S)$ does not depend on the choice of a generic torsor. In the paper we compute the period of a generic torsor of S in terms of the character lattice of the torus S.

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Let T be an algebraic torus defined over a field F (see [3] and [2]) and T^* the character lattice of the torus T over a separable closure F_{sep} of the field F. The continuous action of the absolute Galois group Γ_F of the field F on the lattice T^* factors through a finite group G, called a *decomposition group* of the torus T. A torus P is called *quasi-split*, if the lattice P^* is *permutation*, i.e, if there exists a $\mathbb{Z}$-basis of the lattice P^* that is invariant under G.

Let T be a torus with the decomposition group G. A *resolution* of T is an exact sequence of tori

$$1 \to T \to P \xrightarrow{\varphi} S \to 1 \tag{1}$$

with the decomposition group G and a quasi-split torus P. The *class of the resolution* (1) is the element of the group $\mathrm{Ext}^1_G(T^*, S^*)$ corresponding to the exact sequence of G-lattices of characters

$$0 \to S^* \to P^* \to T^* \to 0$$

(see [1, Ch. XIV, §1]).

Let K/F be a field extension. We consider T-torsors (principal homogeneous spaces) over the field K. There is a bijective correspondence between the set of isomorphism classes of such torsors and the first Galois cohomology group $H^1(K,T) := H^1\big(\Gamma_K, T(K_{\mathrm{sep}})\big)$. The *period* of a torsor is the order of the corresponding element in this group. In the present paper we compute periods of generic torsors in terms of character lattices.

1. Generic torsors

Let T be a torus over a field F. Choose a resolution (1) of the torus T. For any field extension K/F and any point $s \in S(K)$, the pre-image $\varphi^{-1}(s)$ is a

Date: October, 2009.

The work has been supported by the NSF grant DMS #0652316.

T-torsor over K. Conversely, for any T-torsor E over K, there exists a point $s \in S(K)$ such that $E \simeq \varphi^{-1}(s)$. Indeed, the right group in the exact sequence

$$S(K) \to H^1(K,T) \to H^1(K,P)$$

is trivial since P is a quasi-split torus. The generic fiber of the morphism φ, that is a T-torsor over the function field $F(S)$ of the torus S, is called a *generic* T-torsor.

Proposition 1.1. *For any torus T over F, the period of the generic T-torsor is divisible by the period of every T-torsor over any field extension of F.*

Proof. Since the natural map $H^1(F,T) \to H^1(F(t),T)$ is injective, replacing F by the field $F(t)$, we may assume that the field F is infinite. Let a be the class of the T-torsor (1) over S in the group $H^1_{ét}(S,T)$. Then the class of the generic torsor in $H^1\big(F(S),T\big)$ is equal to the inverse image of the element a with respect to the morphism $\operatorname{Spec} F(S) \to S$ of the generic point. Since the group $H^1\big(F(S),T\big)$ is equal to the colimit of the groups $H^1_{ét}(U,T)$ over all open subsets $U \subset S$, there is a nonempty open subset U in S such that the period of the inverse image $a|_U$ of the element a in $H^1_{ét}(U,T)$ coincides with the period of the generic T-torsor.

Let K be a field extension of F and E a T-torsor over K. Choose a point $s : \operatorname{Spec}(K) \to S$ in $S(K)$ such that the class of E in $H^1(K,T)$ is the inverse image of the element a with respect to s. As K is infinite, the set $P(K)$ is dense in P. Since φ is surjective, $\varphi\big(P(K)\big)$ is dense in S, therefore, there is a point s' in the intersection of $\varphi\big(P(K)\big)s$ and U. Replacing s by s', we may assume that $s \in U(K)$, i.e., s is a morphism of $\operatorname{Spec}(K)$ to U. The class E in $H^1(K,T)$ is the inverse image of the element $a|_U$ with respect to s. Therefore, the period of the torsor E divides the period of the torsor $a|_U$ that is equal to the period of the generic T-torsor. $\square$

It follows from Proposition 1.1 that the period $e(T)$ of the generic T-torsor does not depend on the choice of the resolution (1). By Proposition 1.1, $e(T)$ is divisible by the period of every T-torsor over any field extension of F.

2. Computation of the period of a generic torsor

Let

$$1 \to T \to R \xrightarrow{f} S \to 1 \tag{2}$$

be an exact sequence of tori with a decomposition group G. For any subgroup $H \subset G$ consider the pairing induced by the product in cohomology:

$$\widehat{H}^0(H,T_*) \otimes H^1(H,S^*) \to \operatorname{Ext}^1_H(T^*,S^*) \xrightarrow{\operatorname{cor}_{G/H}} \operatorname{Ext}^1_G(T^*,S^*), \tag{3}$$

where T_* is the co-character group of the torus T and $\operatorname{cor}_{G/H}$ is the corestriction homomorphism.

Proposition 2.1. *The following are equivalent:*

(1) *For any field extension K/F, the homomorphism $f_K : R(K) \to S(K)$ is surjective.*
(2) *The generic point of the torus S in $S\big(F(S)\big)$ belongs to the image of the map $f_{F(S)}$.*
(3) *The exact sequence* (2) *admits a rational splitting.*
(4) *There exists a commutative diagram of homomorphisms of algebraic tori with the decomposition group G:*

$$\begin{tikzcd} 1 \arrow[r] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & T \arrow[r] & R \arrow[r, "f"] & S \arrow[r] & 1 \end{tikzcd}$$

with the exact rows and a quasi-split torus P.
(5) *The class of the exact sequence* (2) *in* $\mathrm{Ext}^1_G(T^*, S^*)$ *belongs to the image of the map*

$$\coprod_{H \subset G} \hat{H}^0(H, T_*) \otimes H^1(H, S^*) \to \mathrm{Ext}^1_G(T^*, S^*),$$

induced by the pairing (3)*, where the direct sum is taken over all subgroups H in the group G.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): By assumption, there is a point $x \in R\big(F(S)\big)$ such that the image of x in $S\big(F(S)\big)$ coincides with the generic point. The point x yields a rational morphism $g : S \dashrightarrow R$ such that the composition $S \dashrightarrow R \xrightarrow{f} S$ is the identity.

(3) $\Rightarrow$ (4): Let L/F be a splitting field of the tori with Galois group G. By assumption, there is a rational splitting $g : S \dashrightarrow R$ of f. Let $U \subset S$ be the domain of definition of the map g. The character lattice S^* is identified with the factor group $L[S]^\times/L^\times$. Moreover, S^* is a sublattice in $\Lambda := L[U]^\times/L^\times$ and the factor lattice Λ/S^* is permutation (see the proof of [2, Prop. 5]). Let M and P be the tori with characters lattices Λ and Λ/S^* respectively. The morphism $U \to R$ yields a homomorphism of G-lattices $R^* \to \Lambda$ and, therefore, a tori homomorphism $M \to R$. By construction, the composition $M \to R \to S$ coincides with the map $M \to S$, induced by the inclusion of S^* into Λ.

(4) $\Rightarrow$ (1): Let K/F be a field extension. The triviality of the group $H^1(K, P)$ implies that the map $M(K) \to S(K)$ is surjective. Therefore, f_K is also surjective.

(4) $\Leftrightarrow$ (5): The diagram in (4) exists if and only if there is a quasi-split torus P and a G-homomorphism $\alpha : T^* \to P^*$ such that the class of the exact sequence (2) lies in the image of the map $\alpha^* : \mathrm{Ext}^1_G(P^*, S^*) \to \mathrm{Ext}^1_G(T^*, S^*)$. The lattice P^* is a direct sum of lattices of the form $\mathbb{Z}[G/H]$, where H is a subgroup in G. In the case $P^* = \mathbb{Z}[G/H]$, the map α is given by an element

$a \in (T_*)^H$ and $\mathrm{Ext}^1_G(P^*, S^*) = H^1(H, S^*)$. The map α^* coincides with multiplication by the class of the element a in $\hat{H}^0(H, T_*)$ with respect to the pairing (3). □

Theorem 2.2. *Let T be a torus with the decomposition group G. Then the period $e(T)$ of the generic T-torsor is equal to the order of the class of the identity endomorphism of the lattice T^* in the cokernel of the map*

$$\coprod_{H \subset G} \hat{H}^0(H, T_*) \otimes \hat{H}^0(H, T^*) \to \hat{H}^0\big(G, \mathrm{End}(T^*)\big). \tag{4}$$

Proof. Choose a resolution (1) of the torus T with a free G-module P^*. Fix a natural number n. Let the torus R be defined by the commutative diagram of tori homomorphisms

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & P & \longrightarrow & S & \longrightarrow & 1 \\ & & {\scriptstyle n}\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \longrightarrow & R & \xrightarrow{\alpha} & S & \longrightarrow & 1 \end{array}$$

with exact rows. The class of the T-torsor in the group $H^1(F(S), T)$, given by the generic fiber of the morphism α, is equal to nth multiple of the class of the generic T-torsor. Hence $e(T)$ divides n if and only if α has a rational splitting. The statement of the theorem follows now from Proposition 2.1 since $H^1(H, S^*) = \hat{H}^0(H, T^*)$ and $\mathrm{Ext}^1_G(T^*, S^*) = \hat{H}^0\big(G, \mathrm{End}(T^*)\big)$. □

3. Flasque and coflasque resolutions of a torus

Let S be a torus with the decomposition group G. The torus S is called *flasque* (respectively, *coflasque*) if $\hat{H}^{-1}(H, S^*) = 0$ (respectively, $\hat{H}^1(H, S^*) = 0$)) for any subgroup $H \subset G$.

A *flasque resolution* of a torus T is an exact sequence of tori

$$1 \to S \to P \to T \to 1 \tag{5}$$

with the decomposition group G, a flasque torus S and a quasi-split torus P. A *coflasque resolution* of a torus T is an exact sequence of tori

$$1 \to T \to P' \to S' \to 1 \tag{6}$$

with the decomposition group G, a coflasque torus S' and a quasi-split torus P'. Flasque and coflasque resolutions of T exist by [2, Lemme 3].

Theorem 3.1. *Let* (6) *be a coflasque resolution of a torus T. Then the period $e(T)$ of the generic T-torsor is equal to the order of the class of the resolution* (6) *in the group* $\mathrm{Ext}^1_G(T^*, S'^*)$.

Proof. As in the proof of Theorem 2.2, the number $e(T)$ coincides with the order of the class of the resolution in the cokernel of the pairing from Proposition 2.1(5). Since $H^1(H, S'^*) = 0$ for any subgroup $H \subset G$, the cokernel coincides with $\mathrm{Ext}^1_G(T^*, S'^*)$. □

The number $e(T)$ can be also computed using a flasque resolution of the torus T.

Theorem 3.2. *Let* (5) *be a flasque resolution of a torus T. Then the period $e(T)$ of the generic T-torsor is equal to the order of the class of the resolution* (5) *in the group* $\mathrm{Ext}^1_G(S^*, T^*)$.

Proof. By Theorem 3.1, it suffices to show that the orders of the classes $r \in \mathrm{Ext}^1_G(S^*, T^*)$ and $r' \in \mathrm{Ext}^1_G(T^*, S'^*)$ of the flasque and coflasque resolutions respectively coincide. Consider the diagram with the exact row and column:

$$\begin{array}{ccccc} & & & & \mathrm{Ext}^1_G(S^*, P'^*) \\ & & & & \downarrow \\ & & \mathrm{Hom}_G(T^*, T^*) & \xrightarrow{\beta} & \mathrm{Ext}^1_G(S^*, T^*) \\ & & {\scriptstyle\beta'}\downarrow & & {\scriptstyle\alpha}\downarrow \\ \mathrm{Ext}^1_G(P^*, S'^*) & \longrightarrow & \mathrm{Ext}^1_G(T^*, S'^*) & \xrightarrow{\alpha'} & \mathrm{Ext}^2_G(S^*, S'^*) \end{array}$$

According to [2, §1], the groups $\mathrm{Ext}^1(P^*, S'^*)$ and $\mathrm{Ext}^1_G(S^*, P'^*)$ are trivial, hence, the homomorphisms α and α' are injective. Since $r = -\beta(1_{T^*})$ and $r' = \beta'(1_{T^*})$ by [1, Ch. XIV, §1], and also the square in the diagram is anti-commutative in view of [1, Ch. V, . 4.1], then $\alpha(r) = \alpha'(r')$, i.e., the elements r and r' have the same order. □

4. Examples

4.1. **Relative Brauer group.** Let L/F be a finite separable field extension, G the Galois group of a normal closure L' of the field L and $C = \mathrm{Gal}(L'/L)$. Consider the torus $T = R_{L/F}(\mathbb{G}_{m,L})/\,\mathbb{G}_m$ (see [3, Ch. III, §5]). The exact sequence of tori

$$1 \to \mathbb{G}_m \to R_{L/F}(\mathbb{G}_{m,L}) \to T \to 1,$$

that is a flasque resolution of the torus T, yields an exact sequence

$$H^1\big(K, R_{L/F}(\mathbb{G}_{m,L})\big) \to H^1(K,T) \to H^2(K,\mathbb{G}_m) \to H^2\big(K, R_{L/F}(\mathbb{G}_{m,L})\big)$$

for any field extension K/F. Furthermore, $H^2(K,\mathbb{G}_m) = \mathrm{Br}(K)$ is the Brauer group of the field K. By Hilbert Theorem 90 and Faddeev-Shapiro Lemma, $H^1\big(K, R_{L/F}(\mathbb{G}_{m,L})\big) = 0$ and

$$H^2\big(K, R_{L/F}(\mathbb{G}_{m,L})\big) = H^2(KL, \mathbb{G}_m) = \mathrm{Br}(KL),$$

where $KL = K \otimes_F L$. Hence,

$$H^1(K,T) = \mathrm{Br}(KL/K) := \mathrm{Ker}\big(\mathrm{Br}(K) \to \mathrm{Br}(KL)\big)$$

is the relative Brauer group of the extension KL/K.

The character group of the torus T coincides with the kernel of the augmentation $\mathbb{Z}[G/C] \to \mathbb{Z}$. From the exact sequence

$$H^0(G, \mathbb{Z}[G/C]) \to H^0(G,\mathbb{Z}) \to H^1(G, I) \to 0$$

we find that $H^1(G,I) = \mathbb{Z}/[L:F]\mathbb{Z}$, moreover, the canonical generator of the group $H^1(G,I)$ coincides up to sign with the class of the flasque resolution in the group $\operatorname{Ext}^1_G(\mathbb{Z},S^*) = H^1(G,I)$. Therefore, the period of the generic element in the group $\operatorname{Br}(KL/K)$ for $K = F(T)$ is equal to $[L:F]$.

4.2. **Norm 1 elements.** Let L/F be a finite Galois field extension with Galois group G. Consider the torus $S = R^{(1)}_{L/F}(\mathbb{G}_{m,L})$ of elements of norm 1 in the extension L/F (see [3, Ch. VI, §8]). The co-character lattice S_* is isomorphic to the kernel of the augmentation $I := \operatorname{Ker}\big(\mathbb{Z}[G] \to \mathbb{Z}\big)$. Let P be a permutation torus with the co-character group $\mathbb{Z}[G\times G]$ and $\varphi : P \to S$ the surjective tori homomorphism induced by the G-homomorphisms of lattices $\mathbb{Z}[G\times G] \to I$ taking (g,g') to $g-g'$. Set $T = \operatorname{Ker}(\varphi)$. Thus, we have the resolution (1) of the torus T. The exactness of the sequence of cohomology groups for this resolution yields an isomorphism

$$H^1(K,T) \simeq S(K)/RS(K) = \hat{H}^{-1}\big(G,(KL)^\times\big)$$

for any field extension K/F, where $RS(K)$ is the subgroup of the group $S(K)$ of norm 1 elements in the extension KL/K, generated by elements of the form $g(x)/x$ for all $x \in (KL)^\times$ and $g \in G$. Using the exact sequence for T^* dual to(1), we get the isomorphisms:

$$\hat{H}^0\big(G,\operatorname{End}(T^*)\big) \simeq \hat{H}^0\big(G,\operatorname{End}(\mathbb{Z}[G]/\mathbb{Z})\big) \simeq \hat{H}^0\big(G,\operatorname{End}(\mathbb{Z})\big) = \mathbb{Z}/|G|\mathbb{Z}.$$

For any subgroup $H \subset G$ we have:

$$\hat{H}^0(H,T_*) = \hat{H}^{-1}(H,I) = \hat{H}^{-2}(H,\mathbb{Z}) = H^{ab} := H/[H,H],$$

$$\hat{H}^0(H,T^*) = H^1(H,\mathbb{Z}[G]/\mathbb{Z}) = H^2(H,\mathbb{Z}) = \widehat{H} := \operatorname{Hom}(H,\mathbb{Q}/\mathbb{Z}).$$

The image of the pairing $H^{ab} \otimes \widehat{H} \to \mathbb{Z}/|G|\mathbb{Z}$ coincides with $k\mathbb{Z}/|G|\mathbb{Z}$, where k is the period of the group H^{ab}, i.e., the smallest natural number k such that $(H^{ab})^k = 1$. By Theorem 2.2, the order of the generic element in the group $\hat{H}^{-1}\big(G,(KL)^\times\big)$ for $K = F(S)$ is equal to $|G|/r$, where r is the least common multiple of the periods of groups H^{ab} for all subgroups $H \subset G$. It is not difficult to show that r coincides with the product of the maximal orders of cyclic p-subgroups in G over all prime divisors p of the order of the group G.

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