

MOTIVIC DECOMPOSITION OF COMPACTIFICATIONS OF CERTAIN GROUP VARIETIES

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ABSTRACT. Let D be a central simple algebra of prime degree over a field and let E be an $\mathbf{SL}_1(D)$ -torsor. We determine the complete motivic decomposition of certain compactifications of E . We also compute the Chow ring of E .

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1. INTRODUCTION

Let p be a prime number. For any integer $n \geq 2$, a *Rost motive of degree n* is a direct summand \mathcal{R} of the Chow motive with coefficients in $\mathbb{Z}_{(p)}$ (the localization of the integers at the prime ideal (p)) of a smooth complete geometrically irreducible variety X over a field F such that for any extension field K/F with a closed point on X_K of degree prime to p , the motive \mathcal{R}_K is isomorphic to the direct sum of Tate motives

$$\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}(b) \oplus \mathbb{Z}_{(p)}(2b) \oplus \cdots \oplus \mathbb{Z}_{(p)}((p-1)b),$$

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where $b = (p^{n-1} - 1)/(p - 1)$. The isomorphism class of \mathcal{R} is determined by X , [19, Proposition 3.4]; \mathcal{R} is indecomposable as long as X has no closed points of degree prime to p .

A smooth complete geometrically irreducible variety X over F is a *p-generic splitting variety* for an element $s \in H_{\text{ét}}^n(F, \mathbb{Z}/p\mathbb{Z}(n-1))$, if s vanishes over a field extension K/F if and only if X has a closed point of degree prime to p over K . A *norm variety* of s is a *p-generic splitting variety* of dimension $p^{n-1} - 1$.

A Rost motive living on a *p-generic splitting variety* of an element s is determined by s up to isomorphism and called the Rost motive of s . In characteristic 0, any *symbol* s admits a norm variety possessing a Rost motive. This played an important role in the proof of the Bloch-Kato conjecture (see [31]). It is interesting to understand the complement to the Rost motive in the motive of a norm variety X for a given s ; this complement, however, depends on X and is not determined by s anymore.

For $p = 2$, there are nice norm varieties known as norm quadrics. Their complete motivic decomposition is a classical result due to M. Rost. A norm quadric X can be viewed as a compactification of the affine quadric U given by $\pi = c$, where π is a quadratic $(n-1)$ -fold Pfister form and $c \in F^\times$. The summands of the complete motivic decomposition of X are given by the degree n Rost motive of X and shifts of the degree $n-1$ Rost motive of the projective Pfister quadric $\pi = 0$. It is proved in [16, Theorem A.4] that $\text{CH}(U) = \mathbb{Z}$ if the equation $\pi = c$ has no solutions over F . In the present paper we extend these results to arbitrary prime p (and $n = 3$).

For arbitrary p , there are nice norm varieties in small degrees. For $n = 2$, these are the Severi-Brauer varieties of degree p central simple F -algebras. Any of them admits a degree 2 Rost motive which is simply the total motive of the variety.

The first interesting situation occurs in degree $n = 3$. Let D be a degree p central division F -algebra, $G = \mathbf{SL}_1(D)$ the special linear group of D , and E a principle homogeneous space under G . The affine variety E is given by the equation $\text{Nrd} = c$, where Nrd is the reduced norm of D and $c \in F^\times$. Any smooth compactification of E is a norm variety of the element $s := [D] \cup (c) \in H_{\text{ét}}^3(F, \mathbb{Z}/p\mathbb{Z}(2))$. It has been shown by N. Semenov in [26] for $p = 3$ (and $\text{char } F = 0$) that the motive of a certain smooth equivariant compactification of E decomposes in a direct sum, where one of the summands is the Rost motive of s , another summand is a motive ε vanishing over any field extension of F splitting D , and each of the remaining summands is a shift of the motive of the Severi-Brauer variety of D . All these summands (but ε) are indecomposable and ε was expected to be 0.

Another proof of this result (covering arbitrary characteristic) has been provided in [30] along with the claim that $\varepsilon = 0$, but the proof of the claim was incomplete.

In the present paper we prove the following main result (see Theorem 10.3):

Theorem 1.1. *Let F be a field, D a central division F -algebra of prime degree p , X a smooth compactification of an $\mathbf{SL}_1(D)$ -torsor, and $M(X)$ its Chow motive with $\mathbb{Z}_{(p)}$ -coefficients. Assume that $M(X)$ over the function field of the Severi-Brauer variety S of D is isomorphic to a direct sum of Tate motives. Then $M(X)$ (over F) is isomorphic to the direct sum of the Rost motive of X and several shifts of $M(S)$. This is the unique decomposition of $M(X)$ into a direct sum of indecomposable motives.*

We note that the compactification in [26] (for $p = 3$) has the property required in Theorem 1.1 (see Example 10.6).

In Section 6 we show that the condition that $M(X)$ is *split* (i.e., isomorphic to a finite direct sum of Tate motives) over $F(S)$ is satisfied for all smooth $G \times G$ -equivariant compactifications of $G = \mathbf{SL}_1(D)$. Moreover, we prove that the motive $M(X)$ is split for all smooth equivariant compactifications X of split semisimple groups (see Theorem 6.8).

We also compute the Chow ring of G in arbitrary characteristic as well as the Chow ring of E in characteristic 0 (see Theorem 9.7 and Corollary 10.8):

Theorem 1.2. *Let D be a central division algebra of prime degree p and $G = \mathbf{SL}_1(D)$.*

1) *There is an element $h \in \mathrm{CH}^{p+1}(G)$ such that*

$$\mathrm{CH}(G) = \mathbb{Z} \cdot 1 \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}.$$

2) *Let E be a nonsplit G -torsor. If $\mathrm{char} F = 0$, then $\mathrm{CH}(E) = \mathbb{Z}$.*

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2. K -COHOMOLOGY

Let X be a smooth variety over F . We write $A^i(X, K_n)$ for the K -cohomology groups as defined in [25]. In particular, $A^i(X, K_i)$ is the Chow group $\mathrm{CH}^i(X)$ of classes of codimension i algebraic cycles on X .

Let G be a simply connected semisimple algebraic group. The group $A^1(G, K_2)$ is additive in G , i.e., if G and G' are two simply connected group, then the projections of $G \times G'$ onto G and G' yield an isomorphism (see [13, Part II, Proposition 7.6 and Theorem 9.3])

$$A^1(G, K_2) \oplus A^1(G', K_2) \xrightarrow{\sim} A^1(G \times G', K_2).$$

The following lemma readily follows.

Lemma 2.1. 1) *The map*

$$A^1(G, K_2) \rightarrow A^1(G \times G, K_2) = A^1(G, K_2) \oplus A^1(G, K_2)$$

induced by the product homomorphism $G \times G \rightarrow G$ is equal to $(1, 1)$.

2) *The map $A^1(G, K_2) \rightarrow A^1(G, K_2)$ induced by the morphism $G \rightarrow G$, $x \mapsto x^{-1}$ is equal to -1 .*

Proof. 1) It suffices to note that the isomorphism

$$A^1(G \times G', K_2) \xrightarrow{\sim} A^1(G, K_2) \oplus A^1(G', K_2)$$

inverse to the one mentioned above, is given by the pull-backs with respect to the group embeddings $G, G' \hookrightarrow G \times G'$.

2) The composition of the embedding of varieties $G \hookrightarrow G \times G$, $g \mapsto (g, g^{-1})$ with the product map $G \times G \rightarrow G$ is trivial. \square

If G is an absolutely simple simply connected group, then $A^1(G, K_2)$ is an infinite cyclic group with a canonical generator q_G (see [13, Part II, §7]).

3. BGQ SPECTRAL SEQUENCE

Let X be a smooth variety over F . We consider the Brown-Gersten-Quillen *coniveau spectral sequence*

$$(3.1) \quad E_2^{s,t} = A^s(X, K_{-t}) \Rightarrow K_{-s-t}(X)$$

converging to the K -groups of X with the topological filtration [23, §7, Th. 5.4].

Example 3.2. Let $G = \mathbf{SL}_n$. By [29, §2], we have $\mathrm{CH}(G) = \mathbb{Z}$. It follows that all the differentials of the BGQ spectral sequence for G coming to the zero diagonal are trivial.

Lemma 3.3 ([20, Theorem 3.4]). *If δ is a differential of finite order in the spectral sequence (3.1) on the q -th page $E_q^{*,*}$, then for every prime divisor p of the order of δ , the integer $p - 1$ divides $q - 1$.*

Let p be a prime integer, D a central division algebra over F of degree p and $G = \mathbf{SL}_1(D)$. As D is split by a field extension of degree p , it follows from Example 3.2 that all Chow groups $\mathrm{CH}^i(G)$ are p -periodic for $i > 0$ and the order of every differential in the BGQ spectral sequence for G coming to the zero diagonal divides p . The edge homomorphism $K_1(G) \rightarrow E_2^{0,-1} = A^0(G, K_1) = F^\times$ is a surjection split by the pull-back with respect to the structure morphism $G \rightarrow \mathrm{Spec} F$. Therefore, all the differentials starting at $E_*^{0,-1}$ are trivial.

It follows then from Lemma 3.3 that the only possibly nontrivial differential coming to the terms $E_q^{i,-i}$ for $q \geq 2$ and $i \leq p + 1$ is

$$\partial_G : A^1(G, K_2) = E_p^{1,-2} \rightarrow E_p^{p+1,-p-1} = \mathrm{CH}^{p+1}(G).$$

By [29, Theorem 6.1] (see also [22, Theorem 5.1]), $K_0(G) = \mathbb{Z}$, hence the factors

$$K_0(G)^{(i)}/K_0(G)^{(i+1)} = E_\infty^{i,-i}$$

of the topological filtration on $K_0(G)$ are trivial for $i > 0$. It follows that the map ∂_G is surjective. As the group $A^1(G, K_2)$ is cyclic with the generator q_G , the group $\mathrm{CH}^{p+1}(G)$ is cyclic of order dividing p . It is shown in [33, Theorem 4.2] (see also Theorem 8.2) that the differential ∂_G is nontrivial. We have proved the following lemma.

Lemma 3.4. *If D is a central division algebra of degree p , then $\mathrm{CH}^{p+1}(G)$ is a cyclic group of order p generated by $\partial_G(q_G)$. \square*

4. SPECIALIZATION

Let A be a discrete valuation ring with residue field F and quotient field L . Let \mathcal{X} be a smooth scheme over A and set $X = \mathcal{X} \otimes_A F$, $X' = \mathcal{X} \otimes_A L$. By [11, Example 20.3.1], there is a *specialization* ring homomorphism

$$\sigma : \mathrm{CH}^*(X') \rightarrow \mathrm{CH}^*(X).$$

Example 4.1. Let X be a variety over F , $L = F(t)$ the rational function field. Consider the valuation ring $A \subset L$ of the parameter t and $\mathcal{X} = X \otimes_F A$. Then $X' = X_L$ and we have a specialization ring homomorphism $\sigma : \mathrm{CH}^*(X_L) \rightarrow \mathrm{CH}^*(X)$.

A section of the structure morphism $\mathcal{X} \rightarrow \text{Spec } A$ gives two rational points $x \in X$ and $x' \in X'$. By definition of the specialization, $\sigma([x']) = [x]$.

Let F be a field of finite characteristic. By [2, Ch. IX, §2, Propositions 5 and 1], there is a complete discretely valued field L of characteristic zero with residue field F . Let A be the valuation ring and D a central simple algebra over F . By [14, Theorem 6.1], there is an Azumaya algebra \mathcal{D} over A such that $D \simeq \mathcal{D} \otimes_A F$. The algebra $D' = \mathcal{D} \otimes_A L$ is a central simple algebra over L . Then we have a specialization homomorphism

$$\sigma : \text{CH}^*(\mathbf{SL}_1(D')) \rightarrow \text{CH}^*(\mathbf{SL}_1(D))$$

satisfying $\sigma([e']) = [e]$, where e and e' are the identities of the groups.

5. A SOURCE OF SPLIT MOTIVES

We work in the category of Chow motives over a field F , [9, §64]. We write $M(X)$ for the motive (with integral coefficients) of a smooth complete variety X over F .

A motive is *split* if it is isomorphic to a finite direct sum of Tate motives $\mathbb{Z}(a)$ (with arbitrary shifts a). Let X be a smooth proper variety such that the motive $M(X)$ is split, i.e., $M(X) = \coprod_i \mathbb{Z}(a_i)$ for some a_i . The *generating (Poincaré) polynomial* $P_X(t)$ of X is defined by

$$P_X(t) = \sum_i t^{a_i}.$$

Note that the integer a_i is equal to the rank of the (free abelian) Chow group $\text{CH}^i(X)$.

Example 5.1. Let G be a split semisimple group and $B \subset G$ a Borel subgroup. Then

$$P_{G/B}(t) = \sum_{w \in W} t^{l(w)},$$

where W is the Weyl group of G and $l(w)$ is the length of w (see [8, §3]).

Proposition 5.2 (P. Brosnan, [4, Theorem 3.3]). *Let X be a smooth projective variety over F equipped with an action of the multiplicative group \mathbb{G}_m . Then*

$$M(X) = \coprod_i M(Z_i)(a_i),$$

where the Z_i are the (smooth) connected components of the subscheme of $X^{\mathbb{G}_m}$ of fixed points and $a_i \in \mathbb{Z}$. Moreover, the integer a_i is the dimension of the positive eigenspace of the action of \mathbb{G}_m on the tangent space \mathcal{T}_z of X at an arbitrary closed point $z \in Z_i$. The dimension of Z_i is the dimension of $(\mathcal{T}_z)^{\mathbb{G}_m}$.

Let T be a split torus of dimension n . The choice of a \mathbb{Z} -basis in the character group T^* allows us to identify T^* with \mathbb{Z}^n . We order \mathbb{Z}^n (and hence T^*) lexicographically.

Suppose T acts on a smooth variety X and let $x \in X$ be an T -fixed rational point. Let $\chi_1, \chi_2, \dots, \chi_m$ be all characters of the representation of T in the tangent space \mathcal{T}_x of X at x . Write a_x for the number of positive (with respect to the ordering) characters among the χ_i 's.

Corollary 5.3. *Let X be a smooth projective variety over F equipped with an action of a split torus T . If the subscheme X^T of T -fixed points in X is a disjoint union of finitely many rational points, the motive of X is split. Moreover,*

$$P_X(t) = \sum_{x \in X^T} t^{a_x}.$$

Proof. Induction on the dimension of T . □

Example 5.4. Let T be a split torus of dimension n and X a smooth projective toric variety (see [12]). Let σ be a cone of dimension n in the fan of X and $\{\chi_1, \chi_2, \dots, \chi_n\}$ a (unique) \mathbb{Z} -basis of T^* generating the dual cone σ^\vee . The standard T -invariant affine open set corresponding to σ is $V_\sigma := \text{Spec } F[\sigma^\vee]$. The map $V_\sigma \rightarrow \mathbb{A}^n$, taking x to $(\chi_1(x), \chi_2(x), \dots, \chi_n(x))$ is a T -equivariant isomorphism, where $t \in T$ acts on the affine space \mathbb{A}^n by componentwise multiplication by $\chi_i(t)$. The only one T -equivariant point $x \in V_\sigma$ corresponds to the origin under the isomorphism, so we can identify the tangent space \mathcal{T}_x with \mathbb{A}^n , and the χ_i 's are the characters of the representation of T in the tangent space \mathcal{T}_x . Let a_σ be the number of positive χ_i 's with respect to a fixed lexicographic order on T^* . Every T -fixed point in X belongs to V_σ for a unique σ . It follows that the motive $M(X)$ is split and

$$P_X(t) = \sum_{\sigma} t^{a_\sigma},$$

where the sum is taken over all dimension n cones in the fan of X .

6. COMPACTIFICATIONS OF ALGEBRAIC GROUPS

A *compactification* of an affine algebraic group G is a projective variety containing G as a dense open subvariety. A $G \times G$ -equivariant compactification of G is a projective variety X equipped with an action of $G \times G$ and containing the homogeneous variety $G = (G \times G) / \text{diag}(G)$ as an open orbit. Here the group $G \times G$ acts on G by the left-right translations.

Let G be a split semisimple group over F . Write G_{ad} for the corresponding adjoint group. The group G_{ad} admits the so-called *wonderful* $G_{ad} \times G_{ad}$ -equivariant compactification \mathbf{X} (see [3, §6.1]). Let $T \subset G$ be a split maximal torus and T_{ad} the corresponding maximal torus in G_{ad} . The closure \mathbf{X}' of T_{ad} in \mathbf{X} is a toric T_{ad} -variety with fan consisting of all Weyl chambers in $(T_{ad})_* \otimes \mathbb{R} = T_* \otimes \mathbb{R}$ and their faces.

Let B be a Borel subgroup of G containing T and B^- the opposite Borel subgroup. There is an open $B^- \times B$ -invariant subscheme $\mathbf{X}_0 \subset \mathbf{X}$ such that the intersection $\mathbf{X}'_0 := \mathbf{X}_0 \cap \mathbf{X}'$ is the standard open T_{ad} -invariant subscheme of the toric variety \mathbf{X}' corresponding to the negative Weyl chamber Ω that is a cone in the fan of \mathbf{X}' . Note that the Weyl group W of G acts simply transitively on the set of all Weyl chambers.

A $G \times G$ -equivariant compactification X of G is called *toroidal* if X is normal and the quotient map $G \rightarrow G_{ad}$ extends to a morphism $\pi : X \rightarrow \mathbf{X}$ (see [3, §6.2]). The closed subscheme $X' := \pi^{-1}(\mathbf{X}')$ of X is a projective toric T -variety. Note that the diagonal subtorus $\text{diag}(T) \subset T \times T$ acts trivially on X' . The fan of X' is a subdivision of the fan consisting of the Weyl chambers and their faces. The scheme X is smooth if and only if so is X' .

Conversely, if F is a perfect field, given a smooth projective toric T -variety with a W -invariant fan that is a subdivision of the fan consisting of the Weyl chambers and their faces, there is a unique smooth $G \times G$ -equivariant toroidal compactification X of G with the toric variety X' isomorphic to the given one (see [3, §6.2] and [15, §2.3]). By [5] and [7], such a smooth toric variety exists for every split semisimple group G . In other words, the following holds.

Proposition 6.1. *Every split semisimple group G over a perfect field admits a smooth $G \times G$ -equivariant toroidal compactification.* \square

Let X be a smooth $G \times G$ -equivariant toroidal compactification of G over F . Recall that the toric T -variety X' is smooth projective. Set $X_0 := \pi^{-1}(\mathbf{X}_0)$ and $X'_0 := \pi^{-1}(\mathbf{X}'_0) = X' \cap X_0$. Then the T -invariant subset $X'_0 \subset X'$ is the union of standard open subschemes V_σ of X' (see Example 5.4) corresponding to all cones σ in the negative Weyl chamber Ω . The subscheme $(V_\sigma)^T$ reduces to a single rational point if σ is of largest dimension. In particular, the subscheme $(X'_0)^T$ of T -fixed points in X'_0 is a disjoint union of k rational points, where k is the number of cones of maximal dimension in Ω . It follows that $|(X')^T| = k|W|$, the number of all cones of maximal dimension in the fan of X' .

Let U and U^- be the unipotent radicals of B and B^- respectively.

Lemma 6.2 ([3, Proposition 6.2.3]). 1) *Every $G \times G$ -orbit in X meets X'_0 along a unique T -orbit.*

2) *The map*

$$U^- \times X'_0 \times U \rightarrow X_0, \quad (u, x, v) \mapsto u x v^{-1},$$

is a $T \times T$ -equivariant isomorphism.

3) *Every closed $G \times G$ -orbit in X is isomorphic to $G/B \times G/B$.*

Proposition 6.3. *The scheme $X^{T \times T}$ is the disjoint union of Wx_0W over all $x_0 \in (X'_0)^T$ and Wx_0W is a disjoint union of $|W|^2$ rational points.*

Proof. Take $x \in X^{T \times T}$. Let \mathbf{x} be the image of x under the map $\pi : X \rightarrow \mathbf{X}$. Computing dimensions of maximal tori of the stabilizers of points in the wonderful compactification \mathbf{X} , we see that \mathbf{x} lies in the only closed $G \times G$ -orbit \mathbf{O} in \mathbf{X} (e.g., [10, Lemma 4.2]). By Lemma 6.2(3), applied to the compactification \mathbf{X} of G_{ad} , $\mathbf{O} \simeq G/B \times G/B$. In view of Lemma 6.2(1), $\mathbf{O} \cap \mathbf{X}'_0$ is a closed T -orbit in \mathbf{X}'_0 and therefore, reduces to a single rational T -invariant point in \mathbf{X}'_0 . The group $W \times W$ acts simply transitively on the set of $T \times T$ -fixed point in $G/B \times G/B$. It follows that $|W\mathbf{x}W| = |W|^2$ and $W\mathbf{x}W$ intersects \mathbf{X}'_0 . Therefore, WxW intersects $X^{T \times T} \cap X'_0 = (X'_0)^T$, that is the disjoint union of k rational points. Hence x is a rational point, $x \in W(X'_0)^T W$ and $|WxW| = |W|^2$.

Note that for a point $x_0 \in (X'_0)^T$, the $G \times G$ -orbit of x_0 intersects X'_0 by the T -orbit $\{x_0\}$ in view of Lemma 6.2(1). It follows that different Wx_0W do not intersect and therefore, $X^{T \times T}$ is the disjoint union of Wx_0W over all $x_0 \in (X'_0)^T$. \square

Let X be a smooth $G \times G$ -equivariant toroidal compactification of a split semisimple group G of rank n . By Proposition 6.3, every $T \times T$ -fixed point x in X is of the form $x = w_1 x_0 w_2^{-1}$, where $w_1, w_2 \in W$ and $x_0 \in (X'_0)^T$. Recall that X'_0 is the union of the standard affine open subsets V_σ of the toric T -variety X' over all cones σ of dimension n in the Weyl chamber Ω . Let σ be a (unique) cone in Ω such that $x_0 \in V_\sigma$.

By Lemma 6.2(2), the map

$$f : U^- \times V_\sigma \times U \rightarrow X, \quad (u_1, y, u_2) \mapsto w_1 u_1 x_0 u_2^{-1} w_2^{-1}$$

is an open embedding. We have $f(1, x_0, 1) = x$. Thus, f identifies the tangent space \mathcal{T}_x of x in X with the space $\mathfrak{u}^- \oplus \mathfrak{a} \oplus \mathfrak{u}$, where \mathfrak{u} and \mathfrak{u}^- are the Lie algebras of U and U^- respectively and \mathfrak{a} is the tangent space of V_σ at x' . The torus $T \times T$ acts linearly on the tangent space \mathcal{T}_x leaving invariant \mathfrak{u}^- , \mathfrak{a} and \mathfrak{u} . For convenience, we write $T \times T$ as $S := T_1 \times T_2$ in order to distinguish the components. Let Φ_1^- and Φ_2^- be two copies of the set of negative roots in T_1^* and T_2^* respectively. The set of characters of the S -representation \mathfrak{u}^- (respectively, \mathfrak{u}) is $w_1(\Phi_1^-)$ (respectively, $w_2(\Phi_2^-)$).

Let $\{\chi_1, \chi_2, \dots, \chi_n\}$ be a (unique) \mathbb{Z} -basis of T^* generating the dual cone σ^\vee . By Example 5.4, the set of characters of the S -representation \mathfrak{a} is

$$\{(w_1(\chi_i), -w_2(\chi_i))\}_{i=1}^n \subset S^* = T_1^* \oplus T_2^*.$$

Let Π_1 and Π_2 be (ordered) systems of simple roots in Φ_1 and Φ_2 respectively. Consider the lexicographic ordering on $S^* = T_1^* \oplus T_2^*$ corresponding to the basis $\Pi_1 \cup \Pi_2$ of S^* . As $\chi_i \neq 0$, we have $(w_1(\chi_i), -w_2(\chi_i)) > 0$ if and only if $w_1(\chi_i) > 0$. For every $w \in W$, write $b(\sigma, w)$ for the number of all i such that $w(\chi_i) > 0$. Note that the number of positive roots in $w(\Phi^-)$ is equal to the length $l(w)$ of w . By Corollary 5.3, we have

$$(6.4) \quad P_X(t) = \sum_{w_1, w_2 \in W, \sigma \subset \Omega} t^{l(w_1) + b(\sigma, w_1) + l(w_2)} = \left(\sum_{w \in W, \sigma \subset \Omega} t^{l(w) + b(\sigma, w)} \right) \cdot P_{G/B}(t),$$

as by Example 5.1,

$$P_{G/B}(t) = \sum_{w \in W} t^{l(w)}.$$

We have proved the following theorem.

Theorem 6.5. *Let X be a smooth $G \times G$ -equivariant toroidal compactification of a split semisimple group G . Then the motive $M(X)$ is split into a direct sum of $s|W|$ Tate motives, where s is the number of cones of maximal dimension in the fan of the associated toric variety X' . Moreover,*

$$P_X(t) = \left(\sum_{w \in W, \sigma \subset \Omega} t^{l(w) + b(\sigma, w)} \right) \cdot P_{G/B}(t).$$

In particular, the motive $M(X)$ is divisible by $M(G/B)$.

Example 6.6. Let G be a semisimple adjoint group and X the wonderful compactification of G . Then the negative Weyl chamber Ω is the cone $\sigma = \Omega$ in the fan of X' . The dual cone σ^\vee is generated by $-\Pi$. Hence $b(w, \sigma)$ is equal to the number of *simple* roots α such that $w(\alpha) \in \Phi^-$.

Example 6.7. Let $G = \mathbf{SL}_3$, $\Pi = \{\alpha_1, \alpha_2\}$. Bisecting each of the six Weyl chambers we get a smooth projective fan with 12 two-dimensional cones. The two cones dual to the ones in the negative Weyl chamber are generated by $\{-\alpha_1, (\alpha_1 - \alpha_2)/3\}$ and $\{-\alpha_2, (\alpha_2 - \alpha_1)/3\}$ respectively. Let X be the corresponding $G \times G$ -equivariant toroidal compactification of G . By (6.4),

$$P_X(t) = (t^5 + t^4 + 4t^3 + 4t^2 + t + 1)(t^3 + 2t^2 + 2t + 1).$$

Now consider arbitrary (not necessarily toroidal) $G \times G$ -equivariant compactifications.

Theorem 6.8. *Let X be a smooth $G \times G$ -equivariant compactification of a split semisimple group G over F . Then the subscheme $X^{T \times T}$ is a disjoint union of finitely many rational points. In particular, the motive $M(X)$ is split.*

Proof. By [3, Proposition 6.2.5], there is a $G \times G$ -equivariant toroidal compactification \tilde{X} of G together with a $G \times G$ -equivariant morphism $\varphi : \tilde{X} \rightarrow X$. Let $x \in X^{T \times T}$. By Borel's fixed point theorem, the fiber $\varphi^{-1}(x)$ has a $T \times T$ -fixed point, so the map $\tilde{X}^{T \times T} \rightarrow X^{T \times T}$ is surjective. By Proposition 6.3, $\tilde{X}^{T \times T}$ is a disjoint union of finitely many rational points, hence so is $X^{T \times T}$. \square

Example 6.9. Let Y be a smooth $H \times H$ -equivariant compactification of the group $H = \mathbf{SL}_n$ over F . In particular the projective linear group \mathbf{PGL}_n acts on Y by conjugation. Let D be a central simple F -algebra of degree n and J the corresponding \mathbf{PGL}_n -torsor. The twist of H by J is the group $G = \mathbf{SL}_1(D)$, hence the twist X of Y is a smooth $G \times G$ -equivariant compactification of G . If E is a G -torsor, one can twist X by E to get a smooth compactification of E . By Theorem 6.8, the motives of these compactifications are split over every splitting field of D .

7. SOME COMPUTATIONS IN $\mathrm{CH}(\mathbf{SL}_1(D))$

Let D be a central simple algebra of prime degree p over F and $G = \mathbf{SL}_1(D)$.

Lemma 7.1. *Let X be a smooth compactification of G . Then D is split by the residue field of every point in $X \setminus G$.*

Proof. Let Y be the projective (singular) hypersurface given in the projective space $\mathbb{P}(D \oplus F)$ by the equation $\mathrm{Nrd} = t^p$, where Nrd is the reduced norm form. The group G is an open subset in Y , so we can identify the function fields $F(X) = F(G) = F(Y)$. Let $x \in X \setminus G$. As x is smooth in X , there is a regular system of local parameters around x and therefore a valuation v of $F(G)$ over F with residue field $F(x)$. Since Y is projective, v dominates a point $y \in Y \setminus G$. Over the residue field $F(y)$ the norm form Nrd is isotropic, hence D is split over $F(y)$. Since v dominates y , the field $F(y)$ is contained in $F(v) = F(x)$. Therefore, D is split over $F(x)$. \square

Lemma 7.2. *If D is a division algebra, then the group $\mathrm{CH}_0(G) = \mathrm{CH}^{p^2-1}(G)$ is cyclic of order p generated by the class of the identity e of G .*

Proof. The group of R -equivalence classes of points in $G(F)$ is equal to $\mathrm{SK}_1(D)$ (see [32, Ch. 6]) and hence is trivial by a theorem of Wang. It follows that we have $[x] = [e]$ in $\mathrm{CH}_0(G)$ for every rational point $x \in G(F)$. If $x \in G$ is a closed point, then $[x'] = [e]$ in $\mathrm{CH}_0(G_K)$, where $K = F(x)$ and x' is a rational point of G_K over x . Taking the norm homomorphism $\mathrm{CH}_0(G_K) \rightarrow \mathrm{CH}_0(G)$ for the finite field extension K/F , we have $[x] = [K : F] \cdot [e]$ in $\mathrm{CH}_0(G)$. It follows that $\mathrm{CH}_0(G)$ is a cyclic group generated by $[e]$.

As $p \cdot \mathrm{CH}_0(G) = 0$ it suffices to show that $[e] \neq 0$ in $\mathrm{CH}_0(G)$. Let Y be the compactification of G as in the proof of Lemma 7.1 and let $Z = Y \setminus G$. As D is a central division algebra, the degree of every closed point of Z is divisible by p by Lemma 7.1.

It follows that the class $[e]$ in $\mathrm{CH}_0(Y)$ does not belong to the image of the push-forward homomorphism i in the exact sequence

$$\mathrm{CH}_0(Z) \xrightarrow{i} \mathrm{CH}_0(Y) \rightarrow \mathrm{CH}_0(G) \rightarrow 0.$$

Therefore, $[e] \neq 0$ in $\mathrm{CH}_0(G)$. \square

Consider the morphism $s : G \times G \rightarrow G$, $s(x, y) = xy^{-1}$. Note that s is flat as the composition of the automorphism $(x, y) \mapsto (xy^{-1}, y)$ of the variety $G \times G$ with the projection $G \times G \rightarrow G$.

Let $h = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$.

Lemma 7.3. *We have $s^*(h) = h \times 1 - 1 \times h$ in $\mathrm{CH}^{p+1}(G \times G)$.*

Proof. By Lemma 2.1, we have $s^*(q_G) = q_G \times 1 - 1 \times q_G$ in $A^1(G \times G, K_2)$. The differentials ∂_G commute with flat pull-back maps, hence we have

$$\begin{aligned} s^*(h) &= s^*(\partial_G(q_G)) = \partial_{G \times G}(s^*(q_G)) = \partial_{G \times G}(q_G \times 1 - 1 \times q_G) = \\ &= \partial_G(q_G) \times 1 - 1 \times \partial_G(q_G) = h \times 1 - 1 \times h. \end{aligned} \quad \square$$

Proposition 7.4. *Let c be an integer with $h^{p-1} = c[e]$ in $\mathrm{CH}^{p^2-1}(G)$. Then*

$$c\Delta_G = \sum_{i=0}^{p-1} h^i \times h^{p-1-i},$$

where Δ_G is the class of the diagonal $\mathrm{diag}(G)$ in $\mathrm{CH}^{p^2-1}(G \times G)$.

Proof. The diagonal in $G \times G$ is the pre-image of e under s . Hence by Lemma 7.3,

$$c\Delta_G = cs^*([e]) = s^*(h^{p-1}) = (h \times 1 - 1 \times h)^{p-1} = \sum_{i=0}^{p-1} h^i \times h^{p-1-i}$$

as $\binom{p-1}{i} \equiv (-1)^i$ modulo p and $ph = 0$. \square

8. ROST'S THEOREM

We have proved in Lemma 3.4 that if D is a central division algebra, then $\partial_G(q_G) \neq 0$ in $\mathrm{CH}^{p+1}(G)$. This result is strengthened in Theorem 8.2 below.

Lemma 8.1. *If there is an element $h \in \mathrm{CH}^{p+1}(G)$ such that $h^{p-1} \neq 0$, then $\partial_G(q_G)^{p-1} \neq 0$.*

Proof. By Lemma 3.4, h is a multiple of $\partial_G(q_G)$. \square

Theorem 8.2 (M. Rost). *Let D be a central division algebra of degree p , $G = \mathbf{SL}_1(D)$. Then $\partial_G(q_G)^{p-1} \neq 0$ in $\mathrm{CH}^{p^2-1}(G) = \mathrm{CH}_0(G)$.*

Proof. Case 1: Assume first that $\mathrm{char}(F) = 0$, F contains a primitive p -th root of unity and D is a cyclic algebra, i.e., $D = (a, b)_F$ for some $a, b \in F^\times$.

Let $c \in F^\times$ be an element such that the symbol

$$u := (a, b, c) \in H_{\acute{e}t}^3(F, \mathbb{Z}/p\mathbb{Z}(3)) \simeq H_{\acute{e}t}^3(F, \mathbb{Z}/p\mathbb{Z}(2))$$

is nontrivial modulo p . Consider a norm variety X of u .

Then u defines a *basic correspondence* in the cokernel of the homomorphism

$$\mathrm{CH}^{p+1}(X) \rightarrow \mathrm{CH}^{p+1}(X \times X)$$

given by the difference of the pull-backs with respect to the projections. A representative in $\mathrm{CH}^{p+1}(X \times X)$ of the basic correspondence is a *special correspondence*. Let $z \in \mathrm{CH}^{p+1}(X_{F(X)})$ be its pull-back. The modulo p degree

$$c(X) := \deg(z^{p-1}) \in \mathbb{Z}/p\mathbb{Z}$$

is independent of the choice of the special correspondence. The construction of $c(X)$ is natural with respect to morphisms of norm varieties (see [24]).

It is shown in [24] that there is an X such that $c(X) \neq 0$. We claim that $c(X') \neq 0$ for every norm variety X' of u . As $F(X')$ splits u and X is p -generic, X has a closed point over $F(X')$ of degree prime to p , or equivalently, there is a prime correspondence $X' \rightsquigarrow X$ of multiplicity prime to p . Resolving singularities, we get a smooth complete variety X'' together with the two morphisms $f : X'' \rightarrow X$ of degree prime to p and $g : X'' \rightarrow X'$. It follows by [28, Corollary 1.19] that X'' is a norm variety of u . Moreover, $c(X'') = \deg(f)c(X) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. As $c(X'') = \deg(g)c(X')$, $c(X')$ is also nonzero. The claim is proved.

Let X be a smooth compactification of the G -torsor E given by the equation $\mathrm{Nrd} = t$ over the rational function field $L = F(t)$ given by a variable t . By the above, since $\{a, b, t\} \neq 0$, we have an element $z \in \mathrm{CH}^{p+1}(X_{L(X)})$ such that $\deg(z^{p-1}) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$. The torsor E is trivial over $L(X)$, i.e. $E_{L(X)} \simeq G_{L(X)}$. Then the restriction of z to the torsor gives an element $y \in \mathrm{CH}^{p+1}(G_{L(X)})$ with $y^{p-1} \neq 0$. The field extension $L(X)/F$ is purely transcendental. By Section 4 and Lemma 7.2, every specialization homomorphism $\sigma : \mathrm{CH}^{p-1}(G_{L(X)}) \rightarrow \mathrm{CH}^{p-1}(G)$ is an isomorphism taking the class of the identity to the class of the identity. Specializing, we get an element $h \in \mathrm{CH}^{p+1}(G)$ with $h^{p-1} \neq 0$. It follows from Lemma 8.1 that $\partial_G(q_G)^{p-1} \neq 0$.

Case 2: Suppose that $\mathrm{char}(F) = 0$ but F may not contain p -th roots of unity and D is an arbitrary division algebra of degree p (not necessarily cyclic). There is a finite field extension K/F of degree prime to p containing a primitive p -th root of unity and such that the algebra $D \otimes_F K$ is cyclic (and still nonsplit). By Case 1, $\partial_G(q_G)_K^{p-1} \neq 0$ over K . Therefore $\partial_G(q_G)^{p-1} \neq 0$.

Case 3: F is an arbitrary field. Choose a field L of characteristic zero and a central simple algebra D' of degree p over L as in Section 4 and let $G' = \mathbf{SL}_1(D')$. By Case 2, there is an element $h' \in \mathrm{CH}^{p+1}(G')$ such that $(h')^{p-1} \neq 0$. Applying a specialization σ (see Section 4), we have $h^{p-1} \neq 0$ for $h = \sigma(h')$. By Lemma 8.1 again, $\partial_G(q_G)^{p-1} \neq 0$. \square

Let D be a central division algebra of degree p over F and X a smooth compactification of G . Let $\bar{h} \in \mathrm{CH}^{p+1}(X)$ be an element such that $\bar{h}|_G = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$. Let $i = 0, 1, \dots, p-1$. The element \bar{h}^i defines the following two morphisms of Chow motives:

$$f_i : M(X) \rightarrow \mathbb{Z}((p+1)i), \quad g_i : \mathbb{Z}((p+1)(p-1-i)) \rightarrow M(X).$$

Let

$$R = \mathbb{Z} \oplus \mathbb{Z}(p+1) \oplus \mathbb{Z}(2p+2) \oplus \cdots \oplus \mathbb{Z}(p^2-1).$$

We thus have the following two morphisms:

$$f : M(X) \rightarrow R, \quad g : R \rightarrow M(X).$$

The composition $f \circ g$ is c times the identity, where $c = \deg \bar{h}^{p-1}$. As c is prime to p by Theorem 8.2, switching to the *Chow motives with coefficients in $\mathbb{Z}_{(p)}$* , we have a decomposition

$$(8.3) \quad M(X) = R \oplus N$$

for some motive N .

9. THE CATEGORY OF D -MOTIVES

Let D be a central simple algebra of prime degree p over F . For a field extension L/F , let $N_i^D(L)$ be the subgroup of the Milnor K -group $K_i^M(L)$ generated by the norms from finite field extensions of L that split the algebra D .

Consider the Rost cycle module (see [25]):

$$L \mapsto K_*^D(L) := K_*^M(L)/N_*^D(L),$$

and the corresponding cohomology theory with the ‘‘Chow groups’’

$$\mathrm{CH}_D^i(X) := A^i(X, K_i^D).$$

Note that $\mathrm{CH}_D^i(X) = 0$ if D is split over $F(x)$ for all points $x \in X$.

Let $S = \mathrm{SB}(D)$ be the Severi-Brauer variety of right ideals of D of dimension p . We have $\dim S = p - 1$.

Lemma 9.1. *For a variety X over F , the group $\mathrm{CH}_D(X)$ is naturally isomorphic to the cokernel of the push-forward homomorphism $pr_* : \mathrm{CH}(X \times S) \rightarrow \mathrm{CH}(X)$ given by the projection $pr : X \times S \rightarrow X$.*

Proof. The composition

$$\mathrm{CH}(X \times S) \xrightarrow{pr_*} \mathrm{CH}(X) \rightarrow \mathrm{CH}_D(X)$$

factors through the trivial group $\mathrm{CH}_D(X \times S)$ and therefore, is zero. This defines a surjective homomorphism

$$\alpha : \mathrm{Coker}(pr_*) \rightarrow \mathrm{CH}_D(X).$$

The inverse map is obtained by showing that the quotient map $\mathrm{CH}(X) \rightarrow \mathrm{Coker}(pr_*)$ factors through $\mathrm{CH}_D(X)$.

The kernel of the homomorphism $\mathrm{CH}(X) \rightarrow \mathrm{CH}_D(X)$ is generated by $[x]$ with $x \in X$ such that the algebra $D_{F(x)}$ is split and by $p[x]$ with arbitrary $x \in X$. The fiber of pr over x has a rational point y in the first case and a degree p closed point y in the second. The generators are equal to $pr_*([y])$ in both cases. It follows that they vanish in $\mathrm{Coker} pr_*$. \square

Let $G = \mathrm{SL}_1(D)$.

Corollary 9.2. *The natural map $\mathrm{CH}^i(G) \rightarrow \mathrm{CH}_D^i(G)$ is an isomorphism for all $i > 0$.*

Proof. The algebra D is split over S . More precisely, $D_X = \text{End}_X(I^\vee)$ for the rank p canonical vector bundle I over S (see [27, Lemma 2.1.4]). By [29, Theorem 4.2], the pull-back homomorphism $\text{CH}^*(S) \rightarrow \text{CH}^*(G \times S)$ is an isomorphism. Therefore, $\text{CH}^j(G \times S) = 0$ if $j > p - 1 = \dim(S)$. \square

Let X be a smooth compactification of G . Write $X^k = X \times X \times \cdots \times X$ (k times).

Lemma 9.3. *The restriction homomorphism $\text{CH}_D^*(X^k) \rightarrow \text{CH}_D^*(G^k)$ is an isomorphism.*

Proof. Let $Z = X^k \setminus G^k$. By Lemma 7.1, the residue field of every point in Z splits D , hence $\text{CH}_D(Z) = 0$. The statement follows from the exactness of the localization sequence

$$\text{CH}_D(Z) \rightarrow \text{CH}_D(X^k) \rightarrow \text{CH}_D(G^k) \rightarrow 0. \quad \square$$

It follows from Lemma 9.3 and Corollary 9.2 that $\text{CH}_D^i(X) \simeq \text{CH}^i(G)$ for $i > 0$.

Consider the category of motives of smooth complete varieties over F associated to the cohomology theory $\text{CH}_D^*(X)$ (see [21]). Write $M^D(X)$ for the motive of a smooth complete variety X . We call $M^D(X)$ the D -motive of X . Recall that the group of morphisms between $M^D(X)$ and $M^D(Y)$ for Y of pure dimension d is equal to $\text{CH}_D^d(X \times Y)$. Let \mathbb{Z}^D the motive of the point $\text{Spec } F$.

Recall that we write $M(X)$ for the usual Chow motive of X . We have a functor $N \mapsto N^D$ from the category of Chow motives to the category of D -motives.

Proposition 9.4. *Let N be a Chow motive. Then $N^D = 0$ if and only if N is isomorphic to a direct summand of $N \otimes M(S)$.*

Proof. As $M^D(S) = 0$, we have $N^D = 0$ if N is isomorphic to a direct summand of $N \otimes M(S)$.

Conversely, suppose $N^D = 0$. Let $N = (X, \rho)$, where X is a smooth complete variety of pure dimension d and $\rho \in \text{CH}^d(X \times X)$ is a projector. By Lemma 9.1, we have $\rho = f_*(\theta)$ for some $\theta \in \text{CH}^{d+p-1}(X \times (X \times S))$, where $f : X \times X \times S \rightarrow X \times X$ is the projection. Then

$$f_*((\rho \otimes \text{id}_S) \circ \theta \circ \rho) = \rho$$

and $(\rho \otimes \text{id}_S) \circ \theta \circ \rho$ can be viewed as a morphism $N \rightarrow N \otimes M(S)$ splitting on the right the natural morphism $N \otimes M(S) \rightarrow N$. \square

The morphisms f and g in Section 8 give rise to the morphisms $f^D : M^D(X) \rightarrow R^D$ and $g^D : R^D \rightarrow M^D(X)$ of D -motives.

Proposition 9.5. *The morphism $f^D : M^D(X) \rightarrow R^D$ is an isomorphism in the category of D -motives.*

Proof. As $\text{CH}_D^{p^2-1}(X \times X) \simeq \text{CH}_D^{p^2-1}(G \times G)$ by Lemma 9.3, the composition $g^D \circ f^D$ is multiplication by $c \in \mathbb{Z}$ from Proposition 7.4. By Theorem 8.2, c is not divisible by p . Finally, $p \text{CH}_D(G \times G) = 0$. \square

If D is a central division algebra, it follows from Proposition 9.5 and Corollary 9.2 that for every $i > 0$,

$$(9.6) \quad \text{CH}^i(G) = \text{CH}_D^i(X) = \text{CH}_D^i(R) = \begin{cases} (\mathbb{Z}/p\mathbb{Z})h^j, & \text{if } i = (p+1)j \leq p^2 - 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $h = \partial_G(q_G)$.

We can compute the Chow ring of G .

Theorem 9.7. *Let D be a central division algebra of prime degree p , $G = \mathbf{SL}_1(D)$ and $h = \partial_G(q_G) \in \mathrm{CH}^{p+1}(G)$. Then*

$$\mathrm{CH}(G) = \mathbb{Z} \cdot 1 \oplus (\mathbb{Z}/p\mathbb{Z})h \oplus (\mathbb{Z}/p\mathbb{Z})h^2 \oplus \cdots \oplus (\mathbb{Z}/p\mathbb{Z})h^{p-1}.$$

Proof. If F is a perfect field, G admits a smooth compactification X by Proposition 6.1. The statement follows from (9.6). In general, we proceed as follows.

A variety X over F is called *D -complete* if there is a compactification \bar{X} of X such that D is split by the residue field of every point in $\bar{X} \setminus X$. Note that the restriction map $\mathrm{CH}_D(\bar{X} \times U) \rightarrow \mathrm{CH}_D(X \times U)$ is an isomorphism for every variety U . By the proof of Lemma 7.1, G is a D -complete variety.

We extend the category of D -motives by adding the motives $M^D(X)$ of smooth D -complete varieties X . If X and Y are two smooth D -complete varieties with Y equidimensional of dimension d , we define $\mathrm{Hom}(M^D(X), M^D(Y)) := \mathrm{CH}_D^d(X \times Y)$. The composition homomorphism

$$\mathrm{CH}_D^d(X \times Y) \otimes \mathrm{CH}_D^r(Y \times Z) \rightarrow \mathrm{CH}_D^r(X \times Z)$$

is given by

$$\alpha \otimes \beta \mapsto p_{13*}(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)),$$

where p_{ij} are the three projections of $X \times Y \times Z$ on X , Y and Z , and the push-forward map p_{13*} is defined as the composition

$$p_{13*} : \mathrm{CH}_D^{d+r}(X \times Y \times Z) \simeq \mathrm{CH}_D^{d+r}(X \times \bar{Y} \times Z) \rightarrow \mathrm{CH}_D^r(X \times Z).$$

Here \bar{Y} is a compactification of Y satisfying the condition in the definition of a D -complete variety and the second map is the push-forward homomorphism for the proper projection $X \times \bar{Y} \times Z \rightarrow X \times Z$.

By Proposition 7.4 and Theorem 8.2, the powers of $h = \partial_G(q_G)$ yield the following decomposition of D -motives (with coefficients in $\mathbb{Z}_{(p)}$):

$$M^D(G) \simeq \mathbb{Z}^D \oplus \mathbb{Z}^D(p+1) \oplus \cdots \oplus \mathbb{Z}^D(p^2-1).$$

The result follows as $\mathrm{CH}^i(G) = \mathrm{CH}_D^i(G)$ for $i > 0$ by Corollary 9.2. \square

10. MOTIVIC DECOMPOSITION OF COMPACTIFICATIONS OF $\mathbf{SL}_1(D)$

Let D be a central division F -algebra of degree a power of a prime p and $S = \mathrm{SB}(D)$. We work with motives with $\mathbb{Z}_{(p)}$ -coefficients in this section.

Proposition 10.1. *Let X be a connected smooth complete variety over F such that the motive of X is split over every splitting field of D and D is split over $F(X)$. Then the motive of X is a direct sum of shifts of the motive of S .*

Proof. Note that the variety X is *generically split*, that is, its motive is split over $F(X)$. In particular, X satisfies the nilpotence principle, [30, Proposition 3.1]. Therefore, it suffices to prove the result for motives with coefficients in \mathbb{F}_p : any lifting of an isomorphism of the motives with coefficients in \mathbb{F}_p to the coefficients $\mathbb{Z}_{(p)}$ will be an isomorphism since it will become an isomorphism over any splitting field of D .

For \mathbb{F}_p -coefficients, here is the argument. The (isomorphism class of the) upper motive $U(X)$ is well-defined and, by the arguments as in the proof of [18, Theorem 3.5], the motive of X is a sum of shifts of $U(X)$. Besides, $U(X) \simeq U(S)$, cf. [18, Corollary 2.15]. Finally, $U(S) = M(S)$ because the motive of S is indecomposable, [18, Corollary 2.22]. \square

From now on, the degree of the division algebra D is p . Recall that we work with motives with coefficients in $\mathbb{Z}_{(p)}$. So, we set

$$R = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}(p+1) \oplus \mathbb{Z}_{(p)}(2p+2) \oplus \cdots \oplus \mathbb{Z}_{(p)}(p^2-1)$$

now.

Theorem 10.2. *Let F be a field, D a central division F -algebra of prime degree p , $G = \mathbf{SL}_1(D)$, X a smooth compactification of G , and $M(X)$ its Chow motive with $\mathbb{Z}_{(p)}$ -coefficients. Assume that $M(X)$ is split over every splitting field of D (see Example 6.9). Then the motive $M(X)$ (over F) is isomorphic to the direct sum of R and a direct sum of shifts of $M(S)$.*

Proof. By (8.3), $M(X) = R \oplus N$ for a motive N and by Proposition 9.5, $N^D = 0$. It follows from Proposition 9.4 that N is isomorphic to a direct summand of $N \otimes M(S)$. In its turn, $N \otimes M(S)$ is a direct summand of $M(X \times S)$. In view of Proposition 10.1, $M(X \times S)$ is a direct sum of shifts of $M(S)$. By the uniqueness of the decomposition [6, Corollary 35] and indecomposability of $M(S)$ [18, Corollary 2.22], the motive N is a direct sum of shifts of $M(S)$. \square

Theorem 10.3. *Let E be an $\mathbf{SL}_1(D)$ -torsor and X a smooth compactification of E such that the motive $M(X)$ is split over every splitting field of D (see Example 6.9). Then X satisfies the nilpotence principle. Besides, the motive $M(X)$ is isomorphic to the direct sum of the Rost motive \mathcal{R} of X and a direct sum of shifts of $M(S)$. The above decomposition is the unique decomposition of $M(X)$ into a direct sum of indecomposable motives.*

Proof. By saying that X satisfies the nilpotence principle, we mean that it does it for any coefficient ring, or, equivalently, for \mathbb{Z} -coefficients. However, since the integral motive of X is split over a field extension of degree p , it suffices to check that X satisfies the nilpotence principle for $\mathbb{Z}_{(p)}$ -coefficients, where we can simply refer to [9, Theorem 92.4] and Theorem 10.2 (applied to X over $F(X)$).

It follows that it suffices to get the motivic decomposition of Theorem 10.3 for $\mathbb{Z}_{(p)}$ -coefficients replaced by \mathbb{F}_p -coefficients. For \mathbb{F}_p -coefficients we use the following modification of [17, Proposition 4.6]:

Lemma 10.4. *Let S be a geometrically irreducible variety with the motive satisfying the nilpotence principle and becoming split over an extension of the base field. Let M be a summand of the motive of some smooth complete variety X . Assume that there exists a field extension L/F and an integer $i \in \mathbb{Z}$ such that the change of field homomorphism $\mathrm{Ch}(X_{F(S)}) \rightarrow \mathrm{Ch}(X_{L(S)})$ is surjective and the motive $M(S)(i)_L$ is an indecomposable summand of M_L . Then $M(S)(i)$ is an indecomposable summand of M .*

Proof. It was assumed in [17, Proposition 4.6] that the field extension $L(S)/F(S)$ is purely transcendental. But this assumption was only used to ensure that the change of field homomorphism $\mathrm{Ch}(X_{F(S)}) \rightarrow \mathrm{Ch}(X_{L(S)})$ is surjective. Therefore the old proof works. \square

We apply Lemma 10.4 to our S and X (with $L = F(X)$). First we take $M = M(X)$ and using Theorem 10.2, we extract from $M(X)$ our first copy of shifted $M(S)$. Then we apply Lemma 10.4 again, taking for M the complementary summand of $M(X)$. Continuing this way, we eventually extract from $M(X)$ the same number of (shifted) copies of $M(S)$ as we have by Theorem 10.2 over $F(X)$. Let \mathcal{R} be the remaining summand of $M(X)$. By uniqueness of decomposition, we have $\mathcal{R}_{F(X)} \simeq R$ so that \mathcal{R} is the Rost motive. It is indecomposable (over F), because the degree of every closed point on X is divisible by p .

The uniqueness of the constructed decomposition follows by [1, Theorem 3.6 of Chapter I], because the endomorphism rings of $M(S)$ and of \mathcal{R} are local (see [19, Lemma 3.3]). \square

Remark 10.5. If X is an equivariant toroidal compactification of $\mathbf{SL}_1(D)$, the number of motives $M(S)$ in the decomposition of Theorem 10.3 is equal to $s(p-1)! - 1$, where s is the number of cones of maximal dimension in the fan of the associated toric variety (see Theorem 6.5).

Example 10.6. Let X be the (non-toroidal) equivariant compactification of $\mathbf{SL}_1(D)$ with $p = 3$ considered in [26]. Since $P_X(t) = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$, we have

$$M(X) \simeq \mathcal{R} \oplus M(S)(1) \oplus M(S)(2) \oplus M(S)(3) \oplus M(S)(4) \oplus M(S)(5).$$

Example 10.7. Let X be the toroidal equivariant compactification of $\mathbf{SL}_1(D)$ with $p = 3$ considered in Example 6.7 in the split case. We have

$$M(X) \simeq \mathcal{R} \oplus M(S)(1)^{\oplus 3} \oplus M(S)(2)^{\oplus 5} \oplus M(S)(3)^{\oplus 7} \oplus M(S)(4)^{\oplus 5} \oplus M(S)(5)^{\oplus 3}.$$

Corollary 10.8. *Let E be a nonsplit $\mathbf{SL}_1(D)$ -torsor. Assume that $\mathrm{char} F = 0$. Then $\mathrm{CH}(E) = \mathbb{Z}$.*

Proof. Since $p\mathrm{CH}^{>0}(E) = 0$, it suffices to prove that $\mathrm{CH}^{>0}(E) = 0$ for \mathbb{Z} -coefficients replaced by $\mathbb{Z}_{(p)}$ -coefficients. Below CH stands for Chow group with $\mathbb{Z}_{(p)}$ -coefficients.

We prove that $\mathrm{CH}(E) = \mathrm{CH}_D(E)$ by the argument of Corollary 9.2. It remains to show that $\mathrm{CH}_D^{>0}(E) = 0$.

Let X be a compactification of E as in Theorem 10.3. Since $\mathrm{CH}_D(X)$ surjects onto $\mathrm{CH}_D(E)$ and $\mathrm{CH}_D(S) = 0$, it suffices to check that $\mathrm{CH}_D^{>0}(\mathcal{R}) = 0$. Actually, we have $\mathrm{CH}_D(\mathcal{R}) \simeq \mathrm{CH}_D(E)$ (see Section 9). Moreover, the D -motive of \mathcal{R} is isomorphic to $M^D(E)$.

The Chow group $\mathrm{CH}^{>0}(\mathcal{R})$ has been computed in [19, Appendix RM] (the characteristic assumption is needed here). The generators of the torsion part, provided in [19, Proposition SC.21], vanish in $\mathrm{CH}_D(\mathcal{R})$ by construction. The remaining generators are norms from a degree p splitting field of D so that they vanish in $\mathrm{CH}_D(\mathcal{R})$, too. Hence $\mathrm{CH}_D^{>0}(\mathcal{R}) = 0$ as required. \square

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