NEGLIGIBLE DEGREE TWO COHOMOLOGY OF FINITE GROUPS

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Abstract. For a finite group G, a G-module M and a field F, an element $u \in H^d(G,M)$ is negligible over F if for each field extension L/F and every group homomorphism $\mathrm{Gal}(L_{\mathrm{sep}}/L) \to G$, u belongs to the kernel of the induced homomorphism $H^d(G,M) \to H^d(L,M)$. We determine the group of negligible elements in $H^2(G,M)$ for every abelian group M with trivial G-action.

1. Introduction

The notion of negligible cohomology was introduced by J-P. Serre in [7] (see also [2, Part I, §26]). Let G be a finite group, M a G-module and F a field. A (continuous) group homomorphism $j : \Gamma_L = \mathrm{Gal}(L_{\mathrm{sep}}/L) \to G$ from the absolute Galois group Γ_L of a field extension L of F to G yields a homomorphism

$$j^* : H^d(G,M) \to H^d(L,M)$$

of cohomology groups for every $d \geqslant 0$. An element $u \in H^d(G,M)$ is called *negligible over* F if $u \in \mathrm{Ker}(j^*)$ for all field extensions L/F and all j. All negligible over F elements form a subgroup

$$H^d(G,M)_{\mathrm{neg}} = H^d(G,M)_{\mathrm{neg},F} \subset H^d(G,M).$$

Examples 1.1. 1) Negligible cohomology elements are related to the *embedding problem.* Let K/F be a finite Galois field extension with $G = \mathrm{Gal}(K/F)$. Let

$$1 \to M \to G' \xrightarrow{f} G \to 1 \tag{1.2}$$

be an exact sequence of finite groups with M abelian. The conjugation G'-action on M makes M a G-module. The embedding problem for the exact sequence (1.2) and field extension K/F is to find a Galois G'-algebra K' over F such that the restriction map $G' = \mathrm{Gal}(K'/F) \to \mathrm{Gal}(K/F) = G$ coincides with f. Equivalently, one needs to find a lifting $\Gamma_F \to G'$ of the homomorphism $\Gamma_F \to G$ corresponding to the extension K/F.

Let $u \in H^2(G,M)$ be the class of the exact sequence (1.2) and let $j : \Gamma_L \to G$ be the group homomorphism given by a field extension L/F. Then j extends to a homomorphism $\Gamma_L \to G'$ if and only if the pull-back of the sequence (1.2) under j is split. The latter is equivalent to the triviality of the image of u under $j^* : H^2(G,M) \to H^2(L,M)$. In other words, the class u is negligible if and only if all embedding problems for the exact sequence (1.2) and all G-Galois field extensions L'/L of fields containing F have solutions.

2) Let M be an abelian group which we view as a module over any profinite group with trivial action. The cohomology group $H^d(F,M) = H^d(\Gamma_F,M)$ is the colimit of the

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groups $H^d(G,M)$ over all finite discrete factor groups G of Γ_F. The group $H^d(G,M)_{\rm neg}$ is contained in the kernel of the natural homomorphism $H^d(G,M)\to H^d(F,M)$.

3) Negligible cohomology elements of G are related to the *invariants* of G as follows. Let M be an abelian group with trivial group action. Write $\mathrm{Inv}^d(G,M)$ for the group of degree d (normalized) invariants of G with values in M over a field F (for the definition of the invariant see [2]). We have a homomorphism

$$\mathrm{inv} : H^d(G,M)\to \mathrm{Inv}^d(G,M),$$

taking an element $u\in H^d(G,M)$ to the invariant sending the class of a G-algebra N over a field extension L of F (that is a G-torsor over $\operatorname{Spec} L$) to the image of u under the homomorphism

$$j^* : H^d(G,M)\to H^d(L,M)$$

with respect to the natural group homomorphism $j:\Gamma_L\to G$. By the very definition of negligible elements, $H^d(G,M)_{\rm neg}=\mathrm{Ker}(\mathrm{inv})$.

Let M be a G-module. The groups $H^d(G,M)_{\rm neg}$ are trivial if $d=0$ or 1 (see Corollary 2.2). In the present paper we determine the group $H^2(G,M)_{\rm neg}$ for an arbitrary finite group G and arbitrary abelian group M with trivial G-action. In Section 2 we reduce the problem to the case $M=\mathbb{Z}/p^s\mathbb{Z}$ for a prime integer p and $\mathrm{char}(F)\neq p$.

In Section 3 we consider the case when the base field F contains sufficiently many roots of unity. We identify $\mathbb{Z}/m\mathbb{Z}$ with the group μ_m of m-th roots of unity and compute $H^2(G,\mu_m)_{\rm neg}$ using the Brauer group considerations.

Let p^t be the order of the group of p-primary roots of unity in the field $F(\xi_p)$. In Theorem 4.1 we determine the group $H^2(G,\mathbb{Z}/p^s\mathbb{Z})_{\rm neg}$ in all cases except when $p=2$ and $t=1$. The group of negligible elements in $H^2(G,\mathbb{Z}/p^s\mathbb{Z})$ depends on the character group G^* and the integers p^s and t.

The exceptional case $p=2$ and $t=1$ is more delicate and it requires some computations in the Brauer group. Let $2^{t'}$ be the order of the 2-primary roots of unity in the field $F(\sqrt{-1})$. The group of negligible elements in $H^2(G,\mathbb{Z}/2^s\mathbb{Z})$ depends on the group G^* and the integers s and t' (Theorem 5.2).

We use the following notations in the paper.

F is the base field, $F_{\rm sep}$ is a separable closure of F, $\Gamma_F=\mathrm{Gal}(F_{\rm sep}/F)$ is the *absolute* Galois group of F;

μ_m is the group of m-th roots of unity in $F_{\rm sep}$, $\mu_m(F)=\mu_m\cap F^\times$, ξ_m is a generator of μ_m;

For an abelian group A write $A_{\rm tors}$ for the torsion part of A and set $A[q]:=\mathrm{Ker}(A\xrightarrow{q}A)$, where q is an integer; $A[p^\infty]:=\cup_{s>0}A[p^s]$, where p is a prime integer;

$H^d(F,M):=H^d(\Gamma_F,M)$ for a (discrete) Γ_F-module (Galois module) M.

2. Preliminary results

Let V be a faithful (finite dimensional) representation of the group G over F. The group G acts on the field $F(V)$ of rational functions on V over F making $F(V)/F(V)^G$ a Galois G-extension. The following proposition shows that in the definition of negligible elements it suffices to consider only surjective group homomorphisms j and, moreover, only one (generic) Galois field extension $F(V)/F(V)^G$.

Proposition 2.1. *Let G be a finite group, M a G-module, $u \in H^d(G,M)$ and F a field. Let V be a faithful representation of G. The following conditions are equivalent:*

1. *u is negligible over F, i.e., $u \in H^d(G,M)_{\mathrm{neg}}$;*
2. *$j^*(u) = 0$ for all field extensions L/F and every surjective group homomorphism $j : \Gamma_L \to G$;*
3. *If $K = F(V)^G$ and $j_K : \Gamma_K \to G$ is given by the Galois G-extension $F(V)/K$, then $j_K^*(u) = 0$ in $H^d(K,M)$.*

Proof. (1) ⇒ (2) is trivial.
(2) ⇒ (3) is clear since the map j_K in (3) is surjective.
(3) ⇒ (1): Let N/L be a Galois G-algebra for a field extension L/F and $j : \Gamma_L \to G$ a group homomorphism. We need to show that $j^*(u) = 0$. As the natural homomorphism $H^d(L,M) \to H^d(L(t),M)$, where $L(t)$ is the rational function field over L, is injective, replacing F by $F(t)$ and L by $L(t)$ if necessary, we may assume that the field L is infinite.

The scheme $\mathrm{Spec}(K)$ is the limit of the family of varieties U/G, where $U \subset V$ is a nonempty open G-invariant subscheme such that the morphism $U \to U/G$ is a G-torsor. For every such U write

$$i_U : H^d(G,M) \to H^d_{ét}(U/G,M)$$

for the edge homomorphism in the Hochschild-Serre spectral sequence [4, Ch. III, Th. 2.20]

$$E_2^{p,q} = H^p(G, H^q_{ét}(U,M)) \Rightarrow H^{p+q}_{ét}(U/G,M).$$

Since $j_K^*(u) = 0$ and the étale cohomology commutes with limits, there is U such that $i_U(u) = 0$. As L is infinite, by [2, Part I, Ch.1, §5], there is a morphism $k : \mathrm{Spec}(L) \to U/G$ such that $\mathrm{Spec}(N) \to \mathrm{Spec}(L)$ is the pull-back of $U \to U/G$ with respect to k. Then the composition

$$H^d(G,M) \xrightarrow{i_U} H^d_{ét}(U/G,M) \xrightarrow{k^*} H^d(L,M)$$

coincides with j^*. Since $i_U(u) = 0$ we have $j^*(u) = 0$. □

Corollary 2.2. (*cf.,* [7] *and* [5, Proposition 4.5])

1. *In the notation of the proposition,*
$$H^d(G,M)_{\mathrm{neg}} = \mathrm{Ker}\big(H^d(G,M) \xrightarrow{j^*} H^d(F(V)^G,M)\big).$$
2. *The group $H^d(G,M)_{\mathrm{neg}}$ is trivial if $d \leqslant 1$.*

Proof. (1): This follows immediately from Proposition 2.1
(2): As j is surjective, the inflation map j^* is injective if $d \leqslant 1$. □

In the following proposition we collect some functorial properties of negligible elements.

Proposition 2.3. *Let L/F be a field extension, G a finite group, M a G-module and $f : H \to G$ a homomorphism of finite groups. Then*

1. *The map $f^* : H^d(G,M) \to H^d(H,M)$ takes $H^d(G,M)_{\mathrm{neg}}$ into $H^d(H,M)_{\mathrm{neg}}$;*
2. *$H^d(G,M)_{\mathrm{neg}} \subset H^d(G,M)_{\mathrm{neg},L}$;*

(3) *If L/F is finite, then* $[L:F]\cdot H^d(G,M)_{\mathrm{neg},L}\subset H^d(G,M)_{\mathrm{neg}}$*;*
(4) *If $\alpha: M\to N$ is a G-module homomorphism, then the map $\alpha^*: H^d(G,M)\to H^d(G,N)$ takes $H^d(G,M)_{\mathrm{neg}}$ into $H^d(G,N)_{\mathrm{neg}}$.*

Proof. (1): Let $j:\Gamma_L\to H$ be a group homomorphism for a field extension L of F and $u\in H^d(G,M)_{\mathrm{neg}}$. Then $j^*(f^*(u))=(f\circ j)^*(u)=0$, hence $f^*(u)\in H^d(H,M)_{\mathrm{neg}}$.
(2): Let $K=F(V)^G$ as in Proposition 2.1(3) and set $KL:=L(V)^G$. Let $u\in H^d(G,M)_{\mathrm{neg}}$. By definition, $j_K^*(u)=0$ in $H^d(K,M)$. It follows that $j_{KL}^*(u)=\mathrm{res}_{KL/K}\circ j_K^*(u)=0$ in $H^d(KL,M)$, hence $u\in H^d(G,M)_{\mathrm{neg},L}$ by Corollary 2.2(1).
(3): If L/F is finite and $u\in H^d(G,M)_{\mathrm{neg},L}$, then $\mathrm{res}_{KL/K}\circ j_K^*(u)=j_{KL}^*(u)=0$. Applying the correstriction homomorphism, we get

$$[L:F]\cdot j_K^*(u)=\mathrm{cor}_{KL/K}\circ\mathrm{res}_{KL/K}\circ j_K^*(u)=\mathrm{cor}_{KL/K}\circ j_{KL}^*(u)=0,$$

therefore, $[L:F]\cdot u\in H^d(G,M)_{\mathrm{neg}}$.
(4) is clear. □

Corollary 2.4. *If p is a prime integer such that* $\mathrm{char}(F)\neq p$ *and $p^s\cdot M=0$ for some s, then*

$$H^d(G,M)_{\mathrm{neg}}=H^d(G,M)_{\mathrm{neg},F(\xi_p)}.$$

Proof. Indeed, the degree $[F(\xi_p):F]$ is prime to p. □

From now on assume that M is an abelian group with trivial G-action.

Lemma 2.5. *If M is a torsion free abelian group then $H^2(G,M)_{\mathrm{neg}}=0$.*

Proof. The exact sequence $0\to M\to M\otimes\mathbb{Q}\to M\otimes(\mathbb{Q}/\mathbb{Z})\to 0$ yields the isomorphisms

$$H^2(G,M)\simeq H^1(G,M\otimes(\mathbb{Q}/\mathbb{Z})),\quad H^2(L,M)\simeq H^1(L,M\otimes(\mathbb{Q}/\mathbb{Z}))$$

for every field L. Therefore, $H^2(G,M)_{\mathrm{neg}}\simeq H^1(G,M\otimes(\mathbb{Q}/\mathbb{Z}))_{\mathrm{neg}}=0$ by Corollary 2.2(2). □

The following proposition reduces the computation of negligible elements to the case when M is a torsion group.

Proposition 2.6. *Let M be an abelian group. Then the natural map*

$$H^2(G,M_{\mathrm{tors}})_{\mathrm{neg}}\to H^2(G,M)_{\mathrm{neg}}$$

is an isomorphism.

Proof. If Γ is a profinite group and N is a torsion free abelian group, then $H^1(\Gamma,N)=\mathrm{Hom}(\Gamma,N)=0$ since the image of every (continuous) homomorphism $\Gamma\to N$ is finite. Since the factor group M/M_{tors} is torsion free, it follows that the natural homomorphism $H^2(\Gamma,M_{\mathrm{tors}})\to H^2(\Gamma,M)$ is injective. Therefore, both horizontal maps in the commutative diagram

$$\begin{array}{ccc} H^2(G,M_{\mathrm{tors}}) & \longrightarrow & H^2(G,M) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^2(L,M_{\mathrm{tors}}) & \longrightarrow & H^2(L,M) \end{array}$$

are injective for every field extension L/F and a group homomorphism $j : \Gamma_L \to G$.

Let $u \in H^2(G, M)_{\mathrm{neg}}$. By Lemma 2.5, the group $H^2(G, M/M_{\mathrm{tors}})_{\mathrm{neg}}$ is trivial, hence u comes from an element $w \in H^2(G, M_{\mathrm{tors}})$. The diagram chase shows that $w \in H^2(G, M_{\mathrm{tors}})_{\mathrm{neg}}$, i.e., the map in the statement of the proposition is surjective. □

If $M = \operatorname{colim} M_i$ is a directed colimit of abelian groups M_i then since cohomology of profinite groups commute with directed colimits, we have

$$H^2(G, M)_{\mathrm{neg}} = \operatorname{colim} H^2(G, M_i)_{\mathrm{neg}}.$$

Since every torsion abelian group is the union of finite groups and every finite group is a direct sum of primary cyclic groups, Proposition 2.6 shows that in order to compute $H^2(G, M)_{\mathrm{neg}}$ for an arbitrary abelian group M, it suffices to determine the structure of $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}}$ for all primes p and positive integers s.

If $\mathrm{char}(F) = p > 0$, then $H^d(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}} = H^d(G, \mathbb{Z}/p^s\mathbb{Z})$ since $H^2(L, \mathbb{Z}/p^s\mathbb{Z}) = 0$ for every field extension L/F (see [6, Chapter II, Proposition 4]). In what follows when computing the group $H^d(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}}$ we will assume that $\mathrm{char}(F) \neq p$.

2a. **Cyclic algebras.** Let F be a field and $\Gamma_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$. Write $(\Gamma_F)^*$ for the group of (continuous) characters $\Gamma_F \to \mathbb{Q}/\mathbb{Z}$, i.e.,

$$(\Gamma_F)^* = \mathrm{Hom}(\Gamma_F, \mathbb{Q}/\mathbb{Z}) = H^1(F, \mathbb{Q}/\mathbb{Z}) = H^2(F, \mathbb{Z}).$$

For a character $x \in (\Gamma_F)^*$ and an element $a \in F^\times$ denote by (x, a) the class of the corresponding *cyclic algebra* in the Brauer group $\mathrm{Br}(F)$ (see [3, §2.5]). By definition, $(x, a) = x \cup a$ with respect to the cup-product

$$(\Gamma_F)^* \otimes F^\times = H^2(F, \mathbb{Z}) \otimes H^0(F, F_{\mathrm{sep}}^\times) \to H^2(F, F_{\mathrm{sep}}^\times) = \mathrm{Br}(F).$$

If $x \in (\Gamma_F)^*[2]$, i.e., $2x = 0$, then (x, a) is the class of a *quaternion algebra* split by the quadratic extension $F(a^{1/2})/F$. Conversely, every element in $\mathrm{Br}(F)$ that is split by $F(a^{1/2})/F$ is of the form (x, a) for some $x \in (\Gamma_F)^*[2]$.

Lemma 2.7. *If* $\mathrm{char}(F) \neq 2$*, the kernel of the homomorphism* $(\Gamma_F)^* \to \mathrm{Br}(F)$ *taking a character* x *to* $(x, -1)$ *coincides with* $2(\Gamma_F)^*$*.*

Proof. Let $x \in (\Gamma_F)^*$ and let m be the order of x. Consider the matrix $A \in \mathrm{GL}_m(F)$ defined by $(a_1, a_2, \dots, a_m) \cdot A = (a_2, a_3, \dots, a_m, -a_1)$ for all $a_i \in F$. Note that $A^m = -1$, hence we have a homomorphism $i : \mathbb{Z}/2m\mathbb{Z} \to \mathrm{GL}_m(F_{\mathrm{sep}})$ defined by $i(r + 2m\mathbb{Z}) = A^r$. The upper row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\frac{1}{2}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\frac{1}{2}} & \frac{1}{2m}\mathbb{Z}/\mathbb{Z} & \xrightarrow{2} & \frac{1}{m}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
1 & \longrightarrow & F_{\mathrm{sep}}^\times & \longrightarrow & \mathrm{GL}_m(F_{\mathrm{sep}}) & \longrightarrow & \mathrm{PGL}_m(F_{\mathrm{sep}}) & \longrightarrow & 1,
\end{array}$$

where $k(1+2\mathbb{Z}) = -1$ yields an exact sequence $(\Gamma_F)^* \xrightarrow{2} (\Gamma_F)^* \xrightarrow{\partial} H^2(F, \mathbb{Z}/2\mathbb{Z})$. Identifying $\mathbb{Z}/2\mathbb{Z}$ with μ_2 and $H^2(F, \mathbb{Z}/2\mathbb{Z})$ with the subgroup $H^2(F, \mu_2) = \mathrm{Br}(F)[2]$ of the Brauer group $H^2(F, F_{\mathrm{sep}}^\times) = \mathrm{Br}(F)$ we see that it suffices to show that $\partial(x)$ is equal to the cyclic class $(x, -1)$.

It is shown in [3, §2.5] that the image of x under the composition

$$(\Gamma_F)^* = H^1(F, \mathbb{Q}/\mathbb{Z}) \to H^1(F, \mathrm{PGL}_m(F_{\mathrm{sep}})) \to H^2(F, F_{\mathrm{sep}}^\times) = \mathrm{Br}(F)$$

given by the bottom row of the diagram coincides with $(x, -1)$. □

3. Fields with many roots of unity

Proposition 3.1. *Let G be a finite group and F a field and let m be a positive integer such that* $\mathrm{char}(F)$ *does not divide m and $\mu_m \subset F^\times$. Then*

$$H^2(G, \mu_m)_{\mathrm{neg}} = \mathrm{Ker}\big(H^2(G, \mu_m) \to H^2(G, F^\times)\big),$$

where we view μ_m and $F^\times$ as trivial G-modules.

Proof. Let V be a finite dimensional faithful representation of G such that there is a G-invariant open subset $U \subset V$ with the property that $V \setminus U$ is of codimension at least 2 in V and there is a G-torsor $U \to X$ for a variety X over F. Such representations exist (see [8, Remark 1.4]).

The Hochschild-Serre spectral sequence [4, Ch. III, Th. 2.20]

$$E_2^{p,q} = H^p(G, H^q_{ét}(U, \mathbb{G}_m)) \Rightarrow H^{p+q}_{ét}(X, \mathbb{G}_m)$$

yields an exact sequence

$$\mathrm{Pic}(U)^G \to H^2(G, F[U]^\times) \to \mathrm{Br}(X).$$

The group $\mathrm{Pic}(U)$ is trivial as U is an open subset of the affine space V. By the choice of U every invertible regular function on U is constant, i.e., $F[U]^\times = F^\times$ and hence the map $H^2(G, F^\times) \to \mathrm{Br}(X)$ is injective.

By [4, III, 2.22], the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K)$, where $K = F(X)$, is injective. It follows that the bottom map of the commutative diagram

$$\begin{array}{ccc} H^2(G, \mu_m) & \longrightarrow & H^2(K, \mu_m) \\ \downarrow & & \downarrow \\ H^2(G, F^\times) & \longrightarrow & \mathrm{Br}(K) \end{array}$$

is injective. The right vertical morphism is also injective identifying $H^2(K, \mu_m)$ with $\mathrm{Br}(K)[m]$. Hence the other two homomorphisms in the diagram have equal kernels. Now the statement follows from Corollary 2.2(1). □

Remark 3.2. The proposition also follows from the isomorphism $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, F^\times)$ established in [1].

It follows from Proposition 3.1 that $H^2(G, \mu_m)_{\mathrm{neg}}$ coincides with the image of the connecting homomorphism

$$H^1(G, F^\times/\mu_m) \to H^2(G, \mu_m)$$

for the exact sequence $1 \to \mu_m \to F^\times \to F^\times/\mu_m \to 1$. An element of the group $H^1(G, F^\times/\mu_m)$ is a group homomorphism $G \to F^\times/\mu_m$. Its image is contained in $\mu(F)/\mu_m$. Consider the exact sequence

$$1 \to \mu_m \to \mu(F) \to \mu(F)/\mu_m \to 1. \tag{3.3}$$

We have proved the following statement:

Corollary 3.4. *In the conditions of Proposition 3.1 the group $H^2(G, \mu_m)_{\mathrm{neg}}$ coincides with the image of the connecting homomorphism $H^1(G, \mu(F)/\mu_m) \to H^2(G, \mu_m)$ for the exact sequence (3.3).*

The exact sequence $0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\frac{1}{m}} \mathbb{Q}/\mathbb{Z} \xrightarrow{m} \mathbb{Q}/\mathbb{Z} \to 0$ for an integer $m > 0$ yields an embedding

$$G^*/mG^* \hookrightarrow H^2(G, \mathbb{Z}/m\mathbb{Z}),$$

where $G^* := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z})$ is the *character group* of G. We identify G^*/mG^* with a subgroup of $H^2(G, \mathbb{Z}/m\mathbb{Z})$.

4. Primary case

Let p be a prime integer and F a field such that $\mathrm{char}(F) \neq p$.

Lemma 4.1. *Let $\mu_{p^\infty}(F(\xi_p)) = \mu_{p^t}$ for some t with $1 \leqslant t \leqslant \infty$. Assume that $t \geqslant 2$ if $p = 2$. Then $\mu_{p^\infty}(F(\xi_{p^r})) = \mu_{p^r}$ for every $r \geqslant t$.*

Proof. The image of the injective homomorphism $\chi : \Gamma = \mathrm{Gal}(F(\mu_{p^\infty})/F(\xi_p)) \to \mathbb{Z}_p^\times$ taking an automorphism σ to the unique p-adic unit a such that $\sigma(\xi) = \xi^a$ for all $\xi \in \mu_{p^\infty}$ is contained in $U_t = \{a \in \mathbb{Z}_p^\times \mid a \equiv 1 \ mod\ p^t\}$. Choose an element $\sigma \in \Gamma$ such that $\chi(\sigma) \notin U_{t+1}$. By assumption, U_t is a topological cyclic group generated by σ. It follows that $\mathrm{Im}(\chi) = U_t$ and $F(\xi_{p^r})$ for all $r \geqslant t$ are all intermediate fields between $F(\xi_p)$ and $F(\mu_{p^\infty})$ corresponding to all closed subgroups $U_r \subset U_t$. □

Theorem 4.2. *Let G be a finite group, p a prime integer and s a positive integer. Let F be a field such that $\mathrm{char}(F) \neq p$ and $\mu_{p^\infty}(F(\xi_p)) = \mu_{p^t}$ for some t with $1 \leqslant t \leqslant \infty$.*

(1) *If $t \geqslant s$, then*

$$H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}} = \big(G^*[p^{t-s}] + p^sG^*\big)/p^sG^* \subset G^*/p^sG^* \subset H^2(G, \mathbb{Z}/p^s\mathbb{Z}).$$

(2) *If $t < s$ and $t \geqslant 2$ in the case $p = 2$, then $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}} = 0$.*

Proof. (1): Since $t \geqslant s$, by Corollary 2.4, we may assume that $\mu_{p^s} \subset F^\times$, hence $\mathbb{Z}/p^s\mathbb{Z} \simeq \mu_{p^s}$ as Galois modules. The p-primary component of the exact sequence (3.3) is isomorphic to the upper row of the commutative diagram

$$\begin{CD}
0 @>>> \mathbb{Z}/p^s\mathbb{Z} @>{p^{-s}}>> p^{-t}\mathbb{Z}/\mathbb{Z} @>{p^s}>> p^{s-t}\mathbb{Z}/\mathbb{Z} @>>> 0 \\
@. @| @VVV @VVV @. \\
0 @>>> \mathbb{Z}/p^s\mathbb{Z} @>{p^{-s}}>> \mathbb{Q}/\mathbb{Z} @>{p^s}>> \mathbb{Q}/\mathbb{Z} @>>> 0.
\end{CD}$$

Applying cohomology groups to the diagram and using Corollary 3.4 we see that the group $H^2(G,\mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}}$ coincides with the image of the composition

$$G^*[p^{t-s}] = H^1(G, p^{s-t}\mathbb{Z}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) = G^* \to G^*/p^sG^* \subset H^2(G,\mathbb{Z}/p^s\mathbb{Z}),$$

whence the result.

(2): Let $L = F(\mu_{p^s})$. By Lemma 4.1, we have $\mu_{p^\infty}(L) = \mu_{p^s}$. The first part of the theorem applied to the field L show that $H^2(G,\mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg},L} = 0$. It follows from Proposition 2.3(2) that $H^2(G,\mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}} = 0$. □

5. The case $p = 2$ and $t = 1$

It remains to consider the case $p = 2$ and $t = 1$ and F is a field of characteristic different from 2. The condition $t = 1$ means that -1 is not a square in F.

Proposition 5.1. *Let $b \geqslant a$ be positive integers, L a field such that $\xi_{2^b} \in L(\sqrt{-1})$ and let $\Gamma = \Gamma_L$. Then*

$$\Gamma^*[2^{b-a}] \cap 2\Gamma^* \subset 2^a\Gamma^*.$$

Proof. We prove the statement by induction on a. The case $a = 1$ is obvious.

$a = 2$: Let $x \in \Gamma^*[2^{b-2}] \cap 2\Gamma^*$. Write $x = 2y$ for $y \in \Gamma^*[2^{b-1}]$. Consider the cyclic class $(y,-1) \in \mathrm{Br}(L)$. As $-1 = (\xi_{2^b})^{2^{b-1}}$ in $L' := L(\sqrt{-1})$, we have

$$(y,-1) \otimes_L L' = (y_{L'},-1) = 2^{b-1} \cdot (y_{L'}, \xi_{2^b}) = (2^{b-1}y_{L'}, \xi_{2^b}) = 0$$

in the Brauer group $\mathrm{Br}(L')$ since $2^{b-1}y = 0$. We proved that $(y,-1)$ is split by the extension $L(\sqrt{-1})$ of L, hence $(y,-1)$ is the class of the quaternion algebra $(z,-1)$ for some $z \in \Gamma^*[2]$. It follows that $(y-z,-1) = 0$, hence $y - z \in 2\Gamma^*$ by Lemma 2.7 and therefore, $x = 2y = 2(y-z) \in 4\Gamma^*$.

$a - 1 \Rightarrow a$: Let $x \in \Gamma^*[2^{b-a}] \cap 2\Gamma^*$. By the induction hypothesis, $x = 2^{a-1}y$ for some $y \in \Gamma^*[2^{b-1}]$. Then $2y \in \Gamma^*[2^{b-2}] \cap 2\Gamma^*$ and hence $2y \in 4\Gamma^*$ by the first part of the proof. Finally, $x = 2^{a-2} \cdot 2y \in 2^{a-2} \cdot 4\Gamma^* = 2^a\Gamma^*$. □

Theorem 5.2. *Let G be a finite group and s a positive integer. Let F be a field such that such that $\mathrm{char}(F) \neq 2$ and $-1 \notin F^{\times 2}$. Write $\mu_{2^\infty}(F(\sqrt{-1})) = \mu_{2^{t'}}$ for some t' with $1 \leqslant t' \leqslant \infty$.*

(1) *If $t' \geqslant s$, then*

$$H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\mathrm{neg}} = \big((G^*[2^{t'-s}] \cap 2G^*) + 2^sG^*\big)/2^sG^* \subset G^*/2^sG^* \subset H^2(G,\mathbb{Z}/2^s\mathbb{Z}).$$

(2) *If $t' < s$, then $H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\mathrm{neg}} = 0$.*

Proof. (1): It follows from Theorem 4.2(1) applied to the field $F' := F(\sqrt{-1})$ and Proposition 2.3(2) that

$$H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\mathrm{neg}} \subset H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\mathrm{neg},F'} = \big(G^*[2^{t'-s}] + 2^sG^*\big)/2^sG^*.$$

Applying Corollary 3.4 in the case $m = 2$ we see that $H^2(G, \mathbb{Z}/2\mathbb{Z})_{\text{neg}} = 0$ since $t = 1$. The commutativity of the diagram

$$\begin{CD} G^*/2^sG^* @>>> G^*/2G^* \\ @VVV @VVV \\ H^2(G, \mathbb{Z}/2^s\mathbb{Z}) @>>> H^2(G, \mathbb{Z}/2\mathbb{Z}) \end{CD}$$

shows that $H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} \subset 2G^*/2^sG^*$. It follows that

$$H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} \subset \big((G^*[2^{t'-s}] \cap 2G^*) + 2^sG^*\big)/2^sG^*.$$

Conversely, let $x \in G^*[2^{t'-s}] \cap 2G^*$. We show that the corresponding element in $G^*/2^sG^* \subset H^2(G, \mathbb{Z}/2^s\mathbb{Z})$ is negligible. Let L/F be a field extension and $j : \Gamma_L \to G$ a group homomorphism. Consider the following commutative diagram

$$\begin{CD} G^*/2^sG^* @>{j^*}>> (\Gamma_L)^*/2^s(\Gamma_L)^* \\ @VVV @VVV \\ H^2(G, \mathbb{Z}/2^s\mathbb{Z}) @>{j^*}>> H^2(L, \mathbb{Z}/2^s\mathbb{Z}). \end{CD}$$

By Proposition 5.1 applied to $a = s$ and $b = t'$ we see that the image of x in $(\Gamma_L)^*/2^s(\Gamma_L)^*$ is trivial and hence the image of x in $H^2(L, \mathbb{Z}/2^s\mathbb{Z})$ is also trivial, i.e., x is negligible.

(2): Let $L = F(\mu_{2^s}) = F'(\mu_{2^s})$. By Lemma 4.1 applied to F', we have $\mu_{2^\infty}(L) = \mu_{2^s}$. The first part of the theorem applied to the field L shows that $H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg},L} = 0$. It follows from Proposition 2.3(2) that $H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} = 0$. □

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