

CANONICAL p -DIMENSION OF ALGEBRAIC GROUPS

NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

ABSTRACT. We describe a way to compute the p -relative version of the Berhuy-Reichstein canonical dimension for an arbitrary split semisimple algebraic group over an arbitrary field of an arbitrary characteristic (p is any prime integer). The canonical p -dimension is computed for all split simple groups of classical types.

CONTENTS

| | |
|---|----|
| 1. Agreements and preliminaries | 2 |
| 1.1. Varieties | 2 |
| 1.2. Chow groups | 2 |
| 1.3. Places | 3 |
| 1.4. Places and rational morphisms | 3 |
| 2. Canonical dimension of determination functions | 3 |
| 3. Notion of canonical p -dimension | 4 |
| 4. Canonical (p -)dimension of regular complete varieties | 6 |
| 5. Generically p -split varieties | 8 |
| 6. Canonical p -dimension of algebraic groups | 12 |
| 7. Remarks on $\widetilde{\text{CH}}(G/P)$ | 14 |
| 8. Canonical p -dimension of split simple groups of classical types | 16 |
| 8.1. Type A_{n-1} | 16 |
| 8.2. Type B_n | 17 |
| 8.3. Type C_n | 18 |
| 8.4. Type D_n | 19 |
| 8.5. Appendix: Type G_2 | 21 |
| References | 21 |

The notion of the canonical dimension of an algebraic structure was introduced by Berhuy and Reichstein in [1]. The canonical dimension measures the size of generic

Date: December, 2004.

Key words and phrases. Algebraic groups, torsors, splitting fields, projective homogeneous varieties, Chow groups. *2000 Mathematical Subject Classifications:* 14L17; 14C25.

The James D. Wolfensohn Fund and The Ellentuck Fund support is acknowledged by the first author. The first author was partially supported by the European Community's Human Potential Programme under contract HPRN-CT-2002-00287, KTAGS.

The work of the second author has been supported by the NSF grant DMS #0355166.

splitting fields of the structure. The formal definition is given in Section 2. Here we present two basic examples:

- Let X be a scheme over a field F . A field extension L/F is called a *splitting field* of X , if X has a point over L . A splitting field L is called *generic*, if for any splitting field K of X there exists an F -place $L \rightarrow K$. The *canonical dimension* of X is the minimum of the transcendence degree (over F) of all generic splitting fields of X .
- Let G be an algebraic group over F . The *canonical dimension* of G is the maximum of the canonical dimension of all principal homogeneous varieties (G -torsors), defined over field extensions of F .

When dealing with a given algebraic structure, we usually have finitely many “significant” prime integers involved. For example, such primes associated with an algebraic group G are the *torsion prime integers* of G (see Remark 6.7). In order to locate contribution of a prime integer p to the canonical dimension, we define *canonical p -dimension* in a similar fashion.

It turns out that canonical dimension and p -dimension of an arbitrary regular complete variety X is closely related to the algebraic cycles on X (see Corollaries 4.7 and 4.12). We express canonical p -dimension of a generically cellular variety in terms of its Chow group (see Theorem 5.8).

The main result of the paper is Theorem 6.9, giving a recipe to compute canonical p -dimension of an arbitrary split semisimple algebraic group over an arbitrary field (of arbitrary characteristic). The values of the canonical p -dimension are given for all split simple groups of classical type (see §8).

Acknowledgments: Authors thank V. Chernousov, I. Panin, and M. Rost for helpful remarks. The first author thanks the Princeton Institute for Advanced Study: the results were obtained and the paper written during his stay at the Institute; he was also inspired by the lecture course of A. A. Suslin on the Rost norm varieties.

1. AGREEMENTS AND PRELIMINARIES

1.1. Varieties. We refer as *schemes* to separated schemes of finite type over a field (there is no restrictions on the field, its characteristic is arbitrary).

Variety in the paper is an integral scheme.

For a scheme X , the integer $d(X)$ is defined as the g.c.d. of degree of all closed points on X ; for a prime integer p , $d_p(X)$ is the p -primary part of $d(X)$.

1.2. Chow groups. Let X be a scheme over a field F . We write $\mathrm{CH}(X)$ for the integral Chow group of X (see [6]). Fixing a prime p , we write $\mathrm{Ch}(X)$ for the modulo p Chow group:

$$\mathrm{Ch}(X) = \mathrm{CH}(X)/p \cdot \mathrm{CH}(X) .$$

Furthermore, we write $\mathrm{Ch}(\overline{X})$ (resp. $\mathrm{CH}(\overline{X})$) for the colimit of $\mathrm{Ch}(X_L)$ (resp. $\mathrm{CH}(X_L)$) with L running over all field extensions L/F , and we write $\overline{\mathrm{Ch}}(X)$ (resp. $\overline{\mathrm{CH}}(X)$) for the image of the restriction homomorphism $\mathrm{res}: \mathrm{Ch}(X) \rightarrow \mathrm{Ch}(\overline{X})$ (resp. $\mathrm{CH}(X) \rightarrow \mathrm{CH}(\overline{X})$). The group $\overline{\mathrm{Ch}}(X)$ is called the *reduced* Chow group of X ; the group $\overline{\mathrm{CH}}(X)$ is called the

modulo p reduced Chow group of X . Note that

$$\overline{\text{Ch}}(X) = \overline{\text{CH}}(X) / (\overline{\text{CH}}(X) \cap_p \text{CH}(\overline{X}))$$

is not the same as $\overline{\text{CH}}(X) / p \overline{\text{CH}}(X)$.

1.3. Places. Let K be a field. A *valuation ring* R of K is a subring $R \subset K$, satisfying $K = R \cup (R \setminus \{0\})^{-1}$. Any valuation ring is local; $R = K$ is a trivial example of a valuation ring.

Given two fields K and L , a *place* $K \rightarrow L$ is a local ring homomorphism $\pi: R \rightarrow L$ of a valuation ring $R \subset K$ (an embedding of fields is a trivial example of a place).

If K and L are extensions of a field F , an F -*place* (or *place over F*) is a place $K \rightarrow L$ with π defined and identical on F .

Places are composable: if $K \rightarrow L$ is a place, given by a ring homomorphism π , and $L \rightarrow E$ a place to a third field E , given by a homomorphism ρ of a ring $S \subset L$, then the composition is the place $K \rightarrow E$, given the homomorphism $\rho \circ \pi: \pi^{-1}(S) \rightarrow E$, defined on the valuation ring $\pi^{-1}(S)$. In particular, any place $L \rightarrow E$ can be restricted to any subfield $K \subset L$.

In this paper, a place $K \rightarrow L$ is said to be *geometric*, if it can be represented as a composition of places with valuation rings being discrete valuation rings.

1.4. Places and rational morphisms. Let X and Y be varieties over a field F . If X is complete, then for any valuation ring R of $F(X)$ there exists an F -morphism $\text{Spec } R \rightarrow X$ [8, ch. II th. 4.7]; therefore an F -place $F(X) \rightarrow F(Y)$ produces a rational morphism $Y \rightarrow X$.

Vice versa, if there is a rational morphism $Y \rightarrow X$ and X is regular, then there exists a *geometric F -place* $F(X) \rightarrow F(Y)$. Indeed, since X is regular at the image x of the generic point of Y , there exists a system of local parameters around x , which produces a geometric place $F(X) \rightarrow F(x)$; composing with the embedding $F(x) \subset F(Y)$, we get a required place $F(X) \rightarrow F(Y)$.

2. CANONICAL DIMENSION OF DETERMINATION FUNCTIONS

Let F be a field, \mathbf{Fields}_F the category of all field extensions of F . Let $\mathbf{2}^0$ be the category of the subsets of a 1-elemental set 0 . A *determination function* D over F is a continuous functor $\mathbf{Fields}_F \rightarrow \mathbf{2}^0$, where by continuity we mean that D commutes with the filtered colimits. In other words, D is a rule assigning to each $E \in \mathbf{Fields}_F$ a value $D(E) \in \{\emptyset, 0\}$ such that

- if $D(E) = 0$ for some E , then $D(E') = 0$ for any E' admitting an F -embedding $E \rightarrow E'$;
- (continuity property) if $D(E) = 0$ for some field E covered by a (possibly infinite) filtered family of subfields E_i , then $D(E_i) = 0$ for some E_i .

A field $E \in \mathbf{Fields}_F$ is called a *splitting field* of a determination function D , if $D(E) = 0$. A splitting field E of D is called *generic*, if for any splitting field L there exists an F -place $E \rightarrow L$. If D has at least one generic splitting field, *canonical dimension* $\text{cd}(D)$ of D is defined as the minimum of the transcendence degree (over F) of all generic splitting fields of D ; if D does not admit a generic splitting field, we set $\text{cd}(D) = \infty$.

Lemma 2.1. *For a given determination function D , any its splitting field, which is a subfield of a generic splitting field, is also generic. Besides, any splitting field contains a finitely generated splitting field and $\text{cd}(D) = \infty$ only if D does not admit generic splitting.*

Proof. If E is a generic splitting field and E' a splitting field contained in E , then for any splitting field L , restricting a place $E \rightarrow L$ to E' , we get a place $E' \rightarrow L$; therefore E' is also generic.

Any splitting field contains a finitely generated splitting field by the continuity of the determination function.

If D has a generic splitting field, then, taking a finitely generated splitting subfield, we get a finitely generated generic splitting field, showing that $\text{cd}(D)$ is finite. \square

A determination function D over F is *split*, if $D(F) = 0$. In this case, F is a generic splitting field of D and $\text{cd}(D) = 0$.

Our basic example of a determination function is the determination function associated with a scheme X over F :

$$L \mapsto \begin{cases} \emptyset, & \text{if } X(L) = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Canonical dimension $\text{cd}(X)$ of an F -scheme X is defined as the canonical dimension of the associated determination function.

Example 2.2 ([12, th. 4.3]). Let F be a field of characteristic $\neq 2$. Let X be an anisotropic smooth projective quadric over F . Then $\text{cd}(X) = \dim X - \mathfrak{i}_1(X) + 1$, where $\mathfrak{i}_1(X)$ is the first Witt index of X .

Let \mathbf{pSets} be the category of pointed sets and let k be a field. A functor

$$\mathcal{F}: \mathbf{Fields}_k \rightarrow \mathbf{pSets}$$

is called *continuous*, if it commutes with filtered colimits. If \mathcal{F} is a continuous functor, then for any $F \in \mathbf{Fields}_k$ and $\alpha \in \mathcal{F}(F)$, we get a determination function D_α over F by setting

$$D_\alpha(L) = \begin{cases} 0, & \text{if } \alpha_L \text{ is the distinguished point of the set } \mathcal{F}(L); \\ \emptyset, & \text{otherwise.} \end{cases}$$

Berhuy-Reichstein canonical dimension of a continuous functor \mathcal{F} [1, §10] is the supremum of $\text{cd}(D_\alpha)$ for all F and $\alpha \in \mathcal{F}(F)$. If G is an algebraic group over the field k , *canonical dimension of G* is defined as canonical dimension of the (continuous) functor Tors_G , taking a field F to the set of isomorphism classes $\text{Tors}_G(F)$ of G -torsors over F . We note that canonical dimension of an algebraic group G/k is not the same as canonical dimension of the underlying variety of G (which is always 0 because $G(k) \neq \emptyset$).

3. NOTION OF CANONICAL p -DIMENSION

Let us fix an arbitrary prime p and refer to a splitting field E of a determination function D as *p -generic*,¹ if for any splitting field L of D there exists a finite field extension L'/L

¹Our notion of p -generic splitting is based on the notion of *p -generic splitting varieties* of symbols in a modulo p Milnor's K -group of a field, introduced in [16].

of degree prime to p admitting a place $E \rightarrow L'$. Replacing generic splitting fields by the p -generic ones in the definitions of section 2, we get a modified notion of canonical dimension which we call *canonical p -dimension* and denote cd_p .

We refer to a finite field extension as *p -coprime*, if its degree is not divisible by p .

The following two Lemmas are useful when working with cd_p .

Lemma 3.1 (cf. [12, lemma 3.3]). *Let K be an arbitrary field, p a prime, K'/K a p -coprime field extension, and L/K an arbitrary field extension. Then there exists a field L' , containing K' and L , such that the extension L'/L is also p -coprime.*

Proof. We almost repeat the proof of [12, lemma 3.3], where the case of $p = 2$ was treated. We may assume that K' is generated over K by one element; let $f(t) \in F[t]$ be its minimal polynomial. Since the degree of f is coprime with p , there exists an irreducible divisor $g \in L[t]$ of f over L such that $\deg(g)$ is coprime with p as well. We set $L' = L[t]/(g)$. \square

Lemma 3.2. *Let K be a field extension of F of finite transcendence degree over F ; let $K \rightarrow L$ be a geometric F -place and let K' be a p -coprime field extension of K . Then there exists a p -coprime field extension L'/L such that the place $K \rightarrow L$ extends to a place $K' \rightarrow L'$.*

Proof. By Lemma 3.1 it suffices to prove Lemma 3.2 in the case where the place $K \rightarrow L$ is surjective and its valuation ring R is a discrete valuation ring. Also it is clear, that it suffices to consider only two cases: (1) K'/K is purely inseparable and (2) K'/K is separable.

In the first case, the degree $[K' : K]$ is a power of a prime $q \neq p$. We take an arbitrary valuation ring R' of K' , lying over R , i.e., such that $R' \cap K = R$ and the embedding $R \rightarrow R'$ is local (such R' exists in the case of an arbitrary field extension K'/K , [17, ch. VI th. 5']). Let L' be the residue field of R' so that we have a surjective place $K' \rightarrow L'$. We show that L' is also purely inseparable over L (and therefore $[L' : L]$, being a power of the same q , is coprime to p). For this, we take an element $l \in L'$ and show that $l^{q^n} \in L$ for some n : let $k \in R'$ be a preimage of l ; then $k^{q^n} \in K$ for some n and consequently $l^{q^n} \in L$ for the same n .

In the second case we consider all valuation rings R_1, \dots, R_r of K' , lying over R (the number of such valuation rings is finite by [17, ch. VI th. 12 cor. 4]). The residue field of each R_i is a finite extension of L . Moreover, $\sum_{i=1}^r e_i n_i = [K' : K]$ [17, ch. VI th. 20 and p. 63] (the discrete valuation ring assumption and the separability assumption are needed for this equality), where n_i is the degree over L of the residue field of R_i , and e_i is the *reduced ramification index* of R_i over R , [17, def. on pp. 52–53]. It follows that at least one of n_i is not divisible by p . \square

Let us make some first general observations on cd_p . Clearly, a generic splitting field of a determination function is also p -generic; therefore we always have $\text{cd} \geq \text{cd}_p$.

Also it is clear, that cd_p is not interesting, if the determination function in question has a p -coprime splitting field. More precisely, one has a simple

Lemma 3.3. *If a determination function D is split by an algebraic field extension E/F of degree non-divisible by p , then $\text{cd}_p(D) = 0$.*

Proof. Taking a finitely generated splitting subfield of E , we get a p -coprime splitting field. It follows by Lemma 3.1, that F is a p -generic splitting field of D . \square

Example 3.4. The computation of $\text{cd}(X)$ for an anisotropic smooth projective quadric X (over a field of characteristic $\neq 2$), given in [12] (see Example 2.2), shows in fact also that $\text{cd}_2(X) = \text{cd}(X)$.

Example 3.5. Let X be a Severi-Brauer variety. If $d(X) = d_p(X)$ (that is, $d(X)$ is a power of the prime p), then $\text{cd}(X) = \text{cd}_p(X) = d_p(X) - 1$, [1, th. 11.4]. Now if $d(X)$ is not a power of a prime, the value of $\text{cd}(X)$ is not known, while $\text{cd}_p(X)$ is still $d_p(X) - 1$ (see Example 5.10).

Example 3.6. The computation of $\text{cd}(\mathbf{SO}_n)$, given in [11], also shows that $\text{cd}_2(\mathbf{SO}_n) = \text{cd}(\mathbf{SO}_n)$ (see also Example 5.11 as well as (8.2) and (8.4)).

Remark 3.7. Let \mathcal{F} and \mathcal{F}' be continuous functors $\mathbf{Fields}_k \rightarrow \mathbf{pSets}$ with a morphism $f: \mathcal{F} \rightarrow \mathcal{F}'$. If the kernel of f is trivial, then for any $F \in \mathbf{Fields}$ and any $\alpha \in \mathcal{F}(F)$ the determination function of α coincides with the determination function of $f(\alpha)$ (cf. [1, lemma 10.2(a)]); therefore $\text{cd}(\mathcal{F}) \leq \text{cd}(\mathcal{F}')$ (cf. [1, lemma 10.2(b)]) and $\text{cd}_p(\mathcal{F}) \leq \text{cd}_p(\mathcal{F}')$ (for any p). If moreover f is surjective (but may not be necessarily injective), then $\text{cd}(\mathcal{F}) = \text{cd}(\mathcal{F}')$ (cf. [1, lemma 10.2(c)]) and $\text{cd}_p(\mathcal{F}) = \text{cd}_p(\mathcal{F}')$.

4. CANONICAL (p -)DIMENSION OF REGULAR COMPLETE VARIETIES

Lemma 4.1. *The function field of a regular variety X is a generic splitting field of X ; in particular, $\text{cd}(X) \leq \dim X$ for regular X .*

Proof. The function field $F(X)$ is a splitting field of X (even in the non-regular case). To check that it is generic, we take an arbitrary splitting field L of X , find a finitely generated splitting subfield of L , take a model Y/F of it, and consider a rational morphism $Y \rightarrow X$, that exists because X has a point over the field $F(Y)$. Since X is regular, there exists a geometric place $F(X) \rightarrow F(Y)$; composing it with the embedding $F(Y) \subset L$, we get the required place $F(X) \rightarrow L$. \square

Remark 4.2. The place $F(X) \rightarrow L$, constructed in the end of proof of Lemma 4.1, is geometric (as defined in §1.3).

We have the following generalization of Lemma 4.1:

Lemma 4.3. *If Y is a closed subvariety of a regular variety X , admitting a dominant rational morphism $X \rightarrow Y$, then the function field of Y is a generic splitting field of X . In particular, $\text{cd}(X) \leq \dim Y$.*

Proof. Clearly, $F(Y)$ is a splitting field of X . The dominant rational morphism produces an embedding of $F(Y)$ into the field $F(X)$, which by Lemma 4.1 is a generic splitting field of X . It follows by Lemma 2.1 that $F(Y)$ is a generic splitting field too. \square

Lemma 4.4. *Let Y be a scheme over a field F , X a variety over F .*

- (1) *If Y admits a dominant rational morphism $X \rightarrow Y$, then the $F(X)$ -scheme $Y_{F(X)}$ has a closed rational point.*

- (2) *If the $F(X)$ -scheme $Y_{F(X)}$ has a closed rational point, then there exists a closed subvariety $Y' \subset Y$, admitting a dominant rational morphism $X \rightarrow Y'$.*

Proof. Existence of a rational morphism $X \rightarrow Y$ is equivalent to existence of a closed rational point on $Y_{F(X)}$. To prove the second statement, we take as Y' the closure of the image of the rational morphism $X \rightarrow Y$. \square

Proposition 4.5. *Any regular complete variety X has a closed subvariety $Y \subset X$ of dimension $\dim Y = \text{cd}(X)$, admitting a dominant rational morphism $X \rightarrow Y$.*

Proof. Let us take a generic splitting field of X , having the transcendence degree $\text{cd}(X)$ over F , replace it by a finitely generated splitting subfield (Lemma 2.1), and consider a projective model T/F of it. Since X is regular, $F(X)$ is a generic splitting field of X (Lemma 4.1) and there exists a place $F(X) \rightarrow F(T)$. Since the variety X is complete, such a place produces a rational morphism $T \rightarrow X$. Replacing T by the closure in $T \times X$ of the graph of such a rational morphism, we get another model of the same field, but now we have a regular morphism $T \rightarrow X$ instead of the rational one we had before.

Since $F(T)$ is a generic splitting field of X , there exists a place $F(T) \rightarrow F(X)$. By completeness of T , it produces a rational morphism $X \rightarrow T$. Composing it with the regular morphism $T \rightarrow X$, we get a rational morphism $X \rightarrow X$; let Y be the closure of its image. Clearly, $\dim Y \leq \dim T = \text{cd}(X)$. On the other hand, by Lemma 4.3, $\dim Y \geq \text{cd}(X)$. Therefore, $\dim Y = \text{cd}(X)$. \square

Combining Lemma 4.3 and Proposition 4.5, we get

Corollary 4.6. *Canonical dimension of a regular complete variety X is the minimum of dimension of closed subvarieties $Y \subset X$, admitting a dominant rational morphism $X \rightarrow Y$.* \square

Taking into account Lemma 4.4, we get the following variation of Corollary 4.6:

Corollary 4.7. *Canonical dimension of a regular complete variety X is the minimum of dimension of closed subvarieties $Y \subset X$, satisfying $Y(F(X)) \neq \emptyset$.* \square

Now we establish variations of Lemma 4.3, Proposition 4.5, and Corollaries 4.6 and 4.7, related to the canonical p -dimension.

Let Y be a scheme over a field F and let X be a variety over F . We say that Y satisfies condition $(*)$ (with respect to X), if there exists an F -variety X' , a dominant rational morphism $X' \rightarrow Y$, and a dominant rational morphism $X' \rightarrow X$ such that the field extension $F(X')/F(X)$ is p -coprime.

Lemma 4.8. *If Y is a closed subvariety of a regular variety X , satisfying condition $(*)$, then the function field of Y is a p -generic splitting field of X . In particular,*

$$\text{cd}_p(X) \leq \dim Y .$$

Proof. Clearly, $F(Y)$ is a splitting field of X . Let now L/F be any splitting field of X . Then we can find a geometric place $F(X) \rightarrow L$ (see Lemma 4.1 with Remark 4.2). Applying Lemma 3.2 to this place and the field extension $F(X')/F(X)$, we get a place $F(X') \rightarrow L'$ for some p -coprime field extension L'/L . Restricting the latter place to the subfield $F(Y) \subset F(X')$, we get a place $F(Y) \rightarrow L'$; therefore, the splitting field $F(Y)$ is p -generic. \square

Lemma 4.9. *Let Y be a scheme over a field F , X a variety over F .*

- (1) *If Y satisfies condition $(*)$, then $d_p(Y_{F(X)}) = 1$ (see §1.1 for definition of d_p).*
- (2) *If $d_p(Y_{F(X)}) = 1$, then there exists a closed subvariety $Y' \subset Y \subset X$, satisfying condition $(*)$. \square*

Proposition 4.10. *Any regular complete variety X has a closed subvariety $Y \subset X$ of $\dim Y = \text{cd}_p(X)$, satisfying condition $(*)$.*

Proof. Let us take a finitely generated p -generic splitting field of X , having the transcendence degree $\text{cd}_p(X)$ over F , and consider a projective model T/F of it. Since $F(X)$ is a generic splitting field of X , while $F(T)$ is a splitting field, there exists a place $F(X) \rightarrow F(T)$. Since the variety X is complete, such a place produces a rational morphism $T \rightarrow X$. Replacing T by the closure in $T \times X$ of the graph of this morphism, we get another model of the same field, but now we have a regular morphism $T \rightarrow X$ instead of the rational one.

Since $F(T)$ is a p -generic splitting field of X , there exists a place $F(T) \rightarrow L'$, where $L' \supset F(X)$ is a field, p -coprime over $F(X)$. Let X'/F be a model of the field L' . By completeness of T , the place $F(T) \rightarrow F(X')$ produces a rational morphism $X' \rightarrow T$. Composing it with the regular morphism $T \rightarrow X$, we get a rational morphism $X' \rightarrow X$; let Y be the closure of its image. Clearly, $\dim Y \leq \dim T = \text{cd}_p(X)$. On the other hand, by Lemma 4.8, $\dim Y \geq \text{cd}_p(X)$. Therefore, $\dim Y = \text{cd}_p(X)$. \square

Lemma 4.8 and Proposition 4.10 together produce

Corollary 4.11. *Canonical p -dimension of a regular complete variety X is the minimum of dimension of the closed subvarieties $Y \subset X$, satisfying $(*)$. \square*

By Lemma 4.9, the following variation of Corollary 4.11 also holds:

Corollary 4.12. *Canonical p -dimension of a regular complete variety X is the minimum of dimension of the closed subvarieties $Y \subset X$ with $d_p(Y_{F(X)}) = 1$. \square*

5. GENERICALLY p -SPLIT VARIETIES

In this section X stands for a smooth complete absolutely irreducible variety over a field F .

Lemma 5.1. *Degree homomorphism $\text{deg} : \text{Ch}_0(\overline{X}) \rightarrow \mathbb{F}_p$ is an isomorphism if and only if $\dim_{\mathbb{F}_p} \text{Ch}_0(\overline{X}) = 1$.*

Proof. The degree homomorphism is non-zero and therefore surjective. \square

Lemma 5.2. *Assume that $\dim_{\mathbb{F}_p} \text{Ch}_0(\overline{X}) = 1$. Let T be an arbitrary F -scheme, let E_1 and E_2 be field extensions of F , and let $f_1 : \text{Spec } E_1 \rightarrow X$ and $f_2 : \text{Spec } E_2 \rightarrow X$ be*

F -morphisms. Then the diagram

$$\begin{array}{ccc}
 & \overline{\text{Ch}}(T \times X) & \\
 (\text{id}_T \times f_1)^* \swarrow & & \searrow (\text{id}_T \times f_2)^* \\
 \overline{\text{Ch}}(T_{E_1}) & & \overline{\text{Ch}}(T_{E_2}) \\
 \searrow & & \swarrow \\
 & \text{Ch}(\overline{T}) &
 \end{array}$$

is commutative.

Proof. Let E be a field extension of F , containing E_1 and E_2 . Replacing T and X by T_E and X_E , we come to the following situation: $E = E_1 = E_2 = F$ and for some closed rational points $x_1, x_2 \in X$, f_i is the embedding $T = T \times \{x_i\} \hookrightarrow T \times X$. We want to show that $f_1^* = f_2^*: \overline{\text{Ch}}(T \times X) \rightarrow \overline{\text{Ch}}(T)$. Since $pr_* \circ f_{i*} = \text{id}$ for $i = 1, 2$, where pr is the projection $T \times X \rightarrow T$, it suffices to show that

$$f_{1*} \circ f_1^* = f_{2*} \circ f_2^*: \overline{\text{Ch}}(T \times X) \rightarrow \overline{\text{Ch}}(T \times X).$$

The composition $f_{i*} \circ f_i^*$ coincides with the multiplication by $[T \times x_i]$. Since by the assumption on $\dim_{\mathbb{F}_p} \text{Ch}_0(\overline{X})$ and Lemma 5.1, the degree homomorphism $\text{deg}: \text{Ch}(\overline{X}) \rightarrow \mathbb{F}_p$ is an isomorphism, we have $[x_1] = [x_2] \in \overline{\text{Ch}}(X)$, and therefore $[T \times x_1] = [T \times x_2]$ in $\overline{\text{Ch}}(T \times X)$. The required assertion follows. \square

Let $i: Y \hookrightarrow X$ be a closed subvariety of X . The closed embedding

$$(\text{id}_Y \times i \times \text{id}_X): Y \times X \rightarrow Y \times X \times X$$

is regular, and we define a pairing

$$\overline{\text{Ch}}(Y) \otimes \overline{\text{Ch}}(X \times X) \rightarrow \overline{\text{Ch}}(Y \times X),$$

by the formula $\alpha \otimes \beta \mapsto (\text{id}_Y \times i \times \text{id}_X)^*(\alpha \times \beta)$.

Proposition 5.3. *Let Y be a closed subvariety of X . Assume that $\overline{\text{Ch}}(X_{F(X)}) = \text{Ch}(\overline{X})$ and $\dim_{\mathbb{F}_p} \text{Ch}_0(\overline{X}) = 1$. Then the above pairing is surjective.*

Proof. We proceed by induction on $\dim Y$. We have a commutative diagram

$$\begin{array}{ccccccc}
 \bigoplus_{Y'} \overline{\text{Ch}}(Y') \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(Y) \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(X \times X) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \bigoplus_{Y'} \overline{\text{Ch}}(Y' \times X) & \longrightarrow & \overline{\text{Ch}}(Y \times X) & \longrightarrow & \overline{\text{Ch}}(X_{F(Y)}) & \longrightarrow & 0
 \end{array}$$

where Y' runs over closed subvarieties of codimension 1 in Y (the upper row is the obvious exact sequence $\bigoplus_{Y'} \overline{\text{Ch}}(Y') \rightarrow \overline{\text{Ch}}(Y) \rightarrow \mathbb{Z} \rightarrow 0$, tensored by $\overline{\text{Ch}}(X \times X)$ over \mathbb{F}_p). The

rows are exact and the left vertical map is surjective by the induction hypothesis. The right vertical map is surjective because the rhombus

$$\begin{array}{ccc}
 & \overline{\text{Ch}}(X \times X) & \\
 \swarrow & & \searrow \\
 \overline{\text{Ch}}(X_{F(Y)}) & & \overline{\text{Ch}}(X_{F(X)}) \\
 \searrow & & \swarrow \\
 & \text{Ch}(\overline{X}) &
 \end{array}$$

is commutative (as guaranteed by the assumption on $\text{Ch}_0(\overline{X})$ and Lemma 5.2 applied to $T = X$). \square

Corollary 5.4. *Under condition of Proposition 5.3, if the push-forward*

$$(i \times \text{id}_X)_* : \overline{\text{Ch}}(Y \times X) \rightarrow \overline{\text{Ch}}(X \times X)$$

is non-zero, then the push-forward $i_ : \overline{\text{Ch}}(Y) \rightarrow \overline{\text{Ch}}(X)$ is also non-zero and, in particular, $\overline{\text{Ch}}_i(X) \neq 0$ for at least one $i \leq \dim Y$.*

Proof. The square

$$\begin{array}{ccc}
 \overline{\text{Ch}}(Y) \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(Y \times X) \\
 \downarrow i_* \otimes \text{id} & & \downarrow (i \times \text{id})_* \\
 \overline{\text{Ch}}(X) \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(X \times X)
 \end{array}$$

is commutative. \square

Definition 5.5. We say that a (complete smooth absolutely irreducible) variety X over F as p -balanced, if the symmetric bilinear form

$$\text{Ch}(\overline{X}) \times \text{Ch}(\overline{X}) \rightarrow \mathbb{F}_p, \quad (\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$$

is non-degenerate (in the sense that its radical is trivial; note that $\dim_{\mathbb{F}_p} \text{Ch}(\overline{X})$ can be infinite).

A variety X over F is called *cellular*, if there is a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

by closed subschemes such that for every $i = 0, 1, \dots, n-1$ the scheme $X_{i+1} \setminus X_i$ is isomorphic to an affine space over F .

Remark 5.6. Let X be a geometrically cellular variety, that is, X_E is cellular for some field extension E/F . We claim that X is p -balanced (for any p). Indeed, the Chow-motive of the cellular variety X_E decomposes into a finite direct sum of twists of the motive of the point (see, e.g., [10, th. 6.5]). Therefore $\text{CH}(X_E) = \text{CH}(\overline{X})$. Moreover, the mutually inverse isomorphisms of the motive of X_E with the above direct sum are given by certain sequences e_0, \dots, e_n and e'_0, \dots, e'_n of homogeneous elements in $\text{CH}(X_E)$, that are bases

of $\text{CH}(X_E)$ mutually dual with respect to the \mathbb{Z} -bilinear form $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$ (simply because they define mutually inverse isomorphisms of motives).

Note that for any p -balanced X and any integer i , one has $\dim_{\mathbb{F}_p} \text{Ch}^i(\overline{X}) = \dim_{\mathbb{F}_p} \text{Ch}_i(\overline{X})$, if at least one of these dimensions is finite. Since $\dim_{\mathbb{F}_p} \text{Ch}^0(\overline{X})=1$, the above relation, taken with $i = 0$, implies that $\dim_{\mathbb{F}_p} \text{Ch}_0(\overline{X}) = 1$ for a p -balanced X .

Definition 5.7. A p -balanced variety X over F is called *p -split*, if $\overline{\text{Ch}}(X) = \text{Ch}(\overline{X})$.

A cellular variety is p -split.

We say that a variety X has a property *generically*, if X over its own function field has this property. This way we get a notion of *generically p -split* variety. According to above remarks, a *generically cellular* variety is generically p -split.

We are ready to prove the main result of the first half of the paper, interpreting the canonical p -dimension of a generically p -split variety in terms of its modulo p reduced Chow group:

Theorem 5.8. *If X is a generically p -split variety (see Definitions 5.7 and 5.5), then*

$$\text{cd}_p(X) = \min\{i \mid \overline{\text{Ch}}_i(X) \neq 0\} .$$

In particular, the formula holds for a generically cellular X .

Proof. Two inequalities are proved separately.

\leq Let i be an integer such that the group $\overline{\text{Ch}}_i(X)$ is non-zero. Then $[Y] \neq 0$ for a closed i -dimensional subvariety $Y \subset X$. We are going to show that $d_p(Y_{F(X)}) = 1$ for such Y , what is enough for our aim by Corollary 4.12.

Since the variety $X_{F(X)}$ is p -split, there exists a prime cycle $Z \subset X_{F(X)}$ such that $\deg([Y_{F(X)}] \cdot [Z]) \neq 0$. Since the product $[Y_{F(X)}] \cdot [Z]$ can be represented by a cycle on the intersection $Y_{F(X)} \cap Z$ (see [6, §8.1]), the scheme $Y_{F(X)}$ has a closed p -coprime point, meaning that $d_p(Y_{F(X)}) = 1$.

\geq Let now Y be a closed subvariety of X , satisfying condition (*) We will show that $\overline{\text{Ch}}_i(X) \neq 0$ for some $i \leq \dim Y$. The desired inequality will then follow by Proposition 4.10.

Let Z be the closure of the image of the rational morphism $X' \rightarrow Y \times X$ (the variety X' with the rational morphisms $X' \rightarrow Y$ and $X' \rightarrow X$ comes from condition (*)). The cycle $(i \times \text{id}_X)_*([Z]) \in \overline{\text{Ch}}(X^2)$ is non-zero, because for the second projection $pr: X^2 \rightarrow X$, we have

$$pr_*(i \times \text{id}_X)_*[Z] = [F(X') : F(X)] \cdot [X] \neq 0 \in \overline{\text{Ch}}(X) .$$

It follows by Corollary 5.4 that $\overline{\text{Ch}}_i(X) \neq 0$ for some $i \leq \dim Y$. \square

Remark 5.9. If we take Y with $\dim Y = \text{cd}_p(X)$ in the beginning of the (\geq) -part of the proof of Theorem 5.8, then, since we have already proved the (\leq) -part of the theorem, we come to the conclusion that the $(\dim Y)$ -dimensional component of the homomorphism $i_*: \overline{\text{Ch}}(Y) \rightarrow \overline{\text{Ch}}(X)$ is non-zero. Since the $(\dim Y)$ -dimensional component of the image of i_* is generated by $[Y] \in \overline{\text{Ch}}(X)$, we see that in fact the class in $\overline{\text{Ch}}(X)$ of Y itself is non-zero.

Example 5.10. Let X be the Severi-Brauer variety of a central simple F -algebra A . Since by Theorem 5.8, $\mathrm{cd}_p(X) = \mathrm{cd}_p(X_L)$ for any p -coprime field extension L/F , $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$, where Y is the Severi-Brauer variety of a division algebra, Brauer-equivalent to the p -primary part of A . Furthermore, $\overline{\mathrm{Ch}}(Y) = \overline{\mathrm{Ch}}^0(Y)$ by [9, prop. 2.1.1]. Therefore, by Theorem 5.8, $\mathrm{cd}_p(Y) = \dim Y$, so that we get

$$\mathrm{cd}_p(X) = \mathrm{cd}_p(Y) = \dim Y = d_p(X) - 1 .$$

Example 5.11. In this example $p = 2$. Let X/F be the orthogonal grassmannian of n -dimensional totally isotropic subspaces of a $(2n + 1)$ -dimensional quadratic form. If $d_2(X) = 2^n$, then $\overline{\mathrm{Ch}}(X) = \overline{\mathrm{Ch}}^0(X)$ by [11, prop. 1.4] and therefore

$$\mathrm{cd}_2(X) = \dim(X) = n(n + 1)/2 .$$

Without any restriction on $d_2(X)$, canonical 2-dimension of X can be expressed as the sum of all i such that the i -th special Schubert class $e_i \in \mathrm{CH}^i(\overline{X})$ is *non-rational*, i.e., does not lie in $\overline{\mathrm{CH}}(X)$ (cf. [18, conj. 6.6]): indeed, by [18, main th. 5.7], the product of all *rational* e_i , reduced modulo 2, is a non-zero element of $\overline{\mathrm{Ch}}(X)$ of the smallest possible dimension.

6. CANONICAL p -DIMENSION OF ALGEBRAIC GROUPS

If P is an algebraic group over a field F , we write $\mathrm{CH}(BP)$ for the P -equivariant Chow ring $\mathrm{CH}_P(\mathrm{Spec} F)$ of the point $\mathrm{Spec} F$ (see [5]).

Let G be a connected algebraic group over F and let P be a subgroup of G . Consider the homomorphism

$$\varphi_G = \varphi_{G,P} : \mathrm{CH}(BP) = \mathrm{CH}_P(\mathrm{Spec} F) \xrightarrow{q^*} \mathrm{CH}_P(G) = \mathrm{CH}(G/P),$$

where $q : G \rightarrow \mathrm{Spec} F$ is the structure morphism.

Remark 6.1. If G is a subgroup of a group G' , then $\varphi_G = i^* \circ \varphi_{G'}$, where $i : G/P \rightarrow G'/P$ is the morphism, induced by embedding of the groups.

Proposition 6.2. *Let $G = \mathbf{GL}_n$. Then the map φ_G is surjective and the left G -action on G/P induces the trivial action on $\mathrm{CH}(G/P)$.*

Proof. The group G is embedded into the affine space of $\mathrm{End}(F^n)$ as a G -equivariant open subset. The map q^* factors as the composite

$$\mathrm{CH}_P(\mathrm{Spec} F) \rightarrow \mathrm{CH}_P(\mathrm{End}(F^n)) \rightarrow \mathrm{CH}_P(G),$$

where the first pull-back map is an isomorphism by the homotopy invariance property and the second restriction map is surjective by the localization. Hence φ_G is surjective.

For a rational point g of G , let $\lambda_g : G \rightarrow G$ is the morphism of the left multiplication by g . It follows from $q \circ \lambda_g = q$ that $\lambda_g^* \circ q^* = q^*$. Since q^* is surjective, λ_g^* is the identity, i.e., G acts trivially on $\mathrm{CH}(G/P)$. \square

Recall that we write $\mathrm{CH}(\overline{G/P})$ for the colimit of $\mathrm{CH}(G_L/P_L)$ over all field extensions L/F . We define a homomorphism $\overline{\varphi}_G$ as the composite

$$\overline{\varphi}_G : \mathrm{CH}(BP) \xrightarrow{\varphi_G} \mathrm{CH}(G/P) \xrightarrow{\mathrm{res}} \mathrm{CH}(\overline{G/P}) .$$

Let E be a (right) G -torsor over a field extension F' of F . Set $K = F'(E)$. Let $\psi_E: \text{CH}(E/P) \rightarrow \text{CH}(G_K/P_K)$ be pull-back map with respect to the morphism $G_K/P_K \rightarrow E/P$, induced by the G -equivariant morphism $G_K \rightarrow E$, taking the identity of G to the generic point of E . We define a homomorphism $\bar{\psi}_E$ as the composite

$$\bar{\psi}_E: \text{CH}(E/P) \xrightarrow{\psi_E} \text{CH}(G_K/P_K) \xrightarrow{\text{res}_K} \text{CH}(\overline{G/P}).$$

We identify G with a subgroup of $S = \mathbf{GL}_n$ for some n .

Lemma 6.3. *Suppose that there is a G -equivariant morphism $E \rightarrow S$ over F and let $h: E/P \rightarrow S/P$ be the induced morphism. Then $\bar{\varphi}_G = \bar{\psi}_E \circ h^* \circ \varphi_S$.*

Proof. The composition $G_K \rightarrow E_K \rightarrow S_K$ of the morphisms induced by $G_K \rightarrow E$ and $E \rightarrow S$, differs from the inclusion $G_K \hookrightarrow S_K$ by a left multiplication by an element of $S(K)$. By Proposition 6.2, the induced pull-back homomorphisms $\text{CH}(S_K/P_K) \rightarrow \text{CH}(G_K/P_K)$ coincide. Composing with the restriction homomorphism $\text{CH}(S/P) \rightarrow \text{CH}(S_K/P_K)$, we get $\psi_E \circ h^* = \text{res}_{K/F} \circ i^*$, where $i: G/P \rightarrow S/P$ is the morphism, induced by the embedding $G \hookrightarrow S$. We have:

$$\bar{\varphi}_G = \text{res} \circ \varphi_G = \text{res}_K \circ \text{res}_{K/F} \circ i^* \circ \varphi_S = \text{res}_K \circ \psi_E \circ h^* \circ \varphi_S = \bar{\psi}_E \circ h^* \circ \varphi_S$$

(for the second equality, see Remark 6.1). \square

Theorem 6.4. 1. *For any G -torsor E (over any field extension of F) we have*

$$\text{Im}(\bar{\varphi}_G) \subset \text{Im}(\bar{\psi}_E).$$

2. *There exists a G -torsor E (over a field extension of F) such that $\text{Im}(\bar{\varphi}_G) = \text{Im}(\bar{\psi}_E)$.*

Proof. 1. We may assume that E is a G -torsor over F . By the Hilbert theorem 90, the S -torsor $(E \times S)/G$ is trivial (where $(E \times S)/G$ stands for the quotient of $E \times S$ by the action $(e, s) \cdot g = (eg, g^{-1}s)$ of G ; the action of G on this quotient is defined by the formula $(e, s) \cdot g = (e, sg)$, so that the embedding $E = E \times 1 \hookrightarrow E \times S$ induces a G -equivariant morphism $E \rightarrow (E \times S)/G$). In particular, there is a G -equivariant morphism $E \rightarrow S$. By Lemma 6.3, $\text{Im}(\bar{\varphi}_G) \subset \text{Im}(\bar{\psi}_E)$.

2. Let $X = S/G$ and $K = F(X)$. Denote by $E \rightarrow \text{Spec } K$ the generic fiber of the projection $S \rightarrow X$. Clearly, E is a G -torsor over K . Denote by $h: E/P_K \rightarrow S/P$ the morphism induced by the canonical G -equivariant morphism $E \rightarrow S$. Since E/P_K is a localization of S/P , the pull-back homomorphism h^* is surjective. By Proposition 6.2, φ_S is also surjective. It follows from Lemma 6.3 that $\text{Im}(\bar{\varphi}_G) = \text{Im}(\bar{\psi}_E)$. \square

Let G be an algebraic group over a field F and let Tors_G be the functor $\mathbf{Fields}_F \rightarrow \mathbf{pSets}$, taking a field K to the set of isomorphism classes $\text{Tors}_G(K)$ of G -torsors over K . For a G -torsor E/K , the determination function of $E \in \text{Tors}_G(K)$ coincides with the determination function of the K -variety E .

Let $P \subset G$ be a subgroup. We assume that P is a special group, that is, the functor Tors_P is trivial.

Lemma 6.5. *The determination functions of the varieties E and E/P coincide.*

Proof. Suppose E/P has a point over K . We need to show that $E(K) \neq \emptyset$. The fiber of the natural morphism $E \rightarrow E/P$ over the point is a P -torsor. Since P is special, this torsor is trivial, i.e., the fiber has a point over K . \square

Remark 6.6. Let G be a split semisimple algebraic group and let P be a parabolic subgroup. The variety G/P is cellular (see, e.g., [2]), therefore $\mathrm{CH}(\overline{G/P}) = \mathrm{CH}(G/P)$.

Remark 6.7. Suppose further that P is a Borel subgroup of G . The image of the composite

$$\mathrm{CH}(BP) \xrightarrow{\varphi_G} \mathrm{CH}(G/P) \xrightarrow{\mathrm{deg}} \mathbb{Z}$$

is a subgroup $t_G \mathbb{Z}$ with a positive integer t_G known as the *torsion index of G* (see [7]). It follows from Theorem 6.4 and Lemma 6.5 that t_G is the l.c.m. of the numbers $d(E)$ over all G -torsors over all field extension. This statement is known as Grothendieck's theorem [7, th. 2]. The prime divisors of the torsion index t_G are called the *torsion primes of G* .

It follows that the canonical dimension of G (resp. $\mathrm{cd}_p(G)$) is the supremum of the canonical dimension of E/P (resp. $\mathrm{cd}_p(E/P)$) over all G -torsors over all field extensions of F .

Let G be a split semisimple group and let $P \subset G$ be a parabolic subgroup. Suppose P is a special group (for example, P is a Borel subgroup of G). Let E be a G -torsor. Note that the variety E/P is projective. In order to apply Theorem 5.8 to the variety E/P , we need the following

Corollary 6.8. *The variety E/P is generically cellular.*

Proof. By Lemma 6.5, the torsor E is split over the function field $L = F(E/P)$, hence $E_L \simeq G_L$ and therefore $(E/P)_L \simeq (G/P)_L$. The latter variety is cellular. \square

Theorems 5.8 and 6.4 yield

Theorem 6.9. *Let G be a split semisimple group and let $P \subset G$ be a special parabolic subgroup (for example, a Borel subgroup). Denote by $\widetilde{\mathrm{CH}}(G/P)$ the image of the graded ring homomorphism $\varphi_G : \mathrm{CH}(BP) \rightarrow \mathrm{CH}(G/P)$. Then $\mathrm{cd}_p(G)$ for a prime p , is equal to the smallest integer i such that $\widetilde{\mathrm{CH}}_i(G/P)$ is not contained in $p \mathrm{CH}_i(G/P)$. \square*

Remark 6.10. The canonical dimension $\mathrm{cd}_p(G)$ is positive if and only if p is a torsion prime of G (see Remark 6.7). Indeed, by Theorem 6.9, $\mathrm{cd}_p(G) = 0$ if and only if $\widetilde{\mathrm{CH}}_0(G/P)$ is not divisible by p in $\mathrm{CH}_0(G/P)$, where P is a Borel subgroup of G . Since $\mathrm{CH}_0(G/P)$ is an infinite cyclic group generated by the class of a rational point, the latter is equivalent to the condition that p does not divide t_G , i.e., p is not a torsion prime of G .

7. REMARKS ON $\widetilde{\mathrm{CH}}(G/P)$

Let P be an arbitrary subgroup of an algebraic group G . Let $P \rightarrow \mathbf{GL}(V)$ be a finite-dimensional representation. The group P acts (on the right) on the product $G \times V$ by $(g, v) \cdot p = (g \cdot p, p^{-1} \cdot v)$. The factor variety $(G \times V)/P$ is a vector bundle over G/P , we denote it by $\mathrm{Bun}(V)$.

We can view V as a P -equivariant vector bundle over the point $\mathrm{Spec} F$. For any $n \geq 0$, the n -th P -equivariant Chern class $c_n(V)$ is an element of $\mathrm{CH}^n(BP)$ (see [5]).

Let T be a split torus. There is a canonical isomorphism

$$\mathcal{S}(\widehat{T}) \xrightarrow{\sim} \mathrm{CH}(BT),$$

(where \widehat{T} is the character group of T , \mathcal{S} stands for the symmetric algebra) defined by the property that the image of a character χ is the first Chern class $c_1(\chi)$ where χ is considered as a 1-dimensional representation of T [5, 3.2].

Let P be a special parabolic subgroup of a split semisimple algebraic group G . Let T be a maximal split torus contained in P and let W_P be the Weyl group of P . Since P is special, the canonical homomorphism

$$\mathrm{CH}(BP) \rightarrow \mathrm{CH}(BT)^{W_P} = \mathcal{S}(\widehat{T})^{W_P}$$

is an isomorphism [5, prop. 6]. Identifying $\mathrm{CH}(BP)$ with $\mathcal{S}(\widehat{T})^{W_P}$, we get a homomorphism

$$\varphi_G : \mathcal{S}(\widehat{T})^{W_P} \rightarrow \mathrm{CH}(G/P)$$

with the image the subring $\widetilde{\mathrm{CH}}(G/P)$.

Lemma 7.1. *Let $\chi_1, \chi_2, \dots, \chi_m \in \widehat{T}$ be all characters (with multiplicities) of a representation $P \rightarrow \mathbf{GL}(V)$. Let $s_n \in \mathcal{S}^n(\widehat{T})^{W_P}$ be the standard symmetric functions on the characters χ_i . Then $\varphi_G(s_n) = c_n(\mathrm{Bun}(V))$.*

Proof. By naturality of the Chern classes, we have $\varphi_G(c_n(V)) = c_n(\mathrm{Bun}(V))$. On the other hand, $c_n(V)$ is the n -th standard symmetric functions on the characters of V . \square

Remark 7.2. Let G be a split semisimple group over an arbitrary field (of an arbitrary characteristic), $B \subset G$ a Borel subgroup, $T \subset B$ a split maximal torus, W the Weyl group of G . The closures X_w of the cells BwB/B of the cellular variety G/B are indexed by the elements $w \in W$ and called *generalized Schubert varieties* of G/B ; moreover, $\dim X_w = l(w)$, where $l: W \rightarrow \mathbb{Z}_{\geq 0}$ is the length function. Taking the unique maximal length element $w_0 \in W$ and setting $X^w = X_{w_0 w}$, we get a different (preferable for us) indexation of the same varieties, for which $\mathrm{codim} X^w = l(w)$. The group $\mathrm{CH}(G/B)$ is free and the classes $[X^w]$, called *generalized Schubert classes*, form its basis.

The following formula for the product of a 1-codimension Schubert class with an arbitrary Schubert class is given in [4, §4.4 cor.2]:

$$[X^{s_\alpha}] \cdot [X^w] = \sum_{\beta} \langle \beta^\vee, \omega_\alpha \rangle \cdot [X^{w \cdot s_\beta}],$$

where α is a simple root, ω_α its fundamental weight, $s_\alpha \in W$ the reflection with respect to α ; β runs over the set of positive roots such that $l(w \cdot s_\beta) = l(w) + 1$, and β^\vee is the dual to β root. Note that the coefficients of this formula depend only on the root system; in particular, they do not depend on the base field and its characteristic. Moreover, this formula completely determines the multiplication table of the basis $[X^w]$, $w \in W$, because the \mathbb{Q} -algebra $\mathrm{CH}(G/B) \otimes \mathbb{Q}$ is generated by $\mathrm{CH}^1(G/B)$ [4].

Remark 7.3. Let $P = B$ be a Borel subgroup of G . We have $W_B = 1$ and therefore the subring $\widetilde{\mathrm{CH}}(G/B)$ is generated by $\phi_G(\widehat{T})$. In the case of simply connected G , for the weight ω_α of a simple root α , one has the formula

$$\phi_G(\omega_\alpha) = -[X^{s_\alpha}], \quad [4, \S 4 \text{ formula (7)}],$$

which determines ϕ_G also in the non simply connected case. This formula also shows that if the group G is simply connected, then $\phi_G(\widehat{T}) = \mathrm{CH}^1(G/B)$, and therefore $\widetilde{\mathrm{CH}}(G/B)$ is the subring of $\mathrm{CH}(G/B)$, generated by $\mathrm{CH}^1(G/B)$.

Remark 7.4. From Theorem 6.9 and Remark 7.3, we see that:

- (1) if G_1 and G_2 are split semisimple groups, then $\mathrm{cd}_p(G_1 \times G_2) = \mathrm{cd}_p(G_1) + \mathrm{cd}_p(G_2)$;
- (2) if $G' \rightarrow G$ is a central isogeny of split semisimple groups, then $\mathrm{cd}_p(G') \leq \mathrm{cd}_p(G)$.

Remark 7.5. Let us consider pairs (Φ, A) , consisting of a root system Φ and a subgroup of the quotient of the weight lattice of Φ by its root lattice. An isomorphism of pairs $(\Phi, A) \rightarrow (\Phi', A')$ is an isomorphism of the root systems $\Phi \rightarrow \Phi'$ such that the induced isomorphism of the lattice quotients maps A to A' . To any split semisimple algebraic group G one attaches an isomorphism class of above pairs, to which we refer as *extended type* of G . Theorem 6.9 with Remarks 7.2 and 7.3 shows that $\mathrm{cd}_p(G)$ (for any p) depends only on the extended type of G ; in particular, it does not depend on the base field and especially on its characteristic (so that computing $\mathrm{cd}_p(G)$ one may always assume that G is defined over \mathbb{C}).

8. CANONICAL p -DIMENSION OF SPLIT SIMPLE GROUPS OF CLASSICAL TYPES

In this section we compute canonical p -dimension of all split simple groups of classical types. We will need the following

Lemma 8.1. *Let R be a commutative ring, $r \in R$, and let A be the factor ring of the polynomial ring $R[x_1, x_2, \dots, x_n]$ modulo the ideal generated by the polynomial $x_1 + x_2 + \dots + x_n - r$. The symmetric group $W = S_n$ acts on A by permuting the x_i . If R has trivial \mathbb{Z} -torsion, then $A^W = R[s_2, s_3, \dots, s_n]$, where s_i are the standard symmetric functions.*

Proof. Consider the natural W -action on the ring $R[x] = R[x_1, x_2, \dots, x_n]$. We have the exact sequence $0 \rightarrow R[x] \xrightarrow{f} R[x] \rightarrow A \rightarrow 0$, where the first map is the multiplication by $f = x_1 + x_2 + \dots + x_n - r$. Applying the W -invariants we get an exact sequence

$$0 \rightarrow R[x]^W \xrightarrow{f} R[x]^W \rightarrow A^W \rightarrow H^1(W, R[x]).$$

The ring $R[x]^W$ coincides with $R[s] = R[s_1, s_2, \dots, s_n]$. The monomials in the variables x_i form a permutation basis of the R -module $R[x]$. By the Faddeev-Shapiro lemma, the group $H^1(W, R[x])$ is a direct sum of the groups $H^1(W', R) = \mathrm{Hom}(W', R)$ for certain subgroups $W' \subset W$. Since R has trivial \mathbb{Z} -torsion, the latter group is trivial. Therefore,

$$A^W = R[s]/(f) = R[s_2, s_3, \dots, s_n]. \quad \square$$

8.1. Type A_{n-1} . A split simple group of type A_{n-1} is isomorphic to $G = \mathbf{SL}(n)/\mu_l$, where l is a divisor of n . Let $P \subset \mathbf{SL}(n)$ be the stabilizer of the line $U = [1 : 0 : \dots : 0] \in \mathbb{P}^{n-1}$ with respect to the natural action of $\mathbf{SL}(n)$ on \mathbb{P}^{n-1} . The semisimple part of P is $\mathbf{SL}(n-1)$ and it intersects μ_l trivially. Hence the parabolic subgroup $P_l = P/\mu_l$ of G is special. We have $G/P_l = \mathbb{P}^{n-1}$.

The intersection T of the group of diagonal matrices $\mathbf{D}(n)$ of $\mathbf{GL}(n)$ with $\mathbf{SL}(n)$ is a maximal torus of $\mathbf{SL}(n)$. The character group \widehat{T} is identified with the factor group of $\mathbb{Z}^n = \widehat{\mathbf{D}(n)}$ with the standard basis x_1, x_2, \dots, x_n by the subgroup generated by $x_1 + x_2 + \dots + x_n$.

The character group of the maximal torus $T_l = T/\mu_l$ of G is the subgroup of \widehat{T} consisting of all sums $\sum a_i x_i$ such that $\sum a_i$ is divisible by l . Hence, \widehat{T}_l is generated by lx_1 and $x_i - x_1$ for all $i = 2, \dots, n$ with the relation $\sum_{i \geq 2} (x_i - x_1) = -nx_1$.

The Weyl group $W = W_{P_l}$ is the symmetric group S_{n-1} , permuting x_2, \dots, x_n . Applying Lemma 8.1 to the ring $R = \mathbb{Z}[lx_1]$, the element $r = -nx_1$, the variables $x_i - x_1$ and the group W , we get $\mathcal{S}(\widehat{T}_l)^W = \mathbb{Z}[lx_1, s_2, s_3, \dots, s_{n-1}]$, where the s_i are the standard symmetric functions on the $x_i - x_1$, $i \geq 2$.

The group P acts naturally on the space $V = F^n$. The characters of this representation are x_1, x_2, \dots, x_n . The corresponding vector bundle $\text{Bun}(V)$ over $\mathbb{P}^{n-1} = \mathbf{SL}(n)/P = G/P_l$ is the trivial vector bundle of rank n . The line U can be viewed as a 1-dimensional representation of P given by the character x_1 . We have $\text{Bun}(U) = L^\vee$, where L is the canonical line bundle on \mathbb{P}^{n-1} (with the sheaf of sections $\mathcal{O}(1)$). Consider the representation $M = (V/U) \otimes U^\vee$ of the group P with the characters $x_i - x_1$ for all $i = 2, \dots, n$. Note that the group μ_l is contained in the kernel of the representation, hence M is a representation of P_l .

By Lemma 7.1, we have $\varphi_G(lx_1) = lc_1(L^\vee) = -lh$, where $h \in \text{CH}_1(\mathbb{P}^{n-1})$ is the class of a hyperplane, and also $\varphi_G(s_i) = c_i(\text{Bun}(M))$ for all i . Hence the subring $\widetilde{\text{CH}}(\mathbb{P}^{n-1})$ of $\text{CH}(\mathbb{P}^{n-1}) = \mathbb{Z}[h]/(h^n)$ is generated by lh and the Chern classes $c_i(\text{Bun}(M))$. Since

$$\text{Bun}(M) = (\text{Bun}(V)/\text{Bun}(U)) \otimes \text{Bun}(U^\vee) = (\text{Bun}(V)/L^\vee) \otimes L,$$

the class $[\text{Bun}(M)]$ is equal to $n[L] - 1$ in $K_0(\mathbb{P}^{n-1})$. Hence, $c_\bullet(\text{Bun}(M)) = c_\bullet(L)^n = (1+h)^n$. Thus the subring $\widetilde{\text{CH}}(\mathbb{P}^{n-1})$ is generated by lh and $\binom{n}{i}h^i$ for $i = 2, \dots, n-1$.

Let p be a prime integer and let p^k be the largest power of p dividing n . Note that the binomial coefficient $\binom{n}{i}$ is divisible by p unless i is divisible by p^k . The largest value of $i < n$ such that $\binom{n}{i}$ is not divisible by p is $n - p^k$. By Theorem 6.9,

$$\text{cd}_p(\mathbf{SL}(n)/\mu_l) = \begin{cases} p^k - 1, & \text{if } p \text{ divides } l; \\ 1, & \text{otherwise.} \end{cases}$$

Denote by $\text{CSA}_{n,l}(K)$ the set of isomorphism classes of central simple K -algebras of degree n and exponent dividing l . The exact sequence $1 \rightarrow \mu_l \rightarrow \mathbf{SL}(n) \rightarrow \mathbf{SL}(n)/\mu_l \rightarrow 1$ yields a surjective map $\text{Tors}_{\mathbf{SL}(n)/\mu_l}(K) \rightarrow \text{CSA}_{n,l}(K)$ with trivial kernel. By Remark 3.7,

$$\text{cd}_p(\text{CSA}_{n,l}) = \begin{cases} p^k - 1, & \text{if } p \text{ divides } l; \\ 1, & \text{otherwise.} \end{cases}$$

8.2. Type B_n . The only torsion prime is $p = 2$.

Taking a $(2n+1)$ -dimensional vector space, endowed with a completely split quadratic form, let a vector g together with vectors e_i, f_i , $i = 1, 2, \dots, n$ form a basis such that $\{e_i, f_i\}$ are pairwise orthogonal hyperbolic pairs, while g is orthogonal to all e_i, f_i . Let $G = \mathbf{SO}(2n+1)$ be the corresponding special orthogonal group. The inclusion of $\mathbf{D}(n)$ into $\mathbf{SO}(2n+1)$ given by $t(e_i) = t_i e_i$, $t(f_i) = t_i^{-1} f_i$ and $t(g) = g$, where $t = \text{diag}(t_1, \dots, t_n)$, identifies $\mathbf{D}(n)$ with a maximal torus T of $\mathbf{SO}(2n+1)$. In particular, the group \widehat{T} is identified with $\widehat{\mathbb{Z}^n} = \widehat{\mathbf{D}(n)}$. We write x_1, x_2, \dots, x_n for the standard basis of \mathbb{Z}^n .

Let V be the totally isotropic subspace of dimension n generated by all the e_i . Denote by P the stabilizer of V in G , so that $X = G/P$ is the variety of all dimension n

totally isotropic subspaces. The characters of the natural representation $P \rightarrow \mathbf{GL}(V)$ are x_1, x_2, \dots, x_n . The vector bundle $\text{Bun}(V)$ over X is the tautological vector bundle.

The group $W = W_P$ is the symmetric group S_n permuting the x_i . The semisimple part of P is $\mathbf{SL}(n)$, so that P is special.

We have $\mathcal{S}(\widehat{T})^W = \mathbb{Z}[s_1, s_2, \dots, s_n]$, where s_k are the standard symmetric functions on the x_i . By Lemma 7.1, the subring $\widetilde{\text{CH}}(X)$ of $\text{CH}(X)$ is generated by the Chern classes of $\text{Bun}(V)$. These Chern classes are divisible by 2 in $\text{CH}(X)$ [15, ch. III th. 6.11]. Thus, $\widetilde{\text{Ch}}^j(X) = 0$ if $j > 0$. We conclude by Theorem 6.9 that

$$\text{cd}_2 \mathbf{SO}(2n+1) = \frac{n(n+1)}{2}$$

(see also Examples 3.6 and 5.11). The set $\text{Tors}_{\mathbf{SO}(2n+1)}(K)$ is identified with the set of similarity classes $\mathbb{Q}_{2n+1}(K)$ of non-degenerate quadratic forms of dimension $2n+1$ over K . Thus,

$$\text{cd}_2 \mathbb{Q}_{2n+1} = \frac{n(n+1)}{2}.$$

Let $G = \mathbf{Spin}(2n+1)$ be the spinor group. There is an exact sequence

$$1 \rightarrow \mu_2 \rightarrow T' \rightarrow T \rightarrow 1,$$

where T' is a maximal torus of $\mathbf{Spin}(2n+1)$. We have $\widehat{T}' = \widehat{T} + \mathbb{Z}y = \mathbb{Z}^n + \mathbb{Z}y$, where $y = (x_1 + \dots + x_n)/2$. By Lemma 8.1 applied to the ring $R = \mathbb{Z}[y]$, the element $r = 2y$ and the group W , the ring $\mathcal{S}(\widehat{T})^W$ is the polynomial ring $\mathbb{Z}[y, s_2, s_3, \dots, s_n]$.

By Lemma 7.1, $\varphi_G(s_1) = c_1(\text{Bun}(V))$. The latter class coincides with $2e$ where e is a generator of $\text{CH}^1(X)$ [15, ch. III th. 6.11]. Since $s_1 = 2y$ and $\text{CH}^1(X)$ is torsion free, we have $\varphi_G(y) = e$.

As noted above, the images of the s_i in $\text{CH}(X)$ are divisible by 2. Hence the image of $\widetilde{\text{CH}}(X)$ in $\text{Ch}(X) = \text{CH}(X)/2$ is the subring generated by $e \bmod 2$. Let m be the smallest integer such that $2^m > n$. Then $e^{2^m} = 0$ and $e^{2^m-1} \neq 0$ in $\text{Ch}(X)$ [15, ch. III th. 6.11]. Thus,

$$\text{cd}_2 \mathbf{Spin}(2n+1) = \frac{n(n+1)}{2} - 2^m + 1.$$

Let $\overline{\mathbb{Q}}_{2n+1}(K)$ be the subset of $\mathbb{Q}_{2n+1}(K)$ consisting of all classes of forms with trivial even Clifford invariant. The exact sequence $1 \rightarrow \mu_2 \rightarrow \mathbf{Spin}(2n+1) \rightarrow \mathbf{SO}(2n+1) \rightarrow 1$ yields a surjective map $\text{Tors}_{\mathbf{Spin}(2n+1)}(K) \rightarrow \overline{\mathbb{Q}}_{2n+1}(K)$ with trivial kernel. In particular,

$$\text{cd}_2 \overline{\mathbb{Q}}_{2n+1} = \frac{n(n+1)}{2} - 2^m + 1.$$

8.3. Type C_n . The group $\mathbf{Sp}(2n)$ is special, so that $\text{cd}_p \mathbf{Sp}(2n) = 0$ for all p .

Let $G = \mathbf{PGSp}(2n)$ be the projective symplectic group. The number $p = 2$ is the only torsion prime of G . Instead of applying the general method, we proceed as follows.

The set $\text{Tors}_{\mathbf{PGSp}(2n)}(K)$ is identified with the set of isomorphism classes $\text{ASI}_{2n}(K)$ of central simple K -algebras A of degree $2n$ with a symplectic involution [13, §29.22].

The forgetful functor $\text{ASI}_{2n} \rightarrow \text{CSA}_{2n,2}$ has trivial kernel and is surjective. Therefore, by Remark 3.7 and (8.1),

$$\text{cd}_2 \mathbf{PGL}(2n) = \text{cd}_2 \text{ASI}_{2n} = \text{cd}_2 \text{CSA}_{2n,2} = 2^k - 1,$$

where 2^k is the largest power of 2 dividing $2n$.

8.4. Type D_n . Let $\{e_i, f_i\}$, $i = 1, 2, \dots, n$ be pairwise orthogonal hyperbolic pairs of a hyperbolic quadratic form of dimension $2n$. The inclusion of $\mathbf{D}(n)$ into $\mathbf{SO}(2n)$ given by $t(e_i) = t_i e_i$ and $t(f_i) = t_i^{-1} f_i$, where $t = \text{diag}(t_1, \dots, t_n)$, identifies $\mathbf{D}(n)$ with a maximal torus T' of $\mathbf{SO}(2n)$. In particular, the group \widehat{T}' is identified with $\mathbb{Z}^n = \widehat{\mathbf{D}(n)}$. We write x_1, x_2, \dots, x_n for the standard basis of \mathbb{Z}^n .

Let V be the totally isotropic subspace of dimension n generated by all the e_i and let U be the line Fe_1 . Denote by P the stabilizer of the flag $U \subset V$ in $G = \mathbf{Spin}(2n)$ and set $X = G/P$. The semisimple part of P is isomorphic to $\mathbf{SL}(n-1)$ and intersects trivially the center of G . Hence the image of P in any simple group of type D_n (under a central isogeny of G) is a special group.

Let Y be the connected component of the scheme of maximal (n -dimensional) totally isotropic subspaces such that V is a point of Y . The natural morphism $f : X \rightarrow Y$ is the projective bundle associated with the tautological vector bundle E over Y of rank n . In particular,

$$\dim X = \dim Y + (n-1) = \frac{n(n-1)}{2} + (n-1).$$

Note that Y is isomorphic to the projective homogeneous variety of the group $\mathbf{Spin}(2n-1)$ considered in the type B_{n-1} . The Chern classes of E in $\text{Ch}(Y)$ are divisible by 2 (see the type B_n), hence $\text{Ch}(X) = \text{Ch}(Y)[h]/(h^n)$, where $h = c_1(L)$ for the canonical line bundle L over X .

Similar to the case B_n , the character group of the maximal torus T of $\mathbf{Spin}(2n)$ is equal to $\mathbb{Z}^n + \mathbb{Z}y$, where $y = (x_1 + x_2 + \dots + x_n)/2$. Set $x'_i = x_i - x_1$ for $i = 2, \dots, n$, so that $x'_2 + \dots + x'_n = 2y - nx_1$. The symmetric group $W = W_P$ permutes the x'_i and acts trivially on y and x_1 . Applying Lemma 8.1 to the variables x'_i , the ring $R = \mathbb{Z}[y, x_1]$ and the element $r = 2y - nx_1$ we see that $\mathcal{S}(\widehat{T})^W = \mathbb{Z}[y, x_1, s_2, \dots, s_{n-1}]$, where the s_i are the standard symmetric functions on the x'_i .

Consider the homomorphism (reduced modulo 2)

$$\varphi_G : \mathbb{Z}[y, x_1, s_2, \dots, s_{n-1}] \rightarrow \text{Ch}(X) = \text{Ch}(Y)[h]/(h^n).$$

As in the case A_{n-1} , we have $\text{Bun}(U) = L^\vee$ and therefore $\varphi_G(x_1) = c_1(L^\vee) = -h$. Similar to the case B_n , the class $e = \varphi_G(y)$ is a generator of $\text{Ch}^1(Y)$. Recall that $e^{2^m-1} \neq 0$ and $e^{2^m} = 0$ where m is the smallest integer such that $2^m \geq n$.

Similar to the case A_{n-1} , we observe by Lemma 7.1 that the images of the s_i in $\text{Ch}(X)$ are the Chern classes of the vector bundle $(f^*(E)/L^\vee) \otimes L$. The class of this bundle in $K_0(X)$ is equal to $[f^*(E) \otimes L] - 1$. Since the Chern classes of E are divisible by 2, we can replace E by the trivial bundle of rank n and replace $[f^*(E) \otimes L]$ by $n[L]$. As in the case A_{n-1} , we see that $\varphi_G(s_i) = \binom{n}{i} h^i$.

The subring $\widetilde{\text{Ch}}(X) = \text{Im}(\varphi_G)$ is generated by h and e . The largest degree nontrivial monomial in h and e is $h^{n-1}e^{2^m-1}$. By Theorem 6.9,

$$\text{cd}_2 \mathbf{Spin}(2n) = \dim X - (n-1) - (2^m - 1) = \frac{n(n-1)}{2} - 2^m + 1.$$

Let $\overline{Q}_{2n}(K)$ be the subset of the set $Q_{2n}(K)$ of isomorphism classes of non-degenerate quadratic forms of dimension $2n$ consisting of all classes of forms with trivial discriminant and Clifford invariant. The exact sequence $1 \rightarrow \mu_2 \rightarrow \mathbf{Spin}(2n) \rightarrow \mathbf{SO}(2n) \rightarrow 1$ yields a surjective map $\text{Tors}_{\mathbf{Spin}(2n)}(K) \rightarrow \overline{Q}_{2n}(K)$ with trivial kernel. In particular,

$$\text{cd}_2 \overline{Q}_{2n} = \frac{n(n-1)}{2} - 2^m + 1.$$

Now let $G = \mathbf{SO}(2n)$. Recall that the character group \widehat{T}' of the maximal torus T' of G is the subgroup of \widehat{T} generated by all the x_i . Thus we have $\mathcal{S}(\widehat{T}') = \mathbb{Z}[x_1, x_2, \dots, x_n]$ and therefore, $\mathcal{S}(\widehat{T}')^W = \mathbb{Z}[x_1, s_1, \dots, s_{n-1}]$. The subring $\widetilde{\text{Ch}}(X)$ is then generated by h . The largest degree nontrivial monomial in h is h^{n-1} . By Theorem 6.9,

$$\text{cd}_2 \mathbf{SO}(2n) = \dim X - (n-1) = \frac{n(n-1)}{2}.$$

Let $Q'_{2n}(K)$ be the subset of the set $Q_{2n}(K)$ consisting of all classes of forms with trivial discriminant. There is a canonical bijection $\text{Tors}_{\mathbf{SO}(2n)}(K) \xrightarrow{\sim} Q'_{2n}(K)$. Therefore,

$$\text{cd}_2 Q'_{2n} = \frac{n(n-1)}{2}.$$

Let $G = \mathbf{PGO}^+(2n)$ be the projective orthogonal group. Let \overline{T} be the image of the maximal torus T under the canonical isogeny $\mathbf{Spin}(2n) \rightarrow G$. The character group \overline{T} is the subgroup of \widehat{T} generated by all the simple roots. Thus we have $\mathcal{S}(\overline{T}) = \mathbb{Z}[2x_1, x'_2, \dots, x'_n]$ and therefore, $\mathcal{S}(\overline{T})^W = \mathbb{Z}[2x_1, s_1, \dots, s_{n-1}]$. The subring $\widetilde{\text{Ch}}(X)$ is then generated by $\binom{n}{i}h^i$. Let 2^k be the largest power of 2 dividing n . Note that the binomial coefficient $\binom{n}{i}$ is even unless i is divisible by 2^k . The largest value of $i < n$ such that $\binom{n}{i}$ is odd is $n - 2^k$. The largest degree nontrivial monomial in h is $\binom{n}{n-2^k}h^{n-2^k}$. By Theorem 6.9,

$$\text{cd}_2 \mathbf{PGO}^+(2n) = \dim X - (n - 2^k) = \frac{n(n-1)}{2} + 2^k - 1.$$

Let $\text{AQP}_{2n}(K)$ be the set of isomorphism classes of central simple algebras of degree $2n$ with a quadratic pair with trivial discriminant [13, §29.F]. The exact sequence $1 \rightarrow \mathbf{PGO}^+(2n) \rightarrow \mathbf{PGO}(2n) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ yields a surjective map $\text{Tors}_{\mathbf{PGO}^+(2n)}(K) \rightarrow \text{AQP}_{2n}(K)$ with trivial kernel. In particular,

$$\text{cd}_2 \text{AQP}_{2n} = \frac{n(n-1)}{2} + 2^k - 1.$$

Suppose now that n is even. There are two isomorphic semispinor groups. We set $\mathbf{Spin}^\sim(2n) = \mathbf{Spin}(2n)/H$, where H is the intersection of $\text{Ker}(y)$ with the center of $\mathbf{Spin}(2n)$. Let T'' be the image of the maximal torus T under the canonical isogeny

$\mathbf{Spin}(2n) \rightarrow G$. The character group of T'' is the subgroup of \widehat{T} generated by all the simple roots and y . Thus we have $\mathcal{S}(\widehat{T}'') = \mathbb{Z}[y, 2x_1, x_2', \dots, x_n']$.

Applying Lemma 8.1 to the elements x_i' , the ring $R = \mathbb{Z}[y, 2x_1]$, and the element $r = 2y - nx_1$, we see that $\mathcal{S}(\widehat{T}'')^W = \mathbb{Z}[y, 2x_1, s_2, \dots, s_{n-1}]$.

The subring $\widetilde{\text{Ch}}(X)$ is then generated by e and $\binom{n}{i}h^i$. The largest degree nontrivial monomial in h and e is $\binom{n}{n-2^k}h^{n-2^k}e^{2^m-1}$. By Theorem 6.9,

$$\text{cd}_2 \mathbf{Spin}^\sim(2n) = \dim X - (n - 2^k) - (2^m - 1) = \frac{n(n-1)}{2} + 2^k - 2^m.$$

Let $\text{AQP}'_{2n}(K)$ be the set of isomorphism classes of central simple algebras of degree $2n$ with a quadratic pair with trivial discriminant and trivial component of the Clifford algebra. The exact sequence $1 \rightarrow \mu_2 \rightarrow \mathbf{Spin}^\sim(2n) \rightarrow \mathbf{PGO}^+(2n) \rightarrow 1$ yields a surjective map $\text{Tors}_{\mathbf{Spin}^\sim(2n)}(K) \rightarrow \text{AQP}'_{2n}(K)$ with trivial kernel. In particular,

$$\text{cd}_2 \text{AQP}'_{2n} = \frac{n(n-1)}{2} + 2^k - 2^m.$$

8.5. Appendix: Type G_2 . The only torsion prime is 2. Since $\text{Tors}_G \simeq \overline{\mathbb{Q}}_8$ for a split simple G of type G_2 , we have $\text{cd}_2(G) = \text{cd}_2(\overline{\mathbb{Q}}_8) = 3$ (see §8.4).

REFERENCES

- [1] G. Berhuy and Z. Reichstein. *On the notion of canonical dimension for algebraic groups*. Adv. Math., to appear.
- [2] V. Chernousov, S. Gille, A. Merkurjev. *Motivic decomposition of isotropic projective homogeneous varieties*. Duke Math. J., to appear.
- [3] M. Demazure. *Invariants symétriques des groupes de Weyl et torsion*. Invent. Math. **21** (1973), 287–301.
- [4] M. Demazure. *Désingularisation des variétés de Schubert généralisées*. Ann. Sci. Ecole Norm. Sup. (4) **7** (1974), 53–88.
- [5] D. Edidin and W. Graham. *Equivariant intersection theory*. Invent. Math. **131** (1998), no. 3, 595–634.
- [6] W. Fulton. *Intersection Theory*. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
- [7] A. Grothendieck. *Torsion homologique et sections rationnelles*. Exp. **5** in: *Anneaux de Chow et applications*. Séminaire C. Chevalley 1958. Paris (1958).
- [8] R. Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, Heidelberg, Berlin, 1977.
- [9] N. A. Karpenko. *The Grothendieck Chow-motifs of Severi-Brauer varieties*. Algebra i Analiz **7** (1995), no. 4, 196–213 (in Russian). Engl. transl.: St. Petersburg Math. J. **7** (1996), no. 4, 649–661.
- [10] N. A. Karpenko. *Cohomology of relative cellular spaces and isotropic flag varieties*. Algebra i Analiz **12** (2000), no. 1, 3–69 (in Russian). Engl. transl.: St. Petersburg Math. J. **12** (2001), no. 1, 1–50.
- [11] N. A. Karpenko. *Canonical dimension of orthogonal groups*. Transform. Groups, to appear.
- [12] N. A. Karpenko and A. S. Merkurjev. *Essential dimension of quadrics*. Invent. Math. **153** (2003), 361–372.
- [13] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol. *The Book of Involutions*. American Mathematical Society Colloquium Publications, **44**. American Mathematical Society, Providence, RI, 1998, xxii+593 pp.
- [14] B. Köck. *Chow motif and higher Chow theory of G/P* . Manuscripta Math. **70** (1991), 363–372.
- [15] M. Mimura and H. Toda. *Topology of Lie groups. I and II*. Translations of Mathematical Monographs, vol. 91, American Mathematical Society, Providence, RI, 1991, Translated from the 1978 Japanese edition by the authors.

- [16] M. Rost. *Norm varieties and algebraic cobordism*. Li, Ta Tsien (ed.) et al., Proceedings of the International Congress of mathematicians, ICM 2002, Beijing, China, August 20-28, 2002. Vol. II: Invited lectures. Beijing: Higher Education Press. 77–85 (2002).
- [17] P. Samuel and O. Zariski. *Commutative Algebra. Volume II*. Princeton (New Jersey), London, New York, Toronto; D. Van Nostrand Company, Inc, 1958.
- [18] A. Vishik. *On the Chow groups of quadratic grassmannians*. Linear Algebraic Groups and Related Structures (Preprint Server) **168** (Nov. 30, 2004), 23 p.

LABORATOIRE DE MATHÉMATIQUES DE LENS, FACULTÉ DES SCIENCES JEAN PERRIN, UNIVERSITÉ D'ARTOIS, RUE JEAN SOUVRAZ SP 18, 62307 LENS CEDEX, FRANCE

E-mail address: `karpenko@euler.univ-artois.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CA 90095-1555, USA

E-mail address: `merkurev@math.ucla.edu`