ALGEBRAIC ORIENTED COHOMOLOGY THEORIES

ALEXANDER MERKURJEV

Abstract. For every smooth projective variety over a field F we define its fundamental polynomial in $\mathbb{Z}[\mathbf{b}] = \mathbb{Z}[b_1, b_2, \dots]$ and prove that the fundamental polynomials generate the Lazard ring $\mathrm{Laz} \subset \mathbb{Z}[\mathbf{b}]$. Using description of invariant prime ideals in Laz, due to Landweber, we assign to every smooth projective variety X the numbers $n_p(X)$ for every prime integer p. Inequality $n_p(Y) > n_p(X)$ for some prime p is an obstruction for existence of a morphism $Y \to X$ over F.

1. Introduction

Let $\mathbf{Sm}(F)$ be the category of smooth quasi-projective varieties over a field F. M. Levine and F. Morel have defined in [7] an oriented cohomology theory over F as a contravariant functor A^* from the category $\mathbf{Sm}(F)$ to the category of graded commutative rings satisfying certain properties (see Section 2). Examples of the oriented cohomology theories are K^* given by the Grothendieck rings of varieties in $\mathbf{Sm}(F)$ (Example 2.3) and H^* given by the Chow rings (Example 2.2).

It is proved in [7] that if $\operatorname{char} F = 0$ (resolution of singularities is used) then there exists a universal oriented *algebraic cobordism* cohomology theory Ω^*. For every oriented cohomology theory A^* there is unique morphism of cohomology theories $\Omega^* \to A^*$ commuting with the push-forward homomorphisms. For a variety $X \in \mathbf{Sm}(F)$ the group $\Omega^*(X)$ is generated by the classes $[f]$ corresponding to projective morphisms $f : Y \to X$ in $\mathbf{Sm}(F)$. The homomorphism $\Omega^*(X) \to A^*(X)$ takes the class $[f]$ to $f_A(1_Y)$, where f_A is the push-forward homomorphism in A^*. Thus, the image of the morphism $\Omega^*(X) \to A^*(X)$, which we denote by $A^*_c(X)$, can be defined just in terms of the theory A: the group $A^*_c(X)$ is generated by the elements $f_A(1_Y)$ for all projective morphisms $f : Y \to X$ in $\mathbf{Sm}(F)$.

To every oriented cohomology theory A one has associated a commutative formal group law Φ^A over the *coefficient ring* $A^*(\mathrm{pt})$. The formal group law Φ^Ω is the universal one and the coefficient ring $\Omega^*(\mathrm{pt})$ is the Lazard ring.

In the present paper we consider oriented cohomology theories on $\mathbf{Sm}(F)$ for arbitrary fields F and don't refer to the problem of resolution of singularities and existence of the cobordism theory. The idea is to consider "large" oriented cohomology theories A^* such that the natural homomorphism $\Omega^*(X) \to A^*(X)$ is injective at least for $X = \mathrm{pt}$ and work inside A^* instead of Ω^*.

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How to construct "large" cohomology theories? In Section 4 we consider an operation (we call it *tilde operation*), which assigns to every oriented cohomology theory A^* another theory $\widetilde{A}^*$ defined by

$$\widetilde{A}^*(X) = A^*(X) \otimes \mathbb{Z}[\mathbf{b}] = A^*(X)[\mathbf{b}],$$

where $\mathbb{Z}[\mathbf{b}] = \mathbb{Z}[b_1, b_2, \dots]$ is the polynomial ring in infinitely many variables. We define the push-forward homomorphisms in $\widetilde{A}^*$ in such a way that the inverse Todd genus of the natural embedding $A^* \hookrightarrow \widetilde{A}^*$ is the universal one (with the coefficients b_i). We prove that the two theories $\widetilde{H}^*$ and $\widetilde{K}^*$ are large enough so that the coefficient rings $\widetilde{H}^*_c(\mathrm{pt})$ and $\widetilde{K}^*_c(\mathrm{pt})$ are both isomorphic to the Lazard ring. In Sections 6 and 7 we follow closely the method of [11].

For every projective variety $X \in \mathbf{Sm}(F)$ we define the *fundamental polynomial* $\mathbf{F}^H_X \in \mathbb{Z}[\mathbf{b}]$ and prove that for every field F the fundamental polynomials of all projective $X \in \mathbf{Sm}(F)$ generate the same ring - the Lazard ring Laz considered as a subring of $\mathbb{Z}[\mathbf{b}]$. The fundamental polynomials $\mathbf{F}^H_X$ do not change under field extensions (and therefore can computed over an algebraic closure of F); nevertheless, they keep track of an arithmetic information on X. Namely, all the coefficients of $\mathbf{F}^H_X$ are divisible by the greatest common divisor of the degrees $[F(x) : F]$ of all closed points of X. For example, existence of division algebras of a given dimension over an extension of F explains the fact that the fundamental polynomial of the projective space $\mathbb{P}^n_F$ is divisible by $n+1$ in $\mathbb{Z}[\mathbf{b}]$ (Example 3.8), the well known fact in topology (see [10, Ch. VII]).

In Section 9 we prove that the characteristic classes of vector bundles over $X \in \mathbf{Sm}(F)$ take values in the subgroup $A^*_c(X) \subset A^*(X)$. We use this result in Section 10 where we study the Landweber-Novikov operations on Laz. In Section 11 we introduce ideals $I(X) \subset \mathrm{Laz}$ for every projective variety $X \in \mathbf{Sm}(F)$, consisting of the fundamental polynomials of all projective varieties $Y \in \mathbf{Sm}(F)$ such that there is a morphism $Y \to X$ over F. We prove that the ideal $I(X)$ is invariant under the Landweber-Novikov operations and so are all the associated prime ideals. Invariant prime ideas were described by Landweber in [5]. Based on this description one can associate to every projective variety $X \in \mathbf{Sm}(F)$ and every prime integer p an integer $n_p(X) \in \{0, 1, \dots, \infty\}$. Inequality $n_p(Y) > n_p(X)$ for some prime p is an obstruction for existence of a morphism $Y \to X$ over F.

Although the paper is purely algebraic, the most of the constructions are borrowed from topology. The class $[-T_X] \in K_0(X)$ of the tangent bundle T_X over X is a replacement for the stable normal bundle of X. The tilde operation is analogous to the smash product with the Thom spectrum MU. The embedding of the Lazard ring into $\mathbb{Z}[\mathbf{b}]$ is the Hurewicz homomorphism $\pi_*(MU) \to H_*(MU)$. The Landweber-Novikov operations are induced by those on the spectrum MU.

2. Definition of an oriented cohomology theory

Let F be a field, and let $\mathbf{Sm}(F)$ be the category of smooth quasi-projective varieties over F. Let A^* be a functor from $\mathbf{Sm}(F)^{op}$ to the category $\mathbf{GrRings}$ of $\mathbb{Z}$-graded commutative rings. For a morphism $f : Y \to X$ in $\mathbf{Sm}(F)$ the (*pull-back*) ring homomorphism $A^*(f)$ is denoted by f^A.

An *oriented cohomology theory over* F (see [7]) is a functor

$$A^* : \mathbf{Sm}(F)^{op} \to \mathbf{GrRings}$$

together with a graded (*push-forward*) group homomorphism

$$f_A : A^*(Y) \to A^{*+d}(X)$$

for every projective morphism $f : Y \to X$ in $\mathbf{Sm}(F)$ of pure codimension d, satisfying the following:

(i) (Additivity) Let $Z = X \coprod Y$ where $X, Y \in \mathbf{Sm}(F)$, and let $i : X \hookrightarrow Z$, $j : Y \hookrightarrow Z$ be the closed embeddings. Then the homomorphism

$$i_A + j_A : A^*(X) \oplus A^*(Y) \to A^*(Z)$$

is an isomorphism.

(ii) For a pair of projective morphisms $f : Y \to X$ and $g : Z \to Y$, one has $(f \circ g)_A = f_A \circ g_A$.

(iii) Let $E \to X$ be a vector bundle over $X \in \mathbf{Sm}(F)$ of rank r, and let $\mathbb{P}(E) \to X$ be the associated projective bundle. Then $A^*\big(\mathbb{P}(E)\big)$ is a free $A^*(X)$-module with basis $1, \xi, \xi^2, \dots, \xi^{r-1}$, where $\xi = s^A s_A(1_{\mathbb{P}(E)})$, and s is the zero section of the tautological line bundle over $\mathbb{P}(E)$.

(iv) (Transverse property) Let

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array} \tag{2.1}$$

be a transverse Cartesian square in $\mathbf{Sm}(F)$ with f a projective morphism, i.e. the sequence of tangent bundles over Y'

$$0 \to T_{Y'} \xrightarrow{df' \oplus dh} f'^* T_{X'} \oplus h^* T_Y \xrightarrow{dg - df} (fh)^* T_X \to 0$$

is exact. Then $f'_A \circ h^A = g^A \circ f_A$.

(v) (Homotopy invariance) Let $p : V \to X$ be an affine bundle (a torsor for a vector bundle over X). Then $p^A : A^*(X) \to A^*(V)$ is an isomorphism.

(vi) (Projection formula) Let $f : Y \to X$ be a projective morphism in $\mathbf{Sm}(F)$. Then for every $a \in A^*(X)$ and $b \in A^*(Y)$, $f_A\big(b \cdot f^A(a)\big) = f_A(b) \cdot a$.

The ring $A^*(\mathrm{pt})$, where $\mathrm{pt} = \mathrm{Spec}(F)$, is called the *coefficient ring of* A^*. For every $X \in \mathbf{Sm}(F)$, $A^*(X)$ is an algebra over $A^*(\mathrm{pt})$.

Example 2.2. ([7, Ex. 1.2]) The *Chow cohomology theory* H^* assigns to every variety $X \in \mathbf{Sm}(F)$ the Chow ring $H^*(X) = \mathrm{CH}^*(X)$. The push-forward and pull-back homomorphisms are defined in [2]. The coefficient ring $\mathrm{CH}^*(\mathrm{pt})$ is equal to $\mathbb{Z}$.

Example 2.3. ([7, Ex. 1.3]) The *K-theory* assigns to every variety $X \in \mathbf{Sm}(F)$ the Laurent polynomial ring $K^*(X) = K_0(X)[t, t^{-1}]$ graded by $\deg(t) = -1$, i.e. $K^i(X) = K_0(X)t^{-i}$. If $f : Y \to X$ is a projective morphism of pure codimension d, then for every $a \in K^*(Y)$, $f_K(at^{-i}) = f_*(a)t^{-i-d}$, where f_* is the push-forward homomorphism in algebraic K-theory. We have also $K^*(\mathrm{pt}) = \mathbb{Z}[t, t^{-1}]$.

Lemma 2.4. *Let E be an étale F-algebra, $X \in \mathbf{Sm}(F)$ and let $f : X_E = X \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, E \to X$ be the canonical morphism. Then $f_A(1_{X_E}) = [E : F] \cdot 1_X$.*

Proof. We proceed by induction on $[E : F]$. By the additivity property and projection formula we may assume that E is a field and $X = \mathrm{pt}$. There is a smooth curve W over F and a morphism $g : W \to \mathbb{A}^1_F$ such that $g^{-1}(0) = \mathrm{Spec}\, E$ and $g^{-1}(1) = \mathrm{Spec}\, K$, where K is an étale F-algebra that is not a field (see [8, Lemma 4.8]). The diagram

$$\begin{array}{ccc} \mathrm{Spec}\, E & \xrightarrow{j} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{pt} & \xrightarrow{i_0} & \mathbb{A}^1_F \end{array}$$

is transverse. Hence,

$$f_A(1) = f_A j^A(1) = i_0^A g_A(1).$$

Let $p : \mathbb{A}^1_F \to \mathrm{pt}$ be the structure morphism. By homotopy invariance, $i_0 = (p^A)^{-1}$, hence $f_A(1) = (p^A)^{-1} g_A(1)$.

Similarly, for the structure morphism $h : \mathrm{Spec}\, K \to \mathrm{pt}$, we have $h_A(1) = (p^A)^{-1} g_A(1) = f_A(1)$. By the induction hypothesis, $h_A(1_{X_E}) = [E : F] \cdot 1_X$ as K is not a field, therefore, $f_A(1_{X_E}) = [E : F] \cdot 1_X$. □

Let $X \in \mathbf{Sm}(F)$ and let $p : X \to \mathrm{pt}$ be the structure morphism. If X is projective of dimension d, we define the *fundamental class* $[X]^A$ *of* X *in the theory* A as the element

$$[X]^A = p_A(1_X) \in A^{-d}(\mathrm{pt}).$$

For example, $[\mathrm{pt}]^A = 1$, $[X]^H = 0$ if $d > 0$ and $[X]^K = td(X)t^d$, where $td(X) = p_*([\mathcal{O}_X]) \in \mathbb{Z}$ is the Todd number of X [2, Example 15.2.13].

Proposition 2.5. *Let X and Y be projective varieties in $\mathbf{Sm}(F)$. Then* $[X \times Y]^A = [X]^A \cdot [Y]^A$.

Proof. Consider Cartesian transverse square

$$\begin{array}{ccc} X \times Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{r} & \mathrm{pt}\,. \end{array}$$

We have

$$\begin{aligned} [X \times Y]^A &= (sp)_A(1_{X\times Y}) \\ &= (s_A p_A q^A)(1_Y) \quad \big(\text{property (iv)}\big) \\ &= (s_A s^A r_A)(1_Y) \quad (\text{projection formula}) \\ &= s_A(1_X) \cdot r_A(1_Y) \\ &= [X]^A \cdot [Y]^A. \end{aligned}$$

□

For every smooth variety X we consider the graded subgroup $A^*_c(X)$ in $A^*(X)$ generated by the elements $f_A(1_Y)$ for all projective morphisms $f : Y \to X$ in $\mathbf{Sm}(F)$. Clearly, $A^i_c(X) = 0$ if $i > \dim(X)$. For a projective morphism $g : X \to X'$ the push-forward map g_A takes $A^*_c(X)$ to $A^*_c(X')$.

The subgroup $A^*_c(\mathrm{pt}) \subset A^*(\mathrm{pt})$ is generated by the fundamental classes $[X]^A$ for all smooth projective varieties X. Proposition 2.5 shows that $A^*_c(\mathrm{pt})$ is a subring in $A^*(\mathrm{pt})$.

Example 2.6. $H^*_c(\mathrm{pt}) = H^*(\mathrm{pt}) = \mathbb{Z}$, $\quad K^*_c(\mathrm{pt}) = \mathbb{Z}[t]$.

3. Chern classes

Let $p : L \to X$ be a line bundle over $X \in \mathbf{Sm}(F)$. We define the first Chern class of L in an oriented cohomology theory A^* over F by

$$c_1^A(L) = s^A s_A(1_X) \in A^1(X),$$

where $s : X \to L$ is the zero section of p (see [7]). Since $p \circ t = \mathrm{id}_X$ for every section t of p, we have $t^A = (p^A)^{-1}$ (property (v)). Hence,

$$c_1^A(L) = (p^A)^{-1} s_A(1_X).$$

Example 3.1. The first Chern class of a vector bundle $E \to X$ in K-theory is defined by

$$c_1^K(E) = \big(\mathrm{rank}(E) - [E^\vee]\big)t^{-1} \in K_0(X)t^{-1} = K^1(X).$$

Proposition 3.2. *Let $p : L \to X$ be a line bundle over $X \in \mathbf{Sm}(F)$ and let $i : Y \hookrightarrow X$ be the subscheme of zeros of a section t of p. If Y is a smooth divisor in X, then $i_A(1_Y) = c_1^A(L) \in A^1(X)$.*

Proof. The diagram

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow{s} & L \end{array}$$

where s is the zero section of p, is transverse. Hence,

$$i_A(1_Y) = i_A i^A(1_X) = t^A s_A(1_X) = (p^A)^{-1} s_A(1_X) = c_1^A(L).$$

□

The standard method by Grothendieck (see [7]) gives Chern classes $c_i^A(E) \in A^i(X)$ for every vector bundle $p : E \to X$ of rank r. They satisfy the equation

$$\sum_{i=0}^{r} (-1)^i p^A\big(c_i^A(E)\big)\xi^{r-i} = 0 \in A^r\big(\mathbb{P}(E)\big),$$

where ξ is the first Chern class of the tautological line bundle over $\mathbb{P}(E)$.

A *partition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a sequence of integers (possibly empty) $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k > 0$. The *degree* of α is the integer

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

The integer k is called the *length* $l(\alpha)$ of the partition α. Denote by $p(d)$ the number of all partitions of degree d.

We consider the polynomial ring $\mathbb{Z}[b_1, b_2, \dots] = \mathbb{Z}[\mathbf{b}]$ in infinitely many variables $b_1, b_2, \dots$ as a graded ring with $\deg b_i = i$. For every partition α set

$$b_\alpha = b_{\alpha_1} b_{\alpha_2} \dots b_{\alpha_k}.$$

The monomials b_α form a basis of the polynomial ring over $\mathbb{Z}$, and more precisely, the b_α with $|\alpha| = d$ form a basis of the d-graded component $\mathbb{Z}[\mathbf{b}]_d$. Thus, $\mathbb{Z}[\mathbf{b}]_d$ is a free abelian group of rank $p(d)$.

Let $\mathbb{Z}[c_1, c_2, \dots] = \mathbb{Z}[\mathbf{c}]$ be another polynomial ring with similar grading $\deg c_i = i$. The elements of $\mathbb{Z}[\mathbf{c}]$ are called the *characteristic classes* and the c_n - the *Chern classes.*

For every partition α we define the "smallest" symmetric polynomial

$$P_\alpha(x_1, x_2, \dots) = \sum_{(i_1, i_2, \dots, i_k)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k} = Q_\alpha(\sigma_1, \sigma_2, \dots),$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, where the σ_i are the standard symmetric functions, and set

$$c_\alpha = Q_\alpha(c_1, c_2, \dots).$$

For example, $c_n = c_{(1,1,\dots,1)}$ (n units). The characteristic classes c_α with $|\alpha| = d$ form a basis of $\mathbb{Z}[\mathbf{c}]_d$.

Let A^* be an oriented cohomology theory over a field F. For every element (characteristic class) $c \in \mathbb{Z}[\mathbf{c}]$ and every vector bundle E over a variety $X \in$

$\mathbf{Sm}(F)$ there is a well defined class $c^A(E) \in A^*(X)$. In particular, for every partition α there are *generalized Chern classes*

$$c^A_\alpha(E) \in A^{|\alpha|}(X).$$

We define the *characteristic polynomial of E in the theory A^** by the formula

$$\mathbf{P}^A(E) = \sum_\alpha c^A_\alpha(E) b_\alpha \in A^*(X)[\mathbf{b}].$$

Example 3.3. If L is a line bundle, then $\mathbf{P}^A(L) = \sum_{i \geq 0} c^A_1(L)^i b_i$.

Assume that a vector bundle $E \to X$ has a filtration with line factors $L_1, L_2, \ldots, L_r$. Then it follows from definition of the generalized Chern classes that

$$\mathbf{P}^A(E) = \mathbf{P}^A(L_1) \cdot \mathbf{P}^A(L_2) \cdot \ldots \cdot \mathbf{P}^A(L_r).$$

Hence, by the splitting principle, for an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ over X,

$$\mathbf{P}^A(E) = \mathbf{P}^A(E') \cdot \mathbf{P}^A(E''). \tag{3.4}$$

The value of the i-th Chern class $c_i(E)$ is nilpotent for $i > 0$ (see [7]), hence for every $\alpha \neq \emptyset$, the class $c^A_\alpha(E)$ is also nilpotent. The constant term of the polynomial $\mathbf{P}^A(E)$ is equal to 1, so that the polynomial $\mathbf{P}^A(E)$ is invertible in the polynomial ring $A^*(X)[\mathbf{b}]$. Thus, there is well defined group homomorphism

$$\mathbf{P}^A : K_0(X) \to A^*(X)[\mathbf{b}]^\times, \quad [E] \mapsto \mathbf{P}^A(E).$$

For a variety $X \in \mathbf{Sm}(F)$ we define the *characteristic polynomial of X in the theory A^**:

$$\mathbf{P}^A_X = \mathbf{P}^A(T_X)^{-1} = \mathbf{P}^A(-T_X) \in A^*(X)[\mathbf{b}],$$

where T_X is the tangent bundle of X.

Assume that X is projective. Let $p : X \to \mathrm{pt}$ be the structure morphism. The polynomial

$$\mathbf{F}^A_X = p_A\, \mathbf{P}^A_X = \sum_\alpha p_A c_\alpha(-T_X) b_\alpha \in A^*(\mathrm{pt})[\mathbf{b}]$$

is called the *the fundamental polynomial of X in the theory A^**. The coefficients of the polynomial $\mathbf{F}^A_X$, the elements $p_A c_\alpha(-T_X) \in A^*(\mathrm{pt})$, are called the *characteristic numbers of X in the theory A^**. Clearly, the fundamental class $[X]^A$ is the constant term of the fundamental polynomial $\mathbf{F}^A_X$.

Example 3.5. Let $X \in \mathbf{Sm}(F)$ be a variety of dimension d. Then the polynomial $\mathbf{F}^H_X \in \mathbb{Z}[\mathbf{b}]$ is either zero or homogeneous of degree d. The class of the tangent bundle of the projective space $\mathbb{P}^d_F$ is equal to $[L_{can}]^{d+1} - 1$, where L_{can} is the canonical bundle over $\mathbb{P}^d_F$. Since $c^H_1(L_{can})^d$ is the class of a rational point, the polynomial $\mathbf{F}^H_{\mathbb{P}^d}$ is equal to the degree d part of the power series

$(1+b_1+b_2+\dots)^{-d-1}$. In particular, the b_d-coefficient of $\mathbf{F}^H_{\mathbb{P}^d}$ equals $-(d+1)$. For example,

$$\mathbf{F}^H_{\mathbb{P}^1} = -2b_1,$$
$$\mathbf{F}^H_{\mathbb{P}^2} = -3b_2 + 6b_1^2.$$

Example 3.6. Note that for every vector bundle E over a variety $X \in \mathbf{Sm}(F)$,

$$c^K_\alpha(E) \in K_0(X)^{(|\alpha|)}\, t^{-|\alpha|},$$

where $K_0(X)^{(i)}$ is the i-th term of the topological filtration of $K_0(X)$. Therefore, $c^K_\alpha(E) = 0$ if $|\alpha| > d = \dim X$ (cf. Corollary 9.10). Hence, $\mathbf{F}^K_X \in \mathbb{Z}[t, \mathbf{b}]$ is a homogeneous polynomial (with t of degree 1).The t^d-coefficient of $\mathbf{F}^K_X$ is the Todd number of X. For example,

$$\mathbf{F}^K_{\mathbb{P}^1} = t - 2b_1,$$
$$\mathbf{F}^K_{\mathbb{P}^2} = t^2 - 3tb_1 - 3b_2 + 6b_1^2.$$

We will prove (Proposition 6.2) that $\mathbf{F}^K_X\,|_{t=0} = \mathbf{F}^H_X$ for every $X \in \mathbf{Sm}(F)$.

Thus, every projective variety $X \in \mathbf{Sm}(F)$ has the class $\mathbf{F}^H_X$ in $\mathbb{Z}[\mathbf{b}]$. Clearly, the class does not change under field extensions: for every field extension E/F the varieties X and $X_E = X \times_{\operatorname{Spec} F} \operatorname{Spec} E$ have the same class in $\mathbb{Z}[\mathbf{b}]$. Hence, if X and Y are twisted forms of each other (if they are isomorphic over a separable closure of F), then $\mathbf{F}^H_X = \mathbf{F}^H_Y$.

For a variety $X \in \mathbf{Sm}(F)$ denote by n_X the gcd of $\deg(x) = [F(x) : F]$ over all closed points $x \in X$. By the very definition, for a projective variety X, all the coefficients of $\mathbf{F}^H_X$ (the characteristic numbers) are divisible by n_X. We have proved

Proposition 3.7. (1) *For every projective variety $X \in \mathbf{Sm}(F)$, the polynomial $\mathbf{F}^H_X$ is divisible by n_Y for every twisted form Y of X_E over a field extension E/F.*
(2) *Let n be the* gcd *of all the coefficients of $\mathbf{F}^H_X$ for a projective variety $X \in \mathbf{Sm}(F)$. Then X has a zero-cycle of degree n.*

Example 3.8. For every $d \in \mathbb{N}$ and a field F there is a field extension E/F and a division algebra A over E of dimension $(d+1)^2$. Let Y be the Severi-Brauer variety over E corresponding to A (see [4]). The variety Y is a twisted form of the projective space $\mathbb{P}^d_F$. Since $n_Y = d+1$, the Proposition 3.7 explains why the characteristic polynomial of the projective space $\mathbb{P}^d_F$ in $\mathbb{Z}[\mathbf{b}]$ is divisible by $d+1$.

4. Tilde operation

Let A^* be an oriented cohomology theory over F. We associate to A^* a new cohomology theory $\widetilde{A}^*$ defined by

$$\widetilde{A}^*(X) = A^*(X) \otimes \mathbb{Z}[\mathbf{b}] = A^*(X)[\mathbf{b}].$$

The structure of a graded ring on $\widetilde{A}^*(X)$ is given by the one of the graded ring $A^*(X)$ and by assigning degree $-|\alpha|$ to every b_α. In particular, for every $X \in \mathbf{Sm}(F)$, $\mathbf{P}_X^A \in \widetilde{A}^0(X)$ and, if X is projective, $\mathbf{F}_X^A \in \widetilde{A}^{-d}(\mathrm{pt})$, where $d = \dim(X)$.

The pull-back homomorphism $f^{\widetilde{A}} : \widetilde{A}^*(X) \to \widetilde{A}^*(Y)$ associated to a morphism $f : Y \to X$ is equal to $f^A \otimes \mathrm{id}_{\mathbb{Z}[\mathbf{b}]}$. The push-forward map $f_{\widetilde{A}}$ associated to a projective morphism $f : Y \to X$ is defined by

$$f_{\widetilde{A}}(a) = f_A(a \cdot \mathbf{P}_Y^A) \cdot (\mathbf{P}_X^A)^{-1} = f_A\big(a \cdot \mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1}\big). \tag{4.1}$$

If f is a closed embedding, then $[f^*T_X] - [T_Y]$ is equal to the class of the normal bundle $N_Y X$ of Y in X. Hence $\mathbf{P}_Y^A \cdot f^A(\mathbf{P}_X^A)^{-1} = \mathbf{P}^A(N_Y X)$ and

$$f_{\widetilde{A}}(a) = f_A\big(a \cdot \mathbf{P}^A(N_Y X)\big).$$

Lemma 4.2. *Let $p : L \to X$ be a line bundle. Then*

$$c_1^{\widetilde{A}}(L) = c_1^A(L) \cdot \mathbf{P}^A(L) = \sum_{i \geq 0} c_1^A(L)^{i+1} b_i \in A^*(X)[\mathbf{b}].$$

Proof. Let $s : X \to L$ be the zero section. The normal bundle of s is equal to L. Hence,

$$\begin{aligned} c_1^{\widetilde{A}}(L) &= s^{\widetilde{A}} s_{\widetilde{A}}(1_X) \\ &= s^A s_A\big(\mathbf{P}^A(L)\big) \quad \text{(projection formula)} \\ &= s^A\big(s_A(1) \cdot p^A\, \mathbf{P}^A(L)\big) \\ &= c_1^A(L) \cdot \mathbf{P}^A(L). \end{aligned}$$

□

Proposition 4.3. *The functor $\widetilde{A}^*$ is an oriented cohomology theory.*

Proof. We need to check properties (i)-(vi) in the definition of an oriented cohomology theory.

(i) Let $Z = X \coprod Y$ where $X, Y \in \mathbf{Sm}(F)$, and let $i : X \hookrightarrow Z$, $j : Y \hookrightarrow Z$ be the closed embeddings. The normal bundles $N_X Z$ and $N_Y Z$ are trivial, hence $i_{\widetilde{A}} = i_A \otimes \mathrm{id}_{\mathbb{Z}[\mathbf{b}]}$, $j_{\widetilde{A}} = j_A \otimes \mathrm{id}_{\mathbb{Z}[\mathbf{b}]}$ and obviously $i_{\widetilde{A}} + j_{\widetilde{A}}$ is an isomorphism.

(ii) Let $f : Y \to X$ and $g : Z \to Y$ be two projective morphisms. Then for any $a \in A^*(Z)$,

$$\begin{aligned} (f_{\widetilde{A}} \circ g_{\widetilde{A}})(a) &= f_{\widetilde{A}}\big(g_A(a\, \mathbf{P}_Z^A) \cdot (\mathbf{P}_Y^A)^{-1}\big) \\ &= f_A\big(g_A(a\, \mathbf{P}_Z^A) \cdot (\mathbf{P}_Y^A)^{-1} \cdot \mathbf{P}_Y^A)\big) \cdot (\mathbf{P}_X^A)^{-1} \\ &= (f_A \circ g_A)(a\, \mathbf{P}_Z^A) \cdot (\mathbf{P}_X^A)^{-1} \\ &= (f \circ g)_{\widetilde{A}}(a). \end{aligned}$$

(iii) Let E be a vector bundle of rank r over X and L the tautological line bundle over $\mathbb{P}(E)$. By Lemma 4.2,

$$\widetilde{\xi} = c_1^{\widetilde{A}}(L) = \sum_{i\geq 0} \xi^{i+1} b_i,$$

where $\xi = c_1^A(L)$. Since

$$\xi^r - c_1(E)\xi^{r-1} + \cdots + (-1)^r c_r(E) = 0$$

(see [7]) and the classes $c_i(E)$ are nilpotent, the higher powers ξ^s, $s \geq r$, are trivial modulo the nilradical of $A^*(X)$. Therefore, the matrix expressing the powers of $\widetilde{\xi}$ in terms of powers of ξ is upper triangular modulo the nilradical of $\widetilde{A}^*(X)$ and hence is invertible. Thus, the powers $1, \widetilde{\xi}, \dots, \widetilde{\xi}^{r-1}$ form a basis of $\widetilde{A}^*\big(\mathbb{P}(E)\big)$ over $\widetilde{A}^*(X)$.

(iv) Consider a transverse Cartesian square (2.1). We have

$$\mathbf{P}^A_{Y'} \cdot f'^A(\mathbf{P}^A_{X'})^{-1} = h^A\, \mathbf{P}^A_Y \cdot h^A f^A(\mathbf{P}^A_X)^{-1}$$

and therefore, for every $a \in A^*(Y)$,

$$\begin{aligned}
(f'_{\widetilde{A}} \circ h^{\widetilde{A}})(a) &= f'_A\big(h^A(a) \cdot \mathbf{P}^A_{Y'} \cdot f'^A(\mathbf{P}^A_{X'})^{-1}\big) \\
&= f'_A\big(h^A(a) \cdot h^A\, \mathbf{P}^A_Y \cdot h^A f^A(\mathbf{P}^A_X)^{-1}\big) \\
&= f'_A h^A\big(a \cdot \mathbf{P}^A_Y \cdot f^A(\mathbf{P}^A_X)^{-1}\big) \\
&= g^A f_A\big(a \cdot \mathbf{P}^A_Y \cdot f^A(\mathbf{P}^A_X)^{-1}\big) \\
&= g^A\big(f_A(a \cdot \mathbf{P}^A_Y) \cdot (\mathbf{P}^A_X)^{-1}\big) \\
&= (g^{\widetilde{A}} \circ f_{\widetilde{A}})(a).
\end{aligned}$$

(v) Obvious.

(vi) Let $f : Y \to X$ be a projective morphism in $\mathbf{Sm}(F)$, $a \in A^*(X)$ and $b \in A^*(Y)$. We have

$$\begin{aligned}
f_{\widetilde{A}}\big(b \cdot f^{\widetilde{A}}(a)\big) &= f_A(b \cdot f^A(a) \cdot \mathbf{P}^A_Y) \cdot \big(\mathbf{P}^A_X\big)^{-1} \quad \text{(projection formula)} \\
&= f_A(b \cdot \mathbf{P}^A_Y) \cdot a \cdot \big(\mathbf{P}^A_X\big)^{-1} \\
&= f_{\widetilde{A}}\big(b\big) \cdot a
\end{aligned}$$

□

Remark 4.4. The correspondence $E \mapsto \mathbf{P}^A(E)$ is given by the characteristic class $\mathbf{P}^A = \sum_\alpha c_\alpha b_\alpha$ over $\mathbb{Z}[\mathbf{b}]$. In view of [9], $\mathbf{P}^A$ is the inverse Todd genus of the natural embedding of A^* into $\widetilde{A}^*$ and the formula (4.1) is the Riemann-Roch theorem for this embedding.

Note that if $X \in \mathbf{Sm}(F)$ is projective, the fundamental class $[X]^{\widetilde{A}} \in \widetilde{A}(\mathrm{pt}) = A(\mathrm{pt})[\mathbf{b}]$ coincides with the fundamental polynomial $\mathbf{F}^A_X$. In particular, by Proposition 2.5, $\mathbf{F}^A_{X\times Y} = \mathbf{F}^A_X \cdot \mathbf{F}^A_Y$ for all $X, Y \in \mathbf{Sm}(F)$.

5. Formal group law of a theory

Let A^* be an oriented cohomology theory over F. By [7], there is unique commutative formal group law

$$\Phi^A = \sum_{i,j\geq 0} a^A_{ij} x^i y^j = x + y + \sum_{i,j\geq 1} a^A_{ij} x^i y^j,$$

over the coefficient ring $A^*(\mathrm{pt})$ with $a^A_{ij} \in A^{1-i-j}(\mathrm{pt})$, such that for every two line bundles L and L' over a variety $X \in \mathbf{Sm}(F)$,

$$c_1(L \otimes L') = c_1(L) + c_1(L') + \sum_{i,j\geq 1} a^A_{ij} c_1(L)^i c_1(L')^j \in A^1(X).$$

Example 5.1. Since $c^H_1(L \otimes L') = c^H_1(L) + c^H_1(L')$ for two line bundles L and L' [2, Prop. 2.5(e)], $\Phi^H(x, y) = x + y$ is the *additive group law*. It follows from the description of the first Chern class in K-theory (Example 3.1) that $\Phi^K(x, y) = x + y - xyt$ (called the *multiplicative group law*).

In the rest of the section we prove that the coefficients of the group law Φ^A belong to the subring $A^*_c(\mathrm{pt}) \subset A^*(\mathrm{pt})$.

Lemma 5.2. *Let L_{can} be the canonical line bundle over the projective space $\mathbb{P}^n_F$ over F. Then $p_A\big(c^A_1(L_{can})^i\big) = [\mathbb{P}^{n-i}_F]^A$ for every $i \geq 0$.*

Proof. Induction on n. Let $p : \mathbb{P}^n_F \to \mathrm{pt}$ the structure morphism, $j : \mathbb{P}^{n-1}_F \hookrightarrow \mathbb{P}^n_F$ an embedding, $q = p \circ j$, $\xi = c^A_1(L_{can})$. Then $j^A(\xi) = c^A_1(L'_{can})$, where $L'_{can} = j^*(L_{can})$ is the canonical vector bundle over $\mathbb{P}^{n-1}_F$. Proposition 3.2 gives $\xi = j_A(1_{\mathbb{P}^{n-1}})$, hence, by the induction hypothesis,

$$p_A(\xi^i) = p_A\big(j_A(1) \cdot \xi^{i-1}\big) = p_A j_A\big(j^A(\xi)^{i-1}\big) = q_A\big(c^A_1(L'_{can})^{i-1}\big) = [\mathbb{P}^{n-i}_F]^A.$$

□

Lemma 5.3. (cf. [1, Prop. II.10.6]) *Let V be a smooth hypersurface in $\mathbb{P}^n_F \times \mathbb{P}^m_F$ of type $(1, 1)$ for some n and m. Then*

$$[V]^A = \sum_{i=0}^{n} \sum_{j=0}^{m} a^A_{ij}\ [\mathbb{P}^{n-i}_F]^A \cdot [\mathbb{P}^{m-j}_F]^A \in A^{1-n-m}(\mathrm{pt}).$$

Proof. Let $i : V \hookrightarrow \mathbb{P}^n_F \times \mathbb{P}^m_F$ be the embedding of V as a divisor. The corresponding line bundle is the tensor product $q^*_1 L_1 \otimes q^*_2 L_2$, where L_1 and L_2 are canonical line bundles on $\mathbb{P}^n_F$ and $\mathbb{P}^m_F$ respectively, q_1 and q_2 are projections of $\mathbb{P}^n_F \times \mathbb{P}^m_F$ onto $\mathbb{P}^n_F$ and $\mathbb{P}^m_F$. Hence, by Proposition 3.2,

$$i_A(1_V) = c^A_1(q^*_1 L_1 \otimes q^*_2 L_2) = \Phi^A(\xi, \eta) = \sum_{i,j\geq 0} a^A_{ij}\ \xi^i \eta^j,$$

where $\xi = q^A_1 c^A_1(L_1)$, $\eta = q^A_2 c^A_1(L_2)$.

Let $p : \mathbb{P}^n_F \times \mathbb{P}^m_F \to \mathrm{pt}$, $h_1 : \mathbb{P}^n_F \to \mathrm{pt}$ and $h_2 : \mathbb{P}^m_F \to \mathrm{pt}$ be the structure morphisms. Then

$$\begin{aligned} p_A(\xi^i\eta^j) &= h_{1A}q_{1A}\big(q_1^A c_1^A(L_1)^i \cdot q_2^A c_1(L_2)^j\big) \quad \text{(projection formula)} \\ &= h_{1A}\big(c_1^A(L_1)^i \cdot q_{1A}q_2^A c_1(L_2)^j\big) \quad \text{(transverse property)} \\ &= h_{1A}\big(c_1^A(L_1)^i \cdot h_1^A h_{2A} c_1(L_2)^j\big) \quad \text{(projection formula)} \\ &= h_{1A}\big(c_1^A(L_1)^i\big) \cdot h_{2A}\big(c_1^A(L_2)^j\big) \quad \text{(Lemma 5.2)} \\ &= [\mathbb{P}^{n-i}_F]^A \cdot [\mathbb{P}^{m-j}_F]^A, \end{aligned}$$

and therefore,

$$[V]^A = p_A\big(i_A(1_V)\big) = p_A\Big(\sum_{i,j\geq 0} a^A_{ij}\ \xi^i\eta^j\Big) = \sum_{i=0}^{n}\sum_{j=0}^{m} a^A_{ij}\ [\mathbb{P}^{n-i}_F]^A \cdot [\mathbb{P}^{m-j}_F]^A.$$

□

Corollary 5.4. $a^A_{nm} \in A_c^{1-n-m}(\mathrm{pt})$ *for every n and m.*

Proof. Note first that for every n and m there is a smooth hypersurface V in $\mathbb{P}^n_F \times \mathbb{P}^m_F$ of type $(1,1)$. We can take V given by the equation $\sum_{i=0}^k S_iT_i = 0$, where S_i and T_i are the homogeneous coordinates in $\mathbb{P}^n_F$ and $\mathbb{P}^m_F$ respectively and $k = \min(n,m)$. We prove the statement by induction on $n+m$. By Lemma 5.3 and induction hypothesis, $a_{nm} - [V]^A \in A_c^{1-n-m}(\mathrm{pt})$, whence the result. □

Corollary 5.5. *Let $i(t) = \sum_{k\geq 1} b^A_k t^k$ be the additive inverse power series of Φ^A, that is $\Phi_A\big(t, i(t)\big) = 0$. Then $b^A_k \in A_c^{1-k}(\mathrm{pt})$ for every $k \geq 1$.*

6. K-theory versus Chow theory

A relation between K-theory and (rational) Chow-theory is given by the *Chern character*

$$\mathrm{ch}_X : K^*(X) \to H^*(X) \otimes \mathbb{Q}$$

for every $X \in \mathbf{Sm}(F)$. It is the ring homomorphism defined by

$$\mathrm{ch}([E]t^{-k}) = \mathrm{ch}([E]) = \mathrm{rank}(E) + \sum_{i=1}^{\infty} \frac{1}{i!}\ c^H_{(i)}(E)$$

for a vector bundle $E \to X$ [2, Ch. 15]. In particular, the homomorphism

$$\mathrm{ch}_{\mathrm{pt}} : K^*(\mathrm{pt}) = \mathbb{Z}[t,t^{-1}] \to \mathbb{Q} = H^*(\mathrm{pt}) \otimes \mathbb{Q}$$

is the evaluation at $t = 1$.

For a projective morphism $f : X \to Y$, by the classical Riemann-Roch formula [2, Th. 15.2], for every $a \in K^*(X)$:

$$f_H\big(\mathrm{ch}_X(a) \cdot td(T_X)\big) = \mathrm{ch}_Y\big(f_K(a)\big) \cdot td(T_Y),$$

where $td \in \mathbb{Q}[\mathbf{c}]$ is the (rational) Todd characteristic class.

Let $X \in \mathbf{Sm}(F)$ be a projective variety of dimension d and $p : X \to \mathrm{pt}$ the structure morphism. Applying the Riemann-Roch formula for p and $a = c_\alpha^K(-T_X)$, we get

$$(\mathrm{ch}_X\big(c_\alpha^K(-T_X)\big) \cdot td(T_X)\big) = \mathrm{ch}_{\mathrm{pt}}\big(p_K\big(c_\alpha^K(-T_X)\big)\big) = p_K c_\alpha^K(-T_X)|_{t=1}, \tag{6.1}$$

where $\deg = p_H$ is the degree homomorphism.

Proposition 6.2. (1) *For every projective $X \in \mathbf{Sm}(F)$, $\mathbf{F}_X^K\,|_{t=0} = \mathbf{F}_X^H$.*
(2) *The evaluation homomorphism $\mathbb{Z}[t, \mathbf{b}] \to \mathbb{Z}[\mathbf{b}]$ at $t = 0$ induces ring isomorphism between $\widetilde{K}_c(\mathrm{pt})$ and $\widetilde{H}_c(\mathrm{pt})$. In particular, for every d, the degree d components $\widetilde{K}_c(\mathrm{pt})_d$ and $\widetilde{H}_c(\mathrm{pt})_d$ are free abelian groups of rank at most $p(d)$.*

Proof. (1) Let $d = \dim X$. For every partition α,

$$p_K c_\alpha^K(-T_X) \in K^{|\alpha|-d}(X) = K_0(X) t^{d-|\alpha|}.$$

a) If $|\alpha| > d$, then $c_\alpha^K(-T_X)$ and $c_\alpha^H(-T_X)$ are both zero (Examples 3.5 and 3.6).
b) If $|\alpha| < d$, then $p_K c_\alpha^K(-T_X)|_{t=0} = 0 = \deg c_\alpha^H(-T_X)$.
c) Assume that $|\alpha| = d$. We have

$$\mathrm{ch}(c_\alpha^K) \cdot td^{-1} = c_\alpha^H + \text{ characteristic class of degree} > d.$$

Hence, by (6.1),

$$p_K c_\alpha^K(-T_X)|_{t=0} = p_K c_\alpha^K(-T_X) = p_K c_\alpha^K(-T_X)|_{t=1} = \deg c_\alpha^H(-T_X).$$

(2) By the first statement, the evaluation at $t = 0$ takes $\widetilde{K}_c(\mathrm{pt})$ onto $\widetilde{H}_c(\mathrm{pt})$. We need to prove injectivity of the evaluation. Let $X_1, \dots, X_s \in \mathbf{Sm}(F)$ be projective varieties of the same dimension and $m_1, \dots, m_s \in \mathbb{Z}$ such that

$$\sum_{i=1}^{s} m_i\, \mathbf{F}_{X_i}^K\,|_{t=0} = \sum_{i=1}^{s} m_i\, \mathbf{F}_{X_i}^H = 0.$$

Equivalently,

$$\sum_{i=1}^{s} m_i \deg c^H(-T_{X_i}) = 0 \tag{6.3}$$

for every generalized Chern class $c = c_\alpha$. Since c_α generate $\mathbb{Q}[\mathbf{c}]$, the formula (6.3) holds for every characteristic class $c \in \mathbb{Q}[\mathbf{c}]$. Taking $c = \mathrm{ch}(c_\alpha) \cdot td^{-1}$ and applying formula (6.1) for every X_i, we get

$$\sum_{i=1}^{s} m_i p_K^{(i)} c_\alpha^K(-T_{X_i})|_{t=1} = 0$$

for every α, where $p^{(i)} : X_i \to \mathrm{pt}$ is the structure morphism. But the sum $\sum_{i=1}^{s} m_i p_K^{(i)} c_\alpha^K(-T_{X_i})$ is a monomial in t and hence it is zero for every α. It follows that $\sum_{i=1}^{s} m_i\, \mathbf{F}_{X_i}^K = 0$.

The group $\widetilde{H}_c(\mathrm{pt})_d$ is a subgroup of the free group $\mathbb{Z}[\mathbf{b}]_d$ of rank $p(d)$, whence the last statement of the Proposition. $\square$

7. Hypersurfaces $V(n_1, n_2, \dots, n_k)$

Let $\mathbb{P}$ be the product of projective spaces $\mathbb{P}^{n_1}_F \times \mathbb{P}^{n_2}_F \times \cdots \times \mathbb{P}^{n_k}_F$. We write L_i for the pull-back on $\mathbb{P}$ of the canonical vector bundle over $\mathbb{P}^{n_i}_F$ and by L the tensor product of the L_i. Let

$$V = V(n_1, n_2, \dots, n_k) \subset \mathbb{P}$$

be the scheme of zeros of a section of L. Assume that V is smooth. Let $i : V \hookrightarrow \mathbb{P}$ be the embedding. For an oriented cohomology theory A^* over F, by Proposition 3.2,

$$i_{\widetilde{A}}(1_V) = c_1^{\widetilde{A}}(L) = c_1^A(L) \cdot \mathbf{P}^A(L).$$

Denote by $q : \mathbb{P} \to \mathrm{pt}$ the structure morphism. Then

$$\mathbf{F}_V^A = [V]^{\widetilde{A}} = q_{\widetilde{A}} i_{\widetilde{A}}(1_V) = q_{\widetilde{A}}\big(c_1^A(L) \cdot \mathbf{P}^A(L)\big) = q_A\big(c_1^A(L) \cdot \mathbf{P}^A(L) \cdot \mathbf{P}^A_{\mathbb{P}}\big).$$

The class in $K_0(\mathbb{P})$ of the tangent bundle of $\mathbb{P}$ equals $\sum [L_i]^{n_i+1} - k1$. We have then

$$\mathbf{P}^A_{\mathbb{P}} = \prod_{i=1}^{k} \mathbf{P}^A(L_i)^{-n_i-1}.$$

Thus,

$$\mathbf{F}_V^A = q_A\Big(c_1^A(L) \cdot \mathbf{P}^A(L) \cdot \prod_{i=1}^{k} \mathbf{P}^A(L_i)^{-n_i-1}\Big).$$

Set $\xi_i = c_1^A(L_i)$, $\xi = c_1^A(L)$. Therefore,

$$\mathbf{F}_V^A = q_A\left(\big(\sum_{j\geq 0} \xi^{j+1} b_j\big) \cdot \prod_{i=1}^{k} \big(\sum_{j\geq 0} \xi_i^j b_j\big)^{-n_i-1} \right). \tag{7.1}$$

Note that

$$\xi = \Psi^A(\xi_1, \xi_2, \dots, \xi_k),$$

where Ψ^A is the iterated group law of A.

Assume that $A^* = H^*$, so that $\xi = \sum \xi_i$. We would like to compute the α-characteristic number of V for $\alpha = (n-1)$, where $n = \sum n_i$, that is the coefficient of b_{n-1} in $\mathbf{F}_V^H$. Assume that $n_i > 1$ for at least two values of i, so that $n - 1 \geq n_i + 1$ for all i. Since $\xi_i^{n_i+1} = 0$, we can ignore the second multiple in (7.1). Hence

$$\deg c^H_{(n-1)}(-T_V) = q_A(\xi^n) = \frac{n!}{n_1! n_2! \dots n_k!} q_A(\xi_1^{n_1} \dots \xi_k^{n_k}) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

We have proved

Proposition 7.2. (cf. [10, Lemma VII.6.8], [11]) *Let $V = V(n_1, n_2, \dots, n_k)$ be a smooth hypersurface, $n = \sum n_i$. If $n_i > 1$ for at least two values of i, then*

$$\deg c^H_{(n-1)}(V) = \frac{n!}{n_1! n_2! \dots n_k!}.$$

Now consider K-theory $A^* = K^*$. Let p be a prime integer. Assume that for some s, $n_i = p^{s-1}$ for every i and $k = p$, so that $n = p^s$. We have

$$\xi = \Phi(\xi_1, \xi_2, \dots, \xi_p) = v_1 - v_2 t + \dots + (-1)^p v_p t^{p-1},$$

where v_j are the standard symmetric functions on the ξ_i. Note that the r.h.s. of (7.1) is a polynomial in the b_i with the coefficients of the form $q_K\big(P(v)\big)$, where P is a polynomial over $\mathbb{Z}$.

Lemma 7.3. *Let v be a monomial $v_1^{\alpha_1} \dots v_p^{\alpha_p}$. If $\alpha_i > 0$ for some $i = 1, 2, \dots, p-1$, then $q_K(v)$ is divisible by p.*

Proof. Assume $\alpha_i > 0$, so that $v = v_i u$, where $u = v_i^{-1} v$ is a monomial. For every permutation $\sigma \in S_p$ let $\xi_\sigma = \xi_{\sigma(1)} \xi_{\sigma(2)} \dots \xi_{\sigma(i)}$. Then $v = \sum_{\sigma \in S_p/H} \xi_\sigma u$, where $H = S_i \times S_{p-i}$ is the stabilizer of $\xi_1 \dots \xi_i$ and hence $q_K(v)$ is divisible by p since $q_K(\xi_\sigma u) = q_K(\xi_\tau u)$ for every $\sigma, \tau \in S_p$ and the number $\binom{p}{i}$ of terms in the sum is divisible by p. □

Thus, we can delete all monomials in the v_i's containing v_i for $i = 1, 2, \dots, p-1$ and rewrite (7.1) modulo p:

$$\mathbf{F}_V^K \equiv q_K \left(\Big(\sum_{j \geq 0} [(-1)^p v_p t^{p-1}]^{j+1} b_j\Big) \cdot \prod_{i=1}^{p} \Big(\sum_{j \geq 0} \xi_i^j b_j\Big)^{-p^{s-1}-1} \right) \quad (mod\ p). \tag{7.4}$$

Recall that $\mathbf{F}_V^K$ is a homogeneous polynomial in $\mathbb{Z}[t, \mathbf{b}]$ of degree $\dim(V) = p^s - 1$.

Proposition 7.5. (cf. [11, Lemma, p.121]) *Let $V = V(p^{s-1}, p^{s-1}, \dots, p^{s-1})$ (p terms) be a smooth hypersurface, $\alpha = (p^{s-1}-1, p^{s-1}-1, \dots, p^{s-1}-1)$. Then the b_α-coefficient of $\mathbf{F}_V^K$ is not divisible by p. If $\deg \beta \geq p^s - p$ and b_β-coefficient of $\mathbf{F}_V^K$ is not divisible by p, then $\deg \beta = p^s - p$ and β is a refinement of α.*

Proof. A typical monomial of the r.h.s. of (7.4) is of the form

$$t^{(p-1)(j+1)} b_j b_{\alpha^1} \dots b_{\alpha^p} = t^{(p-1)(j+1)} b_\beta$$

for partitions $\alpha^1, \dots, \alpha^p$. Note that since $v_p \xi_i^{p^{s-1}} = 0$ we may assume that $|\alpha^i| \leq p^{s-1} - 1$ for all i. We have $|\beta| = p^s - 1 - (p-1)(j+1) \leq p^s - p$ and equality holds iff $j = 0$. Hence, if $\deg \beta \geq p^s - p$ and the b_β-coefficient of $\mathbf{F}_V^K$ is not divisible by p, then $\deg \beta = p^s - p$ and $j = 0$. Therefore, $|\alpha^i| = p^{s-1} - 1$ for all i and β is a refinement of α.

It follows from (7.4) that modulo p, the b_α-coefficient of $\mathbf{F}_V^K$ is equal to $(-1)^p t^{p-1} q_K(v_p^{p^{s-1}}) = (-1)^p t^{p-1}$ and hence it is not trivial. □

Define the following partial ordering on the set of all partitions. We write $\alpha \leq \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $l(\alpha) \geq l(\beta)$. We consider largest monomials of polynomials in the b_i' s with respect to this ordering.

We will use the following variant of Bertini theorem [3, Th.II.8.18]:

Theorem 7.6. *Let X be a smooth variety over an infinite field, L a very ample line bundle over X. Then there is a section of L with smooth subscheme of zeros.*

Lemma 7.7. (cf. [11, Proposition, p.125]) *Let F be an infinite field. Then for every prime integer p and every integer $d \geq 1$ there exists a projective variety $M^p_d \in \mathbf{Sm}(F)$ of dimension d such that the polynomial $\mathbf{F}^K_{M^p_d}$ has largest monomial b_d modulo p if $d \neq p^s - 1$ for any s or $t^{p-1}(b_{p^{s-1}-1})^p$ if $d = p^s - 1$ for some $s > 0$.*

Proof. Assume first that $d+1$ is not divisible by p and set $M^p_d = \mathbb{P}^d_F$. By Proposition 6.2, the b_d-coefficients of $\mathbf{F}^K_{M^p_d}$ and $\mathbf{F}^H_{M^p_d}$ coincide. By Example 3.5, this coefficient is equal to $-(d+1)$ and it is not divisible by p.

Assume now that $d+1$ is divisible by p but $d+1 \neq p^s$ for any s. We write $d+1 = p^r(pu+v)$ with $r > 0$ and $0 < v < p$. If $u = 0$, $v > 1$, we set $M^p_d = V\big(p^r, p^r(v-1)\big)$. By Proposition 7.2, the b_d-coefficient of $\mathbf{F}^H_{M^p_d}$ is equal to $\binom{p^r v}{p^r}$ and hence it is not divisible by p.

If $u > 0$, let $M^p_d = V(p^r v, p^{r+1}u)$ and again by Proposition 7.2, the b_d-coefficient of $\mathbf{F}^H_{M^p_d}$ is equal to $\binom{p^r(pu+v)}{p^r v}$ and it is not divisible by p.

If $d+1 = p^s$ for some s, let $M^p_d = V(p^{s-1}, p^{s-1}, \dots, p^{s-1})$ (p terms) be a smooth hypersurface. It exists by Theorem 7.6. Then by Proposition 7.5, the b_α-coefficient of $\mathbf{F}^K_{M^p_d}$ is zero modulo p if $|\alpha| \geq p^s - p$ unless $|\alpha| = p^s - p$ and α refines $(p^{s-1}-1, \dots, p^{s-1}-1)$. □

Corollary 7.8. (cf. [11, Corollary, p.126]) *For a partition α let $M^p_\alpha = M^p_{\alpha_1} \times \cdots \times M^p_{\alpha_r}$. Then for every integer $d \geq 0$, the polynomials $\mathbf{F}^K_{M^p_\alpha}$ (mod p) in $(\mathbb{Z}/p\mathbb{Z})[t, \mathbf{b}]$ with $|\alpha| = d$ are linearly independent.*

Proposition 7.9. *Let F be an infinite field. Then the ring $\widetilde{K}_c(\mathrm{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$ (resp. $\widetilde{H}_c(\mathrm{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$) is a polynomial ring over $\mathbb{Z}/p\mathbb{Z}$ in the variables $\mathbf{F}^K_{M^p_d}$ (resp. $\mathbf{F}^H_{M^p_d}$) (mod p), $d \geq 1$.*

Proof. By Corollary 7.8, $\mathbb{Z}/p\mathbb{Z}$-dimension of the image of $\widetilde{K}_c(\mathrm{pt})_d$ in $(\mathbb{Z}/p\mathbb{Z})[t, \mathbf{b}]$ for every prime integer p is at least $p(d)$. On the other hand, the rank of $\widetilde{K}_c(\mathrm{pt})_d$ is at most $p(d)$ by Proposition 6.2. Hence the classes $\mathbf{F}^K_{M^p_\alpha}$ (*mod* p) form a basis of $\widetilde{K}_c(\mathrm{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}/p\mathbb{Z}$. The statements about $\widetilde{H}_c(\mathrm{pt})$ follow from Proposition 6.2. □

Let J be the ideal in $\widetilde{K}_c(\mathrm{pt}) \otimes \mathbb{Z}/p\mathbb{Z}$ generated by $\mathbf{F}^K_X$ for all projective $X \in \mathbf{Sm}(F)$ of positive dimension. If the field F is infinite, by Proposition 7.9, for every projective $X \in \mathbf{Sm}(F)$ of dimension d,

$$\mathbf{F}^K_X \equiv \lambda\, \mathbf{F}^K_{M^p_d} \ (\mathit{mod}\ J^2) \tag{7.10}$$

for a uniquely determined $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Recall that the b_d-coefficients of $\mathbf{F}_X^K$ and $\mathbf{F}_X^H$ coincide and are equal to $\deg c_{(d)}^H(-T_X)$. Note that the b_d-coefficient of every element of J^2 is trivial for every d.

Proposition 7.11. *For a projective variety $X \in \mathbf{Sm}(F)$ of dimension $d = p^s - 1$, the characteristic number $\deg c_{(d)}^H(-T_X)$ is divisible by p.*

Proof. The characteristic numbers do not change under field extensions, hence we may assume that the field F is infinite. The statement follows from (7.10) since by Lemma 7.7, $\deg c_{(d)}^H(-T_{M_d^p})$ is divisible by p. □

Lemma 7.12. *Let S a set of smooth projective varieties over F. Assume that for every prime integer p and every $d \geq 1$ there is $X \in S$ such that $\deg c_{(d)}^H(-T_X)$ is not divisible by p if $d \neq p^s - 1$ for any s and $\deg c_{(d)}^H(-T_X)$ is not divisible by p^2 if $d = p^s - 1$ for some $s > 0$. Then the ring $\widetilde{K}_c(\mathrm{pt})$ is generated by the $\mathbf{F}_X^K$, $X \in S$.*

Proof. We may assume that F is infinite. Let p be a prime integer. For every $d \geq 1$, there is $X \in S$ such that λ in (7.10) is not zero modulo p. Hence the polynomials $\mathbf{F}_X^K$ generate $\widetilde{K}_c(\mathrm{pt})$ modulo p for every p, whence the statement. □

Proposition 7.13. *The subring $\widetilde{K}_c(\mathrm{pt}) \subset \mathbb{Z}[t, \mathbf{b}]$ is generated by the classes of projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$.*

Proof. Let S be the set of all projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$. Let p be a prime integer and $d \geq 1$. If $d \neq p^s - 1$ for any s, the proof of Lemma 7.7 shows the there is $X \in S$ such that $\deg c_{(d)}^H(-T_X)$ is not divisible by p.

Assume that $d = p^s - 1$ for some s. If $s > 1$, then by Proposition 7.2,

$$\deg c_{(p^s-1)}^H(V_{p^{s-1}, p^s - p^{s-1}}) = \binom{p^s}{p^{s-1}}$$

is not divisible by p^2. If $s = 1$, by Example 3.5,

$$\deg c_{(p-1)}^H(\mathbb{P}^{p-1}) = -p.$$

By Lemma 7.12, the set S generates $\widetilde{K}_c(\mathrm{pt})$. □

Propositions 6.2 and 7.13 imply

Corollary 7.14. *The subring $\widetilde{H}_c(\mathrm{pt}) \subset \mathbb{Z}[\mathbf{b}]$ is generated by the fundamental polynomials of projective spaces $\mathbb{P}_F^n$ and smooth hypersurfaces $V(n, m)$.*

Remark 7.15. It follows from Corollary 7.14 and Proposition 6.2 that the groups $\widetilde{H}_c(\mathrm{pt})$ and $\widetilde{K}_c(\mathrm{pt})$ do not depend on the base field. In Theorem 8.2 we will identify $\widetilde{H}_c(\mathrm{pt})$ with the Lazard subring of $\mathbb{Z}[\mathbf{b}]$.

Proposition 7.16. (1) *The degree d component $\widetilde{K}_c(\mathrm{pt})_d$ is a direct summand of $\mathbb{Z}[t,\mathbf{b}]_d$ of rank $p(d)$.*
(2) *The degree d component $\widetilde{H}_c(\mathrm{pt})_d \subset \mathbb{Z}[\mathbf{b}]_d$ is a free subgroup of (maximal) rank $p(d)$.*

Proof. In view of Remark 7.15 we may assume that the field F is infinite. It follows from the proof of Proposition 7.9 that the monomials $\mathbf{F}^K_{M^p_\alpha}$ are linearly independent in $(\mathbb{Z}/p\mathbb{Z})[t,\mathbf{b}]$ and therefore the map $\widetilde{K}_c(\mathrm{pt})_d \otimes \mathbb{Z}/p \to (\mathbb{Z}/p\mathbb{Z})[t,\mathbf{b}]$ is injective for every prime p. Hence, $\widetilde{K}_c(\mathrm{pt})_d$ is a direct summand of $\mathbb{Z}[t,\mathbf{b}]_d$. The statements about $\widetilde{H}_c(\mathrm{pt})$ follow from Proposition 6.2. □

Remark 7.17. The first statement of the Proposition is an algebraic analog of the Hattori-Stong Theorem [11, Theorem, p.129].

8. Lazard ring

Let Laz be the Lazard ring, the coefficient ring of the universal (one-dimensional, commutative) group law [10, Prop. VII.5.3]. For a commutative ring R, the set of R-points

$$\mathrm{Spec}(\mathrm{Laz})(R) = \mathrm{Mor}\big(\mathrm{Spec}(R), \mathrm{Spec}(\mathrm{Laz})\big) = \mathrm{Hom}_{rings}(\mathrm{Laz}, R)$$

is identified with the set of all formal group laws over R.

Let G denote the scheme $\mathrm{Spec}\,\mathbb{Z}[\mathbf{b}]$. For a commutative ring R the set of R-points $G(R) = \mathrm{Hom}_{rings}(\mathbb{Z}[\mathbf{b}], R)$ can be identified with the set of sequences $(r_1, r_2, \dots)$ of elements of R (r_i is the image of the b_i) and therefore with the set of power series

$$t + r_1t^2 + r_2t^3 + \cdots \in R[[t]].$$

The composition of power series makes G a group scheme over $\mathbb{Z}$.

The group $\mathrm{Spec}(\mathbb{Z}[\mathbf{b}])(R)$ acts on $\mathrm{Spec}(\mathrm{Laz})(R)$ by conjugation

$$({}^f\Phi)(x,y) = f\big(\Phi(f^{-1}(x), f^{-1}(y))\big).$$

Thus, the group scheme G acts on the scheme Spec(Laz). We write

$$\log t = t + m_1t^2 + m_2t^3 + \cdots \in \mathbb{Z}[\mathbf{b}][[t]]$$

for the formal inverse of

$$\exp t = t + b_1t^2 + b_2t^3 + \cdots \in \mathbb{Z}[\mathbf{b}][[t]].$$

It is known that $m_d = \mathbf{F}^H_{\mathbb{P}^d}/(d+1)$ [10, VII, Cor. 6.12].

Lemma 8.1. *For every oriented cohomology theory A^*,*

$$\Phi^{\widetilde{A}}(x,y) = \exp \Phi^A(\log x, \log y).$$

Proof. For a line bundle L,

$$c_1^{\widetilde{A}}(L) = \exp c_1^A(L), \quad c_1^A(L) = \log c_1^{\widetilde{A}}(L).$$

Hence for a pair of line bundles L and L',

$$\begin{aligned} c_1^{\widetilde{A}}(L \otimes L') &= \exp c_1^A(L \otimes L') \\ &= \exp \Phi^A\bigl(c_1^A(L), c_1^A(L')\bigr) \\ &= \exp \Phi^A\bigl(\log c_1^{\widetilde{A}}(L), \log c_1^{\widetilde{A}}(L')\bigr). \end{aligned}$$

□

By Lemma 8.1, the group law

$$\Phi = \exp(\log x + \log y) = x + y + \sum_{i,j\geq 1} a_{ij}\ x^i y^j$$

over $\mathbb{Z}[\mathbf{b}]$ coincides with $\Phi^{\widetilde{H}}$. It defines a ring homomorphism $\mathrm{Laz} \to \mathbb{Z}[\mathbf{b}]$ which is, in fact, injective [10, VII,§5]. We will identify Laz with its image in $\mathbb{Z}[\mathbf{b}]$. The ring Laz is generated by the coefficients a_{ij} and Φ is the universal group law over Laz.

Theorem 8.2. *The subgroup of* $\mathbb{Z}[\mathbf{b}]$ *generated by the fundamental polynomials* $\mathbf{F}_X^H$ *for all* $X \in \mathbf{Sm}(F)$*, coincides with* $\mathrm{Laz} \subset \mathbb{Z}[\mathbf{b}]$*.*

Proof. The differential form

$$\mathrm{d}\,log(x) = (1 + 2m_1x + 3m_2x^2 + \dots)\,\mathrm{d}x = (1 + \mathbf{F}^H_{\mathbb{P}^1}\,x + \mathbf{F}^H_{\mathbb{P}^2}\,x^2 + \dots)\,\mathrm{d}x$$

can be computed out of the formal group law by the formula [10, Prop. VII.5.7]

$$\mathrm{d}\,log(x) = \frac{\mathrm{d}x}{\Phi_y(x,0)}.$$

Hence, the classes of the projective spaces $\mathbb{P}^n_F$ can be expressed in terms of the a_{ij}, so that $\mathbf{F}^H_{\mathbb{P}^n} \in \mathrm{Laz}$. By Lemma 5.3, $\mathbf{F}^H_{V(n,m)} \in \mathrm{Laz}$ for every n and m. It follows from Corollary 7.14 that $\widetilde{H}_c(\mathrm{pt}) \subset \mathrm{Laz}$.

Conversely, the inclusion $\mathrm{Laz} \subset \widetilde{H}_c(\mathrm{pt})$ follows from Corollary 5.4 since Laz is generated by the coefficients a_{ij}. □

Thus, every projective variety $X \in \mathbf{Sm}(F)$ has the class $\mathbf{F}_X^H$ in the Lazard ring Laz.

9. Values of characteristic classes

In this section we prove that the characteristic classes in an oriented cohomology theory A^* over F take values in $A^*_c \subset A^*$. For $X \in \mathbf{Sm}(F)$ let $A^*_{cl}(X)$ be the subgroup in $A^*_c(X)$ generated by the elements $i_A(1_Z)$, where $i : Z \hookrightarrow X$ is a smooth closed subvariety. We write $A^*_{norm}(X)$ the subgroup in $A^*_c(X)$ generated by the subgroups $f_A\bigl(A^*_{cl}(X_E)\bigr)$ for all finite separable field extensions E/F, where $f : X_E \to X$ is the canonical morphism. We have

$$A^*_{cl}(X) \subset A^*_{norm}(X) \subset A^*_c(X).$$

Lemma 9.1. *Let* L *be a very ample line bundle over* $X \in \mathbf{Sm}(F)$*. Then* $c_1^A(L) \in A^*_{norm}(X)$*.*

Proof. If F is infinite, by Bertini theorem 7.6, there exists a section of L with the smooth subscheme of zeros Z. Let $i : Z \hookrightarrow X$ be the closed embedding. By Proposition 3.2,

$$c_1^A(L) = i_A(1_Z) \in A^*_{cl}(X) \subset A^*_{norm}(X).$$

Assume that F is a finite field. We use the following trick from [8, p. 41]. For a prime integer p choose an infinite extension E/F such that the degree of every finite subextension of E/F is a power of p. By Bertini theorem 7.6, applied to the variety X_E, there exists a section of L with the smooth scheme of zeros Z. The variety Z is defined over a finite subextension K/F of E/F of degree p^k. Let $f : X_K \to X$ be the natural morphism and $i : Z \hookrightarrow X_K$ the closed embedding. Then by Lemma 2.4, Proposition 3.2 and the projection formula,

$$\begin{aligned} p^k c_1(L) &= [K:F]c_1(L) = f_A(1)c_1(L) \\ &= f_A\big(c_1(f^*L)\big) = f_A i_A(1_Z) \in f_A\big(A^*_{cl}(X)\big) \subset A^*_{norm}(X). \end{aligned}$$

Applying the same argument to another prime integer q, we get

$$q^m c_1(L) \in A^*_{norm}(X)$$

for some m, hence $c_1(L) \in A^*_{norm}(X)$. □

Corollary 9.2. *Let L be a very ample line bundle over $X \in \mathbf{Sm}(F)$. Then $c_1^A(L) \cdot A^*_{norm}(X) \subset A^*_{norm}(X)$.*

Proof. By projection formula it is sufficient to show that $c_1^A(L) \cdot A^*_{cl}(X) \subset A^*_{norm}(X)$. Let $i : Z \hookrightarrow X$ be a smooth closed subvariety. The restriction $L' = L|_Z$ is very ample over Z. By Lemma 9.1,

$$c_1^A(L) \cdot i_A(1_Z) = i_A\big(i^A c_1(L)\big) = i_A\big(c_1^A(L')\big) \in i_A\big(A^*_{norm}(Z)\big) \subset A^*_{norm}(X).$$

□

Proposition 9.3. *Let L be a line bundle over $X \in \mathbf{Sm}(F)$. Then $c_1^A(L) \cdot A^*_c(X) \subset A^*_c(X)$.*

Proof. Let $f : Y \to X$ be a projective morphism with $Y \in \mathbf{Sm}(F)$ and let $L' = f^*(L)$. Choose very ample line bundles L_1 and L_2 over Y such that $L' = L_1 \otimes L_2^{-1}$. By Lemma 9.1,

$$c_1^A(L_1)^i \cdot c_1^A(L_2)^j \in A^*_{norm}(Y) \subset A^*_c(Y)$$

for all i and j. Then by Proposition 5.4 and Corollary 5.5,

$$c_1^A(L') = \Phi^A\big(c_1^A(L_1), ic_1^A(L_2)\big) \in A^*_c(Y).$$

Finally,

$$c_1^A(L) \cdot f_A(1_Y) = f_A\big(f^A c_1^A(L)\big) = f_A\big(c_1^A(L')\big) \in f_A\big(A^*_c(Y)\big) \subset A^*_c(X).$$

□

Let E be a vector bundle of rank $r > 0$ over X. Consider the projection $p : \mathbb{P}(E) \to X$ and set

$$\xi = c_1^A(L_{can}) \in A^1\big(\mathbb{P}(E)\big),$$

where L_{can} is the canonical line bundle over $\mathbb{P}(E)$.

Lemma 9.4. *For every $i \geq 0$,*

$$p_A(\xi^{r-1+i}) = c_i^A(-E) + \sum_{j>i} a_j d_j^A(E) \in A^*(X),$$

for some $a_j \in A_c^*(\mathrm{pt})$ and characteristic classes d_j of degree j.

Proof. By Jouanolou trick and the splitting principle we may assume that X is affine and E is a subbundle of a trivial bundle E' of rank n with the factor bundle E'/E isomorphic to the direct sum of line bundles $L_1, L_2, \ldots$. Let $l : \mathbb{P}(E) \to \mathbb{P}(E')$ be the closed embedding, $q : \mathbb{P}(E') \to X$ the projection, L'_{can} the canonical line bundle over $\mathbb{P}(E')$,$\zeta = c_1^A(L'_{can}) \in A^1\big(\mathbb{P}(E')\big)$. We can consider l as a composition of closed embeddings of codimension 1 corresponding to the line bundles $q^*L_k \otimes L'_{can}$. Hence, by Proposition 3.2,

$$l_A(\xi^{r-1+i}) = l_A(1 \cdot l^A\zeta^{r-1+i}) = \zeta^{r-1+i} \cdot \prod_k c_1^A\big(q^*L_k \otimes L'_{can}\big). \tag{9.5}$$

We can compute $c_1^A\big(q^*L_k \otimes L'_{can}\big)$ using the formal group law Φ^A:

$$c_1^A\big(q^*L_k \otimes L'_{can})\big) = q^A c_1^A(L_k) + \zeta + \sum_{l,m\geq 1} a_{lm} q^A c_1^A(L_k)^l \zeta^m. \tag{9.6}$$

Applying q_A to (9.5) and using (9.6), we get the formula we need, since by Lemma 5.2, $q_A(\zeta^s) = [\mathbb{P}_F^{n-1-s}]^A \in A_c^*(\mathrm{pt})$, $a_{lm} \in A_c^*(\mathrm{pt})$ (Lemma 5.4) and $\sigma_i(L_j) = c_i^A(E'/E) = c_i^A(-E)$. □

Lemma 9.7. *For every $s \geq 0$, $p_A(\xi^s) \cdot A_c^*(X) \subset A_c^*(X)$.*

Proof. Let $f : Y \to X$ be a projective morphism in $\mathbf{Sm}(F)$, $E' = f^*(E)$. Consider the Cartesian transverse square

$$\begin{array}{ccc} \mathbb{P}(E') & \xrightarrow{g} & \mathbb{P}(E) \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X. \end{array}$$

We have

$$p_A(\xi^s)\cdot f_A(1_Y) = p_A\big(\xi^s\cdot p^A f_A(1_Y)\big) = p_A\big(\xi^s\cdot g_A p'^A(1_Y)\big) = p_A\big(\xi^s\cdot g_A(1)\big) \in A_c^*(X)$$

since by Proposition 9.3, $\xi^s \cdot g_A(1) \in A_c^*\big(\mathbb{P}(E)\big)$.

□

Theorem 9.8. *For every vector bundle E over X and every characteristic class c, $c^A(E) \cdot A_c^*(X) \subset A_c^*(X)$.*

Proof. Since $c(E) = c'(-E)$ for some characteristic class c', it is sufficient to prove that $c^A(-E) \cdot A^*_c(X) \subset A^*_c(X)$ for every c. We may assume that

$$c = c_{\alpha_1} c_{\alpha_2} \dots c_{\alpha_k}$$

for some partition $\alpha = (\alpha_1, \dots, \alpha_k)$. Let $a \in A^*_c(X)$. By Lemma 9.4,

$$(9.9)\ p_A(\xi^{r-1-\alpha_1}) \cdot \dots \cdot p_A(\xi^{r-1-\alpha_k}) a = c^A(-E)a + \sum_{j>i} a_j d^A_j(E) a \in A^*(X),$$

for some $a_j \in A^*_c(\mathrm{pt})$ and characteristic classes d_j of degree bigger than the degree of c. By the reverse induction on the degree of c, we have $d^A_j(E)a \in A^*_c(X)$. By Lemma 9.7, the left hand side of (9.9) also belongs to $A^*_c(X)$. Hence, $c^A(-E)a \in A^*_c(X)$. □

Corollary 9.10. *For every smooth X and every vector bundle E over X, the classes $c^A_\alpha(E)$ belong to $A^*_c(X)$ for every partition α. Moreover, $c^A_\alpha(E) = 0$ if $|\alpha| > \dim(X)$. In particular, if X is projective, the fundamental polynomial $\mathbf{F}^A_X$ is of degree at most* $\dim(X)$.

Proof. The group $A^i_c(X)$ is trivial if $i > \dim(X)$. □

10. Landweber-Novikov operations

Let R be a commutative ring. Assume that the group scheme $G = \operatorname{Spec} \mathbb{Z}[\mathbf{b}]$ acts on $\operatorname{Spec} R$. The co-morphism of the action we denote by

$$\theta_R : R \to R \otimes \mathbb{Z}[\mathbf{b}] = R[\mathbf{b}].$$

For every $r \in R$,

$$\theta(r) = \sum_\alpha s^R_\alpha(r) \otimes b_\alpha$$

for uniquely determined elements $s^R_\alpha(r) \in R$. We call the group endomorphisms

$$s^R_\alpha : R \to R$$

for all partitions α the *Landweber-Novikov operations on R*.

Now consider the natural action of G on $\operatorname{Spec}(\mathrm{Laz})$ (section 8). The corresponding operations s^{Laz}_α we simply denote by s_α.

Let $\varepsilon : \mathrm{Laz} \to \mathbb{Z}$ be the restriction of the augmentation map $\mathbb{Z}[\mathbf{b}] \to \mathbb{Z}$.

Lemma 10.1. *The composition*

$$\mathrm{Laz} \xrightarrow{\theta_{\mathrm{Laz}}} \mathrm{Laz} \otimes \mathbb{Z}[\mathbf{b}] \xrightarrow{\varepsilon \otimes \mathrm{id}} \mathbb{Z}[\mathbf{b}]$$

coincides with the embedding $\mathrm{Laz} \hookrightarrow \mathbb{Z}[\mathbf{b}]$.

Proof. The homomorphism θ_{Laz} corresponds to the group law

$$\exp\big(\Phi(\log x, \log y)\big)$$

on $\mathrm{Laz} \otimes \mathbb{Z}[\mathbf{b}]$, where Φ is the universal group law on Laz. The augmentation of Φ is the additive group law over $\mathbb{Z}$, whence the result. □

Denote by

$$\mu : \mathbb{Z}[\mathbf{b}] \to \mathbb{Z}[\mathbf{b}] \otimes \mathbb{Z}[\mathbf{b}]$$

the co-multiplication ring homomorphism for the group scheme G. Let A^* be an oriented ring cohomology theory over F. Consider the ring homomorphism of cohomology theories

$$\widetilde{\mu} = \mathrm{id}_A \otimes \mu : \widetilde{A}^* \to \widetilde{\widetilde{A}}{}^*.$$

Lemma 10.2. *For every $X \in \mathbf{Sm}(F)$ and $a \in K_0(X)$,*

$$\widetilde{\mu}\big(\mathbf{P}^A(a)\big) = \mathbf{P}^A(a) \cdot \mathbf{P}^{\widetilde{A}}(a).$$

(In the r.h.s. the first term is a polynomial in the b'_α and the second - in the b''_α.)

Proof. The co-multiplication $\mu : \mathbb{Z}[\mathbf{b}] \to \mathbb{Z}[\mathbf{b}] \otimes \mathbb{Z}[\mathbf{b}] = \mathbb{Z}[\mathbf{b}', \mathbf{b}'']$ satisfies

$$\sum_{i \geq 0} t^{i+1} \mu(b_i) = \sum_{j \geq 0} \left(\sum_{k \geq 0} t^{k+1} b'_k \right)^{j+1} b''_j.$$

By the splitting principle and multiplicativity property (3.4), we may assume that $a = [L]$, where L is a line bundle. Hence (with $\xi = c_1(L)$),

$$\begin{aligned}
\widetilde{\mu}\big(\mathbf{P}^A(L)\big) &= \sum_{i \geq 0} \xi^i \mu(b_i) \\
&= \sum_{j \geq 0} \left(\sum_{k \geq 0} \xi^k b'_k \right)^{j+1} \xi^j b''_j \\
&= \sum_{j \geq 0} \mathbf{P}^A(L)^{j+1} \xi^j b''_j \\
&= \mathbf{P}^A(L) \cdot \sum_{j \geq 0} c_1^{\widetilde{A}}(L)^j b''_j \\
&= \mathbf{P}^A(L) \cdot \mathbf{P}^{\widetilde{A}}(L).
\end{aligned}$$

□

Corollary 10.3. *For every projective variety $X \in \mathbf{Sm}(F)$,*

$$\mu(\mathbf{F}_X^A) = \mathbf{F}_X^{\widetilde{A}}.$$

Proof. We apply Lemma 10.2 for $a = [-T_X]$:

$$\mu(\mathbf{F}_X^A) = \mu(p_A \, \mathbf{P}_X^A) = p_A \widetilde{\mu}(\mathbf{P}_X^A) = p_A\big(\mathbf{P}_X^A \cdot \mathbf{P}_X^{\widetilde{A}}\big) = p_{\widetilde{A}}\big(\mathbf{P}_X^{\widetilde{A}}\big) = \mathbf{F}_X^{\widetilde{A}}.$$

□

We can express the Landweber-Novikov operations in terms of characteristic numbers in $\widetilde{H}$. This is an analog of Novikov's formula [1, Th. I.8.3] with the cobordism theory replaced by its approximation $\widetilde{H}$.

Proposition 10.4. *For every projective variety $X \in \mathbf{Sm}(F)$,*

$$s_\alpha(\mathbf{F}_X^H) = p_{\widetilde{H}} c_\alpha^{\widetilde{H}}(-T_X) \in \mathbb{Z}[\mathbf{b}],$$

where $p : X \to \mathrm{pt}$ is the structure morphism.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Laz} & \xrightarrow{\theta_{\mathrm{Laz}}} & \mathrm{Laz} \otimes \mathbb{Z}[\mathbf{b}] & \xrightarrow{\varepsilon \otimes \mathrm{id}} & \mathbb{Z}[\mathbf{b}] \\ {\scriptstyle \theta_{\mathrm{Laz}}}\downarrow & & {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & {\scriptstyle \mu}\downarrow \\ \mathrm{Laz} \otimes \mathbb{Z}[\mathbf{b}] & \xrightarrow{\theta_{\mathrm{Laz}} \otimes \mathrm{id}} & \mathrm{Laz} \otimes \mathbb{Z}[\mathbf{b}', \mathbf{b}''] & \xrightarrow{\varepsilon \otimes \mathrm{id}} & \mathbb{Z}[\mathbf{b}', \mathbf{b}'']. \end{array}$$

By Lemma 10.1 and Corollary 10.3, the composition $\mu \circ (\varepsilon \otimes \mathrm{id}) \circ \theta_{\mathrm{Laz}}$ takes the class $\mathbf{F}_X^H$ to

$$\mathbf{F}_X^{\widetilde{H}} = \sum_\alpha p_{\widetilde{H}} c_\alpha^{\widetilde{H}}(-T_X) b''_\alpha.$$

By Lemma 10.1, another composition $(\varepsilon \otimes \mathrm{id}) \circ (\theta_{\mathrm{Laz}} \otimes \mathrm{id}) \circ \theta_{\mathrm{Laz}}$ takes $\mathbf{F}_X^H$ to

$$\sum_\alpha s_\alpha(\mathbf{F}_X^H) b''_\alpha.$$

□

11. Invariant ideals

Let R be a commutative ring. Assume that the group scheme $G = \operatorname{Spec} \mathbb{Z}[\mathbf{b}]$ acts on $\operatorname{Spec} R$. An ideal $I \subset R$ is called *invariant* if $s_\alpha^R(I) \subset I$ for every α.

Let p be a prime integer. The ideal $p\mathbb{Z}[\mathbf{b}]$ in $\mathbb{Z}[\mathbf{b}]$ is obviously prime and invariant with respect to the action of G on itself by left translations. Therefore, the intersection $I(p) = \mathrm{Laz} \cap p\mathbb{Z}[\mathbf{b}] \subset \mathrm{Laz}$ is a prime invariant ideal in Laz.

Let $n = 0, 1, 2, \ldots, \infty$. We write $I(p, n)$ for the ideal in $I(p)$ generated by all $a \in I(p)$ of degree $\leq p^n - 1$. For example, $I(p, 0) = p\,\mathrm{Laz}$ and $I(p, \infty) = I(p)$.

Thus, for every prime p we have a chain of prime invariant ideals in Laz:

$$p\,\mathrm{Laz} = I(p, 0) \subset I(p, 1) \subset \cdots \subset I(p, n) \subset \cdots \subset I(p, \infty) = I(p).$$

It is known (see [10, Prop. VII.4.21] and [5, Th. 2.7]) that every ideal $I(p, n)$ is prime and invariant and the only nonzero prime invariant ideals in Laz are $I(p, n)$ for all prime p and $n \geq 0$.

Let X be a projective smooth variety over a field F. The set

$$I(X) = \{\mathbf{F}_Y^H \in \mathrm{Laz} \ \text{ for all } Y \in \mathbf{Sm}(F) \text{ such that } \operatorname{Mor}_F(Y, X) \neq \emptyset\}$$

is a graded ideal in Laz. Let $q : X \to \mathrm{pt}$ be a structure morphism. For every projective morphism $f : Y \to X$,

$$q_{\widetilde{H}} f_{\widetilde{H}}(1_Y) = (qf)_{\widetilde{H}}(1_Y) = \mathbf{F}_Y^H.$$

Hence

$$I(X) = q_{\widetilde{H}} \widetilde{H}_c(X).$$

Recall that n_X is the gcd of $\deg(x)$ over all closed points x of a variety X.

Example 11.1. $I(X)_0 = n_X\mathbb{Z}$. If $X(F) \neq \emptyset$, $I(X) = \mathrm{Laz}$.

Theorem 11.2. *For a projective variety $X \in \mathbf{Sm}(F)$ over a field F, the ideal $I(X) \subset \mathrm{Laz}$ is invariant.*

Proof. Let $f : Y \to X$ be a morphism, $q : X \to \mathrm{pt}$ the structure morphism. By Proposition 10.4 and Corollary 9.10,

$$s_\alpha(\mathbf{F}_Y^H) = q_{\widetilde{H}} f_{\widetilde{H}}\big(c^{\widetilde{H}}(-T_Y)\big) \in q_{\widetilde{H}}\big(\widetilde{H}_c(X)\big) = I(X).$$

□

Let P be a minimal prime ideal in Laz containing $I(X)$. By [6, Th. 3.1], P is invariant and hence $P = I(p, n)$ for some prime integer p and $n = 0, 1, \ldots, \infty$. Clearly, P is the only minimal prime ideal containing $I(X)$ and p. We set $n_p(X) = n$. If for a prime integer p there is no invariant prime ideal containing $I(X)$ and p, we set $n_p(X) = \infty$. Thus, for every projective variety X we have the numbers $n_p(X)$ assigned for each prime integer p.

Proposition 11.3. *Let $X \in \mathbf{Sm}(F)$ be a projective variety, p a prime integer. Then the following conditions are equivalent:*

1. *$p \mid n_X$;*
2. *There exists an invariant prime ideal of* Laz *containing $I(X)$ and p.*

Proof. If $I\big(p, n_p(X)\big)$ is the minimal prime ideal, then $I(X)_0 \subset p\mathbb{Z}$, i.e. $p \mid n_X$. Conversely, let $p \mid n_X$ and let $I(p_i, n_i)$ be all minimal prime ideals containing $I(X)$. Since $I(p_i, n_i) \cap \mathbb{Z} = p_i\mathbb{Z}$ and $I(X) \cap \mathbb{Z} = n_X\mathbb{Z}$, the intersection of all the $p_i\mathbb{Z}$ coincides with the radical of $n_X\mathbb{Z}$, hence $p = p_i$ for some i. □

Proposition 11.4. *Let X and Y be projective smooth varieties such that* $\mathrm{Mor}(Y, X) \neq \emptyset$. *Then $n_p(Y) \leq n_p(X)$ for every prime p.*

Proof. We have $I(Y) \subset I(X) \subset I\big(p, n_p(X)\big)$. The minimal prime ideal between $I(Y)$ and $I\big(p, n_p(X)\big)$ is equal $I\big(p, n_p(Y)\big)$, hence $n_p(Y) \leq n_p(X)$. □

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, IL, 1995, Reprint of the 1974 original.

[2] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[3] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[4] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[5] Peter S. Landweber, *Annihilator ideals and primitive elements in complex bordism*, Illinois J. Math. **17** (1973), 273–284.

[6] ———, *Associated prime ideals and Hopf algebras*, J. Pure Appl. Algebra **3** (1973), 43–58.

[7] M. Levine and F. Morel, *Cobordisme algébrique I*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001).

[8] ———, *Algebraic cobordism I*, http://www.math.uiuc.edu/K-theory/0547/ (2002).

[9] I. Panin, *Riemann-roch theorem for oriented cohomology*, http://www.math.uiuc.edu/K-theory/0552/ (2002).
[10] Yuli B. Rudyak, *On Thom spectra, orientability, and cobordism*, Springer-Verlag, Berlin, 1998, With a foreword by Haynes Miller.
[11] Robert E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, N.J., 1968, Mathematical notes.

Alexander Merkurjev, Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
E-mail address: merkurev@math.ucla.edu