MOTIVIC DECOMPOSITION OF CERTAIN SPECIAL LINEAR GROUPS

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Abstract. We compute the motive of the algebraic group \( G = \text{SL}_1(D) \) for a central simple algebra \( D \) of prime degree over a perfect field. As an application we determine certain motivic cohomology groups and differentials in the motivic spectral sequence of \( G \).

1. Introduction

In this paper we study the motive in the triangulated category of geometric mixed effective motives \( \mathbf{DM}_\text{gm}^*(F) \) over a perfect field \( F \) and the motivic cohomology of the algebraic group \( \text{SL}_1(D) \) of reduced norm 1 elements in a central simple algebra \( D \) of prime degree \( l \).

In [13], A. Suslin computed the \( K \)-cohomology groups of the (split) special linear group \( \text{SL}_n \) and the symplectic groups \( \text{Sp}_{2n} \) using higher Chern classes in \( K \)-cohomology. O. Pushin in [15] constructed higher Chern classes in motivic cohomology and found decompositions of the motives of the groups \( \text{SL}_n \) and \( \text{GL}_n \) into direct sums of Tate motives. S. Biglari computed in [1] the motives of certain split reductive groups over \( \mathbb{Q} \). In particular, he showed that

\[
M(\text{SL}_n)_{\mathbb{Q}} \cong \bigoplus_{i=0}^{n-1} \text{Sym}^i (\mathbb{Q}(2)[3] \oplus \mathbb{Q}(3)[5] \oplus \cdots \oplus \mathbb{Q}(n)[2n-1]).
\]

A. Huber and B. Kahn determined the motives over \( \mathbb{Z} \) of split reductive groups in [9].

The motives of non-split algebraic groups are more complicated. The slices of the slice filtration of the motive \( M(\text{GL}_l(D)) \) for a division algebra \( D \) of prime degree were computed by E. Shinder in [17].

In this paper we study the motive of the group \( G = \text{SL}_1(D) \), where \( D \) is a central simple algebra of a prime degree \( l \). As a warm-up, consider the simplest case \( l = 2 \). The variety of \( G \) is then an open subscheme of a 3-dimensional projective isotropic quadric \( X \) given by the homogeneous quadratic equation \( \text{Nrd} = t^2 \), where \( \text{Nrd} \) is the reduced norm form of \( D \). The surface \( Y = X \setminus G \), given by \( \text{Nrd} = 0 \), is isomorphic to \( S \times S \), where \( S \) is the Severi-Brauer variety.

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of $D$ (a conic curve in the case $l = 2$). Computing the motives of $X$ and $Y$ as in \cite[§4]{1}, we get an exact triangle
\[ M(G) \rightarrow \mathbb{Z} \oplus M(S)(1)[2] \oplus \mathbb{Z}(3)[6] \rightarrow M(S)(1)[2] \oplus M(S)(2)[4] \rightarrow M(G)[1]. \]
Canceling out the summands $M(S)(1)[2]$, we obtain an isomorphism
\begin{equation}
M(G) \simeq \mathbb{Z} \oplus N(2)[3],
\end{equation}
where the motive $N$ is defined by the exact triangle
\[ \mathbb{Z}(1)[2] \rightarrow M(S) \rightarrow N \rightarrow \mathbb{Z}(1)[3] \]
with the first morphism dual to the canonical one $M(S) \rightarrow \mathbb{Z}$.

In the general case, when $l$ is an arbitrary prime, since the group $G$ and its motive are split over a field extension of degree $l$, the torsion part of motivic cohomology of $G$ is $l$-torsion. We work over the coefficient ring $\mathbb{Z}[\frac{1}{(l-1)!}]$, just inverting insignificant integers.

As in the case $l = 2$, the motive of $G$ can be computed out of motive of the Severi-Brauer variety $S$ of the algebra $D$. Let $N$ be the motive defined by the exact triangle
\[ \mathbb{Z}(l-1)[2l-2] \rightarrow M(S) \rightarrow N \rightarrow \mathbb{Z}(l-1)[2l-1]. \]
As $(l-1)!$ is invertible in the coefficient ring, one can define symmetric $\text{Sym}^i(M)$ and alternating powers $\text{Alt}^i(M)$ of any motive $M$ for $i = 0, 1, \ldots, l-1$. The main result of the paper is the following theorem generalizing (\ref{mainresult}) and (\ref{mainresult2}) (see Theorem \ref{mainresult3}).

**Theorem.** Let $D$ be a central simple algebra of prime degree $l$ over a perfect field $F$. Then there is an isomorphism
\[ M(\text{SL}_4(D)) \cong \prod_{i=0}^{l-1} \text{Sym}^i(N(2)[3]) = \prod_{i=0}^{l-1} (\text{Alt}^i N)(2i)[3i] \]
in the category $\text{DM}^{eff}_{gm}(F)$ of motives over $F$ with coefficients in $\mathbb{Z}[\frac{1}{(l-1)!}]$.

The most difficult part of the proof is the construction of a morphism $M(G) \rightarrow N(2)[3]$ in $\text{DM}^{eff}_{gm}(F)$. The main players of the proof are the groups $H^{3,2}(G) \simeq \mathbb{Z}$ and the Chow group $\text{CH}^{l+1}(G) = H^{2l+2,l+1}(G) \simeq \mathbb{Z}/l\mathbb{Z}$ (when $D$ is not split). These groups are related by a pair of homomorphisms
\begin{equation}
H^{3,2}(G) \leftarrow \text{Hom}(M(G), N(2)[3]) \rightarrow H^{2l+2,l+1}(G).
\end{equation}
We prove that there is a morphism $M(G) \rightarrow N(2)[3]$ with the images in (\ref{mainresult2}) generating the two side cyclic groups. This is done in Section \ref{sections}.

Using Theorem \ref{mainresult} and the exact triangle (Corollary \ref{corollary})
\[ (\text{Alt}^{l-1} N)(l-1)[2l-2] \rightarrow \text{Alt}^l M(S) \rightarrow (\text{Alt}^l N) \rightarrow (\text{Alt}^{l-1} N)(l-1)[2l-1], \]
we can compute inductively the motivic cohomology of $G$. As an application, in Section \ref{applications} we compute the motivic cohomology $H^{p,q}(G)$ with $2q - p \leq 1$. We also compute certain differentials in the motivic spectral sequence of $G$. 
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2. Motivic cohomology

The base field $F$ is assumed to be perfect. We fix a prime integer $l$ and work over the coefficient ring $\mathbb{Z}$ that is either the ring of integers $\mathbb{Z}$ or $\mathbb{Z}[[l^{-1}]]$, or the localization $\mathbb{Z}(l)$ of $\mathbb{Z}$ by the prime ideal generated by $l$. Note that $\mathbb{Z}/l\mathbb{Z} = \mathbb{Z}/l\mathbb{Z}$.

We write $DM(F) := DM_{gm}(F)$ for the triangulated category of (geometric mixed effective) motives with coefficients in $\mathbb{Z}$ (see [19]). If $p$ and $q \geq 0$ are integers, $\mathbb{Z}(q)[p]$ denotes the Tate motive and $M(X)$ the motive of a smooth variety $X$ over $F$. We have $M(\operatorname{Spec} F) = \mathbb{Z} := \mathbb{Z}(0)[0]$. For a motive $M$ and an integer $q \geq 0$, we write $M(q)$ for $M \otimes \mathbb{Z}(q)$.

For a motive $M$ in $DM(F)$ define the motivic cohomology by

$$H_{p,q}(M) := \operatorname{Hom}(M, \mathbb{Z}(q)[p]),$$

where $\operatorname{Hom}$ is taken in the category $DM(F)$. If $X$ is a smooth variety, simply write $H_{p,q}(X)$ for $H_{p,q}(M(X))$. We have

$$H_{p,q}(X) = 0 \text{ if } p > 2q \text{ or } p > q + \dim(X).$$

In particular, $H_{p,q}(F) = 0$ if $p > q$. Moreover, $H_{p,p}(F) = K_p^M(F)$, the Milnor $K$-groups of $F$ (see [12, Lecture 5]).

The bi-graded group $\prod_{p,q} H_{p,q}(X)$ has a natural structure of a graded commutative ring (with respect to $p$, [12, Theorem 15.9]).

There is a canonical isomorphism between $H^{2p,p}(X)$ and the Chow groups $\operatorname{CH}^p(X)$ of (rational equivalence) classes of algebraic cycles on a smooth variety $X$ of codimension $p$ ([12, Lecture 18]). We also write $\operatorname{CH}^p(M) := H^{2p,p}(M)$ for every motive $M$.

The cancellation theorem (see [21]) states that the canonical morphism

$$\operatorname{Hom}(M, N) \xrightarrow{\sim} \operatorname{Hom}(M(1), N(1))$$

is an isomorphism for every two motives $M$ and $N$.

The natural functor from the category of smooth projective varieties over $F$ to $DM(F)$ extends uniquely to a canonical functor from the category Chow($F$) of Chow motives over $F$ to $DM(F)$ (see [13, Proposition 2.1.4]). The motives in $DM(F)$ coming from Chow($F$) are called pure motives.

Let $M$ be any motive and $X$ a smooth projective variety of pure dimension $d$ over $F$. The two canonical morphisms (given by the diagonal of $X$ in the category of Chow motives)

$$\mathbb{Z}(d)[2d] \to M(X \times X) \to \mathbb{Z}(d)[2d]$$

together with the cancelation theorem define the two mutually inverse isomorphisms (see [3, Appendix B])

$$\operatorname{Hom}(M, M(X)) \cong \operatorname{Hom}(M \otimes M(X), \mathbb{Z}(d)[2d]) = \operatorname{CH}^d(M \otimes M(X)).$$
In particular, if \( Y \) is another smooth projective variety, then
\[
\text{Hom}_{DM(F)}(M(Y), M(X)) \simeq \text{CH}^d(Y \times X) = \text{Hom}_{\text{Chow}(F)}(M(Y), M(X)).
\]

We say that a motive \( M \) is of degree \( d \) if \( M \) is a direct summand of a motive of the form \( M(X)(q)[p] \) with \( 2q - p = d \), where \( X \) is a smooth projective variety. The pure motives are of degree 0. The following statement is an immediate consequence of (2.1).

**Lemma 2.3.** Let \( M \) and \( N \) be motives of degree \( d \) and \( e \) respectively. If \( d > e \), then \( \text{Hom}(M, N) = 0 \).

The coniveau spectral sequence for a smooth variety \( X \) over \( F \),
\[
E_1^{p,q} = \prod_{x \in X^{(p)}} H^{q-p,n-p}F(x) \Rightarrow H^{p+q,n}(X),
\]
where \( X^{(p)} \) is the set of points in \( X \) of codimension \( p \), yields isomorphisms
\[
H^{i+n,n}(X) \simeq A^i(X, K_n) \quad \text{when} \quad n - i \leq 2
\]
with the \( K \)-cohomology groups \( A^i(X, K_n) \) defined in [16].

If \( X \) is a variety over \( F \), we write \( X_{\text{sep}} \) for the variety \( X \otimes_F F_{\text{sep}} \) over a separable closure \( F_{\text{sep}} \) of \( F \).

### 3. Severi-Brauer varieties

Let \( D \) be a central simple algebra of degree \( n \) over \( F \) and \( S \) the Severi-Brauer variety \( SB(D) \) of right ideals in \( D \) of rank \( n \). This is a smooth projective variety of dimension \( n - 1 \) (see [3]). If \( D \) is split, i.e., \( D = \text{End}(V) \) for an \( n \)-dimensional vector space \( V \) over \( F \), then \( S \) is isomorphic to the projective space \( \mathbb{P}(V) \). Therefore, in the split case,
\[
M(S) \simeq \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \cdots \oplus \mathbb{Z}(l - 1)[2l - 2].
\]

Let \( I \to S \) be the tautological vector bundle of rank \( n \) (with the fiber over a right ideal the ideal itself). We have \( D = \text{End}(I)^{\text{op}} = \text{End}(I^\vee) \), where \( I^\vee \) is the vector bundle dual to \( I \). In the split case, when \( S = \mathbb{P}(V) \),
\[
I = V^\vee \otimes L_t = \text{Hom}(V, L_t),
\]
where \( L_t \to \mathbb{P}(V) \) is the tautological line bundle. The sheaf of sections of \( L_t \) is \( O(-1) \).

In the split case, when \( S = \mathbb{P}(V) \), let \( s \in \text{CH}^1(S) \) be the class of a hyperplane section. We have
\[
\text{CH}^i(\mathbb{P}(V)) = \begin{cases} \mathbb{Z}s^i, & i = 0, 1, \ldots, n - 1; \\ 0, & \text{otherwise.} \end{cases}
\]

The ring \( \text{End}(M(S)) = \text{CH}^{n-1}(S \times S) \) is canonically isomorphic to the product \( \mathbb{Z}^n \) of \( n \) copies of \( \mathbb{Z} \) with the idempotents \( s^i \times s^{n-1-i} \).

In the non-split case we have the following statement (see [13], Corollary 8.7.2):
Lemma 3.1. When $D$ is a division algebra of prime degree $l$, the natural map $\text{CH}^i(S) \to \text{CH}^i(S_{\text{sep}})$ is injective and it identifies the Chow group of $S$ as follows

$$\text{CH}^i(S) = \begin{cases} \mathbb{Z}, & i = 0; \\ l\mathbb{Z}s^i, & i = 1, \ldots, l-1; \\ 0, & \text{otherwise.} \end{cases}$$

Any of the two projections $p : S \times S \to S$ is the projective bundle of $I^\vee$, i.e., $S \times S = \mathbb{P}_S(I^\vee)$. Let $L$ be the tautological line bundle of this projective bundle. By the projective bundle theorem \[19,\text{Proposition 3.5.1}], we have:

$$\text{CH}^{l-1}(S \times S) = \text{CH}^{l-1}(S) \cdot 1 \oplus \text{CH}^{l-2}(S) \cdot \xi \oplus \cdots \oplus \text{CH}^0(S) \cdot \xi^{l-1},$$

where $\xi$ is the first Chern class of $L$ in $\text{CH}^1(S \times S)$. Consider the composition

$$\text{type} : \text{CH}^{l-1}(S \times S) = \text{End}(M(S)) \to \text{End}(M(S_{\text{sep}})) \to \mathbb{Z}^l.$$

Proposition 3.3. Let $D$ be a division algebra of degree $l$ and $S = \text{SB}(D)$. Then the ring homomorphism

$$\text{type} : \text{End}(M(S)) \to \mathbb{Z}^l$$

is injective. Its image consists of all tuples $(a_1, a_2, \ldots, a_l)$ such that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_l \pmod{l}.$$

Proof. It follows from Lemma \[3.3] and \[3.2] that $\text{type}$ is injective and $[\text{Im(\text{type})} : l\mathbb{Z}^l] = l$. Therefore, the identity in $\mathbb{Z}^l$ and $l\mathbb{Z}^l$ generate $\text{Im(\text{type})}$. \qed

We will also need the following lemma.

Lemma 3.4. Let $M_1$ and $M_2$ be direct sum of shifts of $M(S)$ (with arbitrary coefficients) and $f : M_1 \to M_2$ a morphism in $\text{DM}(F)$. If $f$ is an isomorphism over a field extension, then $f$ is also an isomorphism.

Proof. Write $M_1$ and $M_2$ as direct sums of the homogeneous (degree $k$) components $M_1^{(k)}$ and $M_2^{(k)}$ respectively. By Lemma \[3.3], the morphism $f$ is given by a triangular matrix with the diagonal terms $f_k : M_1^{(k)} \to M_2^{(k)}$. By assumption, the matrix is invertible over a splitting field $L$, hence all $f_k$ are isomorphisms over $L$. Note that $f_k$ is a shift of a morphism of pure motives that are direct sums of shifts of $M(S)$. By \[i,\text{Corollary 92.7}], all $f_k$ are isomorphisms. Therefore, the triangular matrix is invertible and hence $f$ is an isomorphism. \qed

4. The motive $N$

Let $D$ be a central simple algebra of prime degree $l$ over $F$ and $S = \text{SB}(D)$. The motive $N$ is defined by the triangle

$$\mathbb{Z}(l-1)[2l-2] \to M(S) \xrightarrow{\kappa} N \xrightarrow{\xi} \mathbb{Z}(l-1)[2l-1]$$
in $DM(F)$ with the first morphism of pure motives given by the identity in $\text{CH}^0(S)$. We have

$$N_{\text{sep}} \simeq \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \cdots \oplus \mathbb{Z}(l - 2)[2l - 4],$$

therefore, $\text{Hom}(M(S_{\text{sep}}), N_{\text{sep}}) \simeq \mathbb{Z}^{l-1}$.

Consider the map

$$\text{type} : \text{Hom}(M(S), N) \to \text{Hom}(M(S_{\text{sep}}), N_{\text{sep}}) \simeq \mathbb{Z}^{l-1}.$$ 

For example, $\text{type}(\kappa) = (1, 1, \ldots, 1)$.

**Proposition 4.3.** Let $D$ be a division algebra of degree $l$ and $S = \text{SB}(D)$. Then the homomorphism

$$\text{type} : \text{Hom}(M(S), N) \to \mathbb{Z}^{l-1}$$

is injective. Its image consists of all tuples $(a_1, a_2, \ldots, a_{l-1})$ such that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{l-1} \pmod{l}.$$ 

**Proof.** Let $\varphi \in \text{Ker}(\text{type})$. The triangle \ref{4.1} yields an exact sequence

$$\text{CH}^{l-1}(S) \to \text{End} M(S) \to \text{Hom}(M(S), N) \to H^{2l-1,l-1}(S).$$

The last term is zero as $2l - 1 > 2(l - 1)$. Therefore, $\varphi = \kappa \circ \sigma$ for some $\sigma \in \text{End} M(S)$. By assumption, $\text{type}(\sigma) = (0, \ldots, 0, a)$, where $a \equiv 0$ modulo $l$ in view of Proposition \ref{3.3}. Then $\sigma$ comes from $\text{CH}^{l-1}(S) = l \mathbb{Z}$ by Lemma \ref{3.1} and hence $\varphi = 0$. This proves injectivity. The second statement follows from Proposition \ref{3.3}. \hfill $\square$

**Lemma 4.4.** There is an isomorphism

$$N \otimes M(S) \simeq M(S) \oplus M(S)(1)[2] \oplus \cdots \oplus M(S)(l - 2)[2l - 4].$$

In particular, $N \otimes M(S)$ is a pure motive.

**Proof.** The triangle \ref{4.1} is split after tensoring with $M(S)$. Indeed, the morphism $M(S)(l - 1)[2l - 2] \to M(S) \otimes M(S)$ has a left inverse given by the class of the diagonal in $\text{CH}^{2l-2}(S \times S \times S)$. \hfill $\square$

**Lemma 4.5.** We have $\text{CH}^i(N) = 0$ if $i > l$.

**Proof.** In the exact sequence induced by \ref{4.1}

$$H^{2l-2l+1,i-l+1}(F) \to \text{CH}^i(N) \to \text{CH}^i(S)$$

the first and the last terms are trivial as $2i - 2l + 1 > i - l + 1$ and $\dim(S) < l$. \hfill $\square$

Since $\text{Hom}(\mathbb{Z}(q)[p], \mathbb{Z}) = 0$ if $q > 0$, the natural morphism $M(S) \to \mathbb{Z}$ factors uniquely through a morphism $\nu : N \to \mathbb{Z}$. 


5. Higher Chern classes

Let \( X \) be a smooth variety. The higher Chern classes with values in motivic cohomology were constructed in [41]:

\[
c_j, i : K_j(X) \to H^{2i-j,i}(X).
\]

We will be using the classes

\[
c_i := c_{1, i+1} : K_i(X) \to H^{2i+1, i+1}(X).
\]

**Proposition 5.1** ([41, §4.1]). Let \( L \) be a vector bundle over a smooth variety \( X \) and \( \alpha \in K_i(X) \). Then

\[
c_i(\alpha \cdot [L]) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} c_{i-j}(\alpha) h^j,
\]

where \( h \in CH^1(X) = H^{2,1}(X) \) is the first (classical) Chern class of \( L \).

Let \( E \to X \) be a vector bundle of rank \( n \). We write \( SL(E) \) for the group scheme over \( X \) of determinant 1 automorphisms of \( E \).

Let \( a \) be the generic element of \( SL(E) \) (see [41, §4]). We also write \( a \) for the corresponding element in \( K_1(SL(E)) \). We have \( c_0(a) = 0 \) since \( \det(a) = 1 \).

For a sequence \( i = (i_1, i_2, \ldots, i_k) \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n-1 \), set \( d_i = i_1 + i_2 + \cdots + i_k \) and \( e_i = k \). Let

\[
c_i(\alpha) := c_{i_1}(\alpha)c_{i_2}(\alpha) \cdots c_{i_k}(\alpha) \in H^{2d_i+e_i, d_i+e_i}(SL(E)).
\]

**Proposition 5.2.** Let \( E \to X \) be a vector bundle of rank \( n \). Then the \( H^{*, *}(X) \)-module \( H^{*, *}(SL(E)) \) is free with basis \( \{ c_i(\alpha) \} \) over all sequences \( i \).

**Proof.** This follows from [41, Proposition 3]. \( \square \)

Write \( \widetilde{c_i}(\alpha) \) for the composition

\[
M(SL(E)) \xrightarrow{\text{diag}} M(SL(E)) \otimes M(SL(E)) \xrightarrow{j \circ c_i(\alpha)} M(X)(d_i + e_i)[2d_i + e_i],
\]

where \( j : M(SL(E)) \to M(X) \) is the canonical morphism. The following corollary is deduced from Proposition 5.2 in the same way as in [41, Proposition 4.2].

**Corollary 5.3.** (cf. [41, Proposition 4.4]) The morphisms \( \widetilde{c_i}(\alpha) \) yield an isomorphism

\[
M(SL(E)) \cong \prod_i M(X)(d_i + e_i)[2d_i + e_i]. \quad \square
\]

**Remark 5.4.** The natural composition

\[
M(SL(E)) \xrightarrow{\tilde{c}(\alpha)} M(X)(d_i + e_i)[2d_i + e_i] \to \mathbb{Z}(d_i + e_i)[2d_i + e_i]
\]

coincides with \( c_i(\alpha) \).
Corollary 5.5. There is a canonical isomorphism
\[ M(\text{SL}_n) \simeq \prod_i \mathbb{Z}(d_i + e_i)[2d_i + e_i]. \]

Let \( G = \text{SL}(E) \) where \( E \to X \) is a vector bundle of rank \( n \) over a smooth variety \( X \). Consider the grading on \( M(G) \) with respect to the value \( e(i) \):
\[ M(G)(k) := \prod_{e_i=k} M(X)(d_i + k)[2d_i + k] \]
for \( k = 0, 1, \ldots, n - 1 \). Thus,
\[ M(G) = \prod_{k=0}^{n-1} M(G)(k) \]
and each motive \( M(G)(k) \) has degree \( k \).

Example 5.6. In the split case, we have a natural isomorphism \( M(\text{SL}_l)(1) \simeq N(2)[3] \).

Let \( D \) be a central simple algebra of prime degree \( l \) over \( F \) and \( G = \text{SL}_1(D) \). Let \( S \) be the Severi-Brauer variety of \( D \).

Corollary 5.7 yields

Corollary 5.7. There is a canonical isomorphism
\[ M(G \times S) \simeq \prod_i M(S)(d_i + e_i)[2d_i + e_i]. \]
In particular, \( \text{CH}^*(G \times S) \simeq \text{CH}^*(S) \).

It follows from Corollary 5.3 that \( M(G_{\text{sep}})(1) \simeq N_{\text{sep}}(2)[3] \) and therefore \( \text{Hom}(M(G_{\text{sep}}), M(S_{\text{sep}})(2)[3]) \) is naturally isomorphic to \( \mathbb{Z}^{l-1} \). Consider the map
\[ \text{type} : \text{Hom}(M(G), M(S)(2)[3]) \to \text{Hom}(M(G_{\text{sep}}), M(S_{\text{sep}})(2)[3]) \simeq \mathbb{Z}^{l-1}. \]

By (2.2) and Corollary 5.4, we have
\[ \text{Hom}(M(G), M(S)(2)[3]) = H^{2l+1,l+1}(G \times S) = \prod_{i=1}^{l-1} \text{CH}^i(S)c_{l-i}(\alpha). \]

Lemma 5.3 and (5.8) yield the following proposition.

Proposition 5.9. Let \( D \) be a division algebra of degree \( l \) and \( S = \text{SB}(D) \). Then the homomorphism
\[ \text{type} : \text{Hom}(M(G), M(S)(2)[3]) \to \mathbb{Z}^{l-1} \]
is injective and \( \text{Im}(\text{type}) = l \mathbb{Z}^{l-1} \).

We will need the Chow groups of \( G \) that were computed in [11].
Proposition 5.10. Let $D$ be a central division algebra of prime degree $l$ and $G = \text{SL}_1(D)$. There is an element $h \in \text{CH}^{l+1}(G)$ such that

$$\text{CH}^*(G) = \mathbb{Z} \cdot 1 \oplus (\mathbb{Z} / l \mathbb{Z}) h \oplus (\mathbb{Z} / l \mathbb{Z}) h^2 \oplus \cdots \oplus (\mathbb{Z} / l \mathbb{Z}) h^{l-1}. \quad \square$$

Recall that $D \simeq \text{End}(I^\vee)$ and $G \times S \simeq \text{SL}(I^\vee)$, where $I$ is the tautological vector bundle over $S$ of rank $n$ (see Section 3).

Suppose that the algebra $D$ is split. We can compare the generic matrices $\bar{\alpha}$ in $G = \text{SL}_l$ and $\alpha$ in $G \times S = \text{SL}(I^\vee)$. The bundle $I^\vee \otimes L_t$ over $G \times S$ is trivial, hence

$$\bar{\alpha} \times S = \alpha \otimes L_t \quad \text{in} \quad K_1(G \times S).$$

We have the Chern classes $c_i(\alpha) \in H^{2i+1,l+1}(G \times S)$ and $c_i := c_i(\bar{\alpha}) \in H^{2i+1,l+1}(G)$. We also write $c_i$ for its image in $H^{2i+1,l+1}(G \times S)$ under the pull-back map given by the projection $G \times S \to G$.

By Proposition 6.1, we have

$$c_i = \sum_{j=0}^{i-1} \binom{i}{j} c_{i-j}(\alpha) s^j \quad \text{in} \quad H^{2i+1,l+1}(G \times S)$$

for all $i = 1, 2, \ldots, l - 1$, since the first Chern class of $L_t$ is equal to $-s$, where $s \in \text{CH}^1(S)$ is the class of a hyperplane section, and $c_0(\alpha) = 0$ as $\det(\alpha) = 1$. In particular, $c_1 = c_1(\alpha)$.

The group $H^{3,2}(G) = A^1(G, K_2)$ is infinite cyclic with a canonical generator, and this group does not change under field extensions. (This is true for every absolutely simple simply connected group, see [3, Part II, §9].) Therefore, we can write $H^{3,2}(G) = \mathbb{Z} c_1$ viewing $c_1$ as a generator of $H^{3,2}(G)$.

6. Symmetric and alternating powers

We consider motives with coefficients in $\mathbb{Z} = \mathbb{Z}[\frac{1}{(l-1)!}]$ in this section. Let $i = 0, 1, \ldots, l - 1$. The symmetric group $\Sigma_i$ acts naturally on the $i$-th tensor power $M^{\otimes i}$ of a motive $M$. The elements

$$\tau_i = \frac{1}{i!} \sum_{\sigma \in \Sigma_i} \sigma \quad \text{and} \quad \rho_i = \frac{1}{i!} \sum_{\sigma \in \Sigma_i} \text{sgn}(\sigma) \sigma$$

are idempotents in the group ring of $\Sigma_i$. The motives $\text{Sym}^i(M) := (M, \tau_i)$ and $\text{Alt}^i(M) := (M, \rho_i)$, that are split off $M$ by the projectors $\tau_i$ and $\rho_i$, are called the $i$-th symmetric power and $i$-th alternating power of $M$ respectively. We have $\text{Sym}^0(M) = \mathbb{Z} = \text{Alt}^0(M)$ and $\text{Sym}^1(M) = M = \text{Alt}^1(M)$.

We will need the following properties of symmetric and alternating powers.

Proposition 6.1 ([II, Proposition 2.3]). Let $M$ and $N$ be two motives. Then

1. $\text{Sym}^i(M[1]) \simeq (\text{Alt}^i M)[i]$ and $\text{Alt}^i(M[1]) \simeq (\text{Sym}^i M)[i]$,
2. $\text{Sym}^i(M(q)) \simeq (\text{Sym}^i M)(iq)$,
3. $\text{Sym}^i(M \oplus N) = \bigoplus_{k+m=i} \text{Sym}^k(M) \otimes \text{Sym}^m(N)$ and similarly for $\text{Alt}$. 
Corollary 6.2. We have
\[
\text{Sym}^i(Z(q)[p]) \simeq \begin{cases} 
Z(iq)[ip], & \text{if } p \text{ is even;} \\
0, & \text{if } p > 1 \text{ is odd.}
\end{cases}
\]

Example 6.3. Let \( N \) be the motive \( \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \cdots \oplus \mathbb{Z}(l-2)[2l-4] \) (see (4.2)). Then
\[
\text{Sym}^k(N(2)[3]) = (\text{Alt}^k N)(2k)[3k] = \bigoplus_{e_i=k} \mathbb{Z}(d_i + k)[2d_i + k],
\]
with the notation from Section 5.

Proposition 6.4 ([3, Proposition 15]). Let \( X \to Y \to Z \to X[1] \) be an exact triangle. Then there are sequences of morphisms
\[
\text{Alt}^i X = T_0 \to T_1 \to \cdots \to T_i = \text{Alt}^i Y,
\]
\[
\text{Sym}^i X = V_0 \to V_1 \to \cdots \to V_i = \text{Sym}^i Y
\]
and exact triangles
\[
T_{j-1} \to T_j \to \text{Alt}^{i-j} X \otimes \text{Alt}^i Z \to T_{j-1}[1],
\]
\[
V_{j-1} \to V_j \to \text{Sym}^{i-j} X \otimes \text{Sym}^i Z \to V_{j-1}[1]
\]
for every \( j = 1, 2, \ldots, i \).

Assuming that \( \text{Alt}^k X = 0 \) for \( k > 1 \), we get an exact triangle
\[
X \otimes \text{Alt}^{i-1} Z \to \text{Alt}^i Y \to \text{Alt}^i Z \to (X \otimes \text{Alt}^{i-1} Z)[1].
\]
Applying this to the exact triangle (4.1), we have the following proposition.

Corollary 6.5. There is an exact triangle
\[
(\text{Alt}^{i-1} N)(l-1)[2l-2] \to \text{Alt}^i M(S) \to \text{Alt}^i N \to (\text{Alt}^{i-1} N)(l-1)[2l-1].
\]

This proposition will be used in Section 12 to compute inductively the motivic cohomology of \( \text{Alt}^i N \).

The pure motive \( \text{Alt}^i M(S) \) is a direct summand of \( M(S^i) \) and the latter is a direct sum of shifts of the motive \( M(S) \). If \( D \) is a division algebra, the motive \( M(S) \) is indecomposable [4, Corollary 2.22]. When the coefficient ring \( \mathbb{Z} \) is the local ring \( \mathbb{Z}_{(0)} \), by uniqueness of the decomposition \( \mathbb{Z} \), Corollary 35, \( \text{Alt}^i M(S) \) is a pure motive that is a direct sum of pure shifts of \( M(S) \). Moreover, since in the split case
\[
\text{Alt}^i M(S_{\text{sep}}) = \mathbb{Z}((i(i-1)/2)[i(i-1)] \oplus \text{terms with higher shifts},
\]
we must have
\[
(6.6) \quad \text{Alt}^i M(S) = M(S)((i(i-1)/2)[i(i-1)] \oplus \text{terms with higher shifts}.
\]
We are going to prove the main theorem in the split case. Let $G = \text{SL}_l$ with prime $l$. We have
\[
N = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4] \oplus \cdots \oplus \mathbb{Z}(l-2)[2l-4].
\]
In fact, $N$ is a direct summand of the motive of $S = \mathbb{P}^{l-1}:
\[
M(S) = N \oplus \mathbb{Z}(l-1)[2l-2].
\]
The Chern classes $c_1(\bar{a}), c_2(\bar{a}), \ldots, c_{l-1}(\bar{a})$ with values in the motivic cohomology of $G$, where $\bar{a}$ is the generic matrix in $\text{SL}_l$, define a morphism
\[
\varphi_1 : M(G) \to N(2)[3].
\]
For every $i = 0, 1, \ldots, l-1$, consider the composition
\[
\varphi_i : M(G) \xrightarrow{\text{diag}} M(G^i) \xrightarrow{\varphi^i} N(2)[3] \oplus \cdots \oplus \text{Sym}^i(N(2)[3]),
\]
where the first morphism is given by the diagonal embedding.

**Proposition 7.1.** In the split case $G = \text{SL}_l$, the morphism
\[
\varphi = (\varphi_i) : M(G) \to \prod_{i=0}^{l-1} \text{Sym}^i(N(2)[3])
\]
is an isomorphism.

**Proof.** By Corollary 5.5, Example 6.3 and Lemma 2.3, for every $i = 0, 1, \ldots, l-1$, the morphism $\varphi_i$ is zero on $M(G)^{(j)}$ for $j > i$ and yields an isomorphism $M(G)^{(i)} \cong \text{Sym}^i(N(2)[3])$.
The result follows, as $\varphi$ is given by an invertible triangular matrix. \qed

8. Compactifications of $G$

Let $D$ be a central division algebra over $F$ and $G = \text{SL}_1(D)$. By [??, §6.2] and [??, §6], $G$ admits a smooth projective $G \times G$-equivariant compactification $X$. In other words, $X$ is a projective variety equipped with an action of $G \times G$ and containing $G$ as an open orbit on which the group $G \times G$ acts by the left-right translations. The motive of $X$ is split (i.e., $M(X)$ is a direct sum of Tate motives) over any field extension that splits $D$ by [??, Theorem 6.5].

By Proposition 5.10, the group $\text{CH}^{l+1}(G)$ is cyclic of order $l$. Choose a generator $h \in \text{CH}^{l+1}(G)$ and let $\bar{h} \in \text{CH}^{l+1}(X)$ be any element such that $\bar{h}|_G = h$. Set
\[
R = \mathbb{Z} \oplus \mathbb{Z}(l+1)[2l+2] \oplus \mathbb{Z}(2l+2)[4l+4] \oplus \cdots \oplus \mathbb{Z}(l^2-1)[2l^2-2].
\]

**Proposition 8.2** ([??, §8]). When $\mathbb{Z} = \mathbb{Z}(0)$, the morphism $M(X) \to R$ defined by the powers of $\bar{h}$ has a right inverse. Moreover, we have $M(X) \cong R \oplus T$ for a pure motive $T$ that is a direct sum of pure shifts of $M(S)$, where $S = \text{SB}(D)$.
Note that the composition \( \pi : M(G) \to M(X) \to R \) is given by the powers
\[
h^i \in \text{CH}^{i(p+1)}(G) = \text{Hom}(M(G), \mathbb{Z}(i(p+1)[2i(p+1)]).
\]

9. The morphism \( \theta \)

In this section we construct a morphism \( \theta : M(S)(2)[3] \to M(G) \).
As \( M(G\text{sep})^{(1)} \simeq N\text{sep}(2)[3] \) by Example 5.3, there is a natural isomorphism
\[
\text{Hom}(M(S\text{sep})(2)[3], M(G\text{sep})^{(1)}) \simeq \mathbb{Z}^{l-1}.
\]

Projecting \( M(G\text{sep}) \) onto \( M(G\text{sep})^{(1)} \), we get a composition
\[
type : \text{Hom}(M(S)(2)[3], M(G)) \to \text{Hom}(M(S\text{sep})(2)[3], M(G\text{sep})) \to \mathbb{Z}^{l-1}.
\]

By Corollary 5.4, there is a canonical morphism
\[
\theta : M(S)(2)[3] \to M(G \times S) \to M(G),
\]
where the first morphism is the projection to the \( i \)-component for \( i = 1 \) and
the second morphism is given by the projection \( G \times S \to G \).

Note that by definition of \( \theta \), the composition of \( M(S)(2)[3] \to M(G \times S) \)
with \( \tilde{\epsilon}_k(\alpha) : M(G \times S) \to M(S)(k+1)[2k+1] \) is zero for \( k > 1 \). In view of
(5.11) and Remark 5.3, the composition
\[
M(S\text{sep})(2)[3] \xrightarrow{\theta\text{sep}} M(G\text{sep}) \xrightarrow{\epsilon_i} \mathbb{Z}(i+1)[2i+1]
\]
for \( i = 1, 2, \ldots, l-1 \) coincides with \( i = \binom{i}{i-1} \) times the composition
\[
M(S\text{sep})(2)[3] \to M(G\text{sep} \times S\text{sep}) \xrightarrow{\epsilon_i s^{-1}} \mathbb{Z}(i+1)[2i+1].
\]
The latter is equal to the morphism \( s^{-1} : M(S\text{sep})(2)[3] \to \mathbb{Z}(i+1)[2i+1] \)
that is the identity on the summand \( \mathbb{Z}(i+1)[2i+1] \). It follows that
\[
type(\theta) = (1, 2, \ldots, l-1).
\]

10. A Key Lemma

Let \( D \) be a central simple algebra of degree \( l \) and \( S = \text{SB}(D) \).

**Lemma 10.1.** Let \( Y \) be a variety over \( F \) such that \( D \) is split over the residue field \( F(y) \) for every \( y \in Y \). Then the push-forward homomorphism \( \text{CH}_j(Y \times S) \to \text{CH}_j(Y) \) is surjective for every \( j \).

**Proof.** Let \( y \in Y \) be a point of dimension \( j \). As \( S \) is split over \( F(y) \), there is
a \( F(y) \)-rational point \( y' \in Y \times S \) in the fiber of the projection \( q : Y \times S \to Y \)
over \( y \). We have \([y] = q_!(\{y'\})\).

**Lemma 10.2.** Let \( U \) be a smooth variety such that \( D \) is split over \( F(u) \) for every \( u \in U \). Then \( H^{2j+1,j}(M(U) \otimes N) = 0 \) for every \( j \).
Proof. The last term in the exact sequence induced by the triangle (10.3)
\[ H^{2j,j}(U \times S) \to H^{2k,k}(U) \to H^{2j+1,j}(M(U) \otimes N) \to H^{2j+1,j}(U \times S), \]
where \( k = j - l + 1 \), is zero as \( 2j + 1 > 2j \). The first map is surjective by Lemma 10.3. \qed

Let \( Y \) be a closed subvariety of a smooth variety \( X \). We define the motive \( M_Y(X) \) by the triangle
\[(10.3) \quad M(X \setminus Y) \to M(X) \to M_Y(X) \to M(X \setminus Y)[1]. \]

Lemma 10.4. Let \( X \) be a smooth irreducible variety and \( Y \subset X \) an equidi-
dimensional closed subvariety such that \( S \) is split over \( F(y) \) for every \( y \in Y \). Then \( H^{2i+1,i}(M_Y(X) \otimes N) = 0 \) for every \( i \).

Proof. We proceed by induction on \( \dim(Y) \). Choose a closed subset \( Z \subset Y \) of pure codimension 1 such that \( Y \setminus Z \) is smooth. In the exact triangle
\[ M_{Y \setminus Z}(X \setminus Z) \to M_Y(X) \to M_Z(X) \to M_{Y \setminus Z}(X \setminus Z)[1] \]
the first term is isomorphic to \( M(Y \setminus Z)(c)[2c] \) for \( c = \text{codim}_X(Y) \) since \( Y \setminus Z \) is smooth by [13, Proposition 3.5.4]. In the exact sequence
\[ H^{2i+1,i}(M_Z(X) \otimes N) \to H^{2i+1,i}(M_Y(X) \otimes N) \to H^{2i+1,i}(M_{Y \setminus Z}(X \setminus Z) \otimes N) \]
the last term is isomorphic to \( H^{2i-2c+1,i-c}(M(Y \setminus Z) \otimes N) \) which is zero by Lemma 10.2, and the first term is zero by induction. \( \square \)

Let \( X \) be a smooth \( G \times G \)-equivariant compactification of \( G = \text{SL}_1(D) \) (see Section 8). Set \( Y := X \setminus G \). By [13, Lemma 7.1], \( D \) is split by \( F(y) \) for every \( y \in Y \). Applying Lemma 10.4 to the exact cohomology sequence for the exact triangle (10.3) we get the following corollary.

Corollary 10.5. The natural homomorphism
\[ \text{CH}^i(M(X) \otimes N) \to \text{CH}^i(M(G) \otimes N) \]
is surjective for every \( i \). \( \square \)

Lemma 10.6. The natural homomorphism
\[ \text{CH}^{i+1}(R \otimes N) \to \text{CH}^{i+1}(M(G) \otimes N) \]
induced by \( \pi : M(G) \to R \) is surjective.

Proof. The group \( \text{CH}^{i+1}(M(G) \otimes N) \) is \( l \)-torsion as it is split over a splitting field. Therefore, we may assume that \( Z = \mathbb{Z}_l \). Recall (see Section 8) that \( M(X) = R \oplus T \), where \( R \) is defined in (8.3), and the pure motive \( T \) is a direct sum of shifts of \( M(S) \). Moreover, \( \pi \) is the composition \( M(G) \to M(X) \to R \).
Consider the commutative diagram

\[
\begin{array}{ccc}
\text{CH}^{l+1}(R \otimes N) & \xrightarrow{\alpha} & \text{CH}^{l+1}(M(G) \otimes N) \\
\downarrow{\beta} & & \downarrow{\beta} \\
\text{CH}^{l+1}(M(X) \otimes N) & \leftarrow & \text{CH}^{l}(M(X) \otimes N) \\
\downarrow & & \downarrow{\gamma} \\
\text{CH}^{l+1}(T \otimes N) & \xrightarrow{\gamma} & \text{CH}^{l}(T \otimes N \otimes S),
\end{array}
\]

where \(\gamma\) (and the two homomorphisms above \(\gamma\)) are induced by the morphism \(Z(l-1)[2l-2] \rightarrow M(S)\). By Corollary 10.5, \(\beta\) is surjective. The pure motive \(T\) is a direct sum of shifts of \(M(S)\), so \(T \otimes N\) by Lemma 4.4. It follows from Lemma 10.1 that \(\gamma\) is surjective.

Again by Lemma 4.4,

\[
\text{CH}^{l+1}(M(G) \otimes N \otimes S) = \prod_{i=0}^{l-2} \text{CH}^{2l-i}(G \times S) = 0.
\]

as \(\text{CH}^j(G \times S) = 0\) for \(j \geq l\) in view of Corollary 5.7. Recall that \(M(X) = R \oplus T\). By diagram chase, \(\alpha\) is surjective.

Consider the following key diagram:

\[
\begin{array}{ccc}
\text{Hom}(M(G), M(S)(2)[3]) & \xrightarrow{\xi} & \text{Hom}(M(G), N(2)[3]) \\
\downarrow{l} & & \downarrow{\rho} \\
H^{2l+1}(G \times S) & \xrightarrow{\tau} & H^{3,2}(G) \\
& & \downarrow{\sigma} \\
& & \text{CH}^{l+1}(M(G) \otimes N) \rightarrow 0.
\end{array}
\]

The rows of the diagram are induced by the exact triangle \(\text{Hom}(M(G), M(S)[3]) \xrightarrow{\xi} \text{Hom}(M(G), N(2)[3]) \rightarrow \text{CH}^{l+1}(G) \rightarrow 0\). The left vertical homomorphism is an isomorphism by \(\text{Hom}(M(G), M(S)(2)[3]) \xrightarrow{\xi} \text{Hom}(M(G), N(2)[3]) \rightarrow \text{CH}^{l+1}(G) \rightarrow 0\). The rows are exact since \(\text{CH}^{l+1}(G \times S) \simeq \text{CH}^{l+1}(S) = 0\) by Corollary 5.7. The morphisms \(\rho\) and \(\sigma\) are induced by the morphism \(\nu : N \rightarrow Z\) (see Section 4).

The diagram is commutative. Indeed, both compositions in the right square take a morphism \(\varphi : M(G) \rightarrow N(2)[3]\) to \((\varepsilon(2)[3] \circ \varphi) \otimes \nu\). (The morphisms \(\varepsilon\) and \(\nu\) are defined in Section 3.) Both compositions in the left square take a morphism \(\psi : M(G) \rightarrow M(S)(2)[3]\) to \(\nu(2)[3] \circ \psi\).

Now we can prove the following key lemma.

**Lemma 10.7.** The homomorphism induced by the morphism \(\nu : N \rightarrow Z\)

\[
\sigma : \text{CH}^{l+1}(G) \rightarrow \text{CH}^{l+1}(M(G) \otimes N)
\]

is an isomorphism.
Proof. In the commutative diagram

\[
\begin{array}{c}
\text{CH}^{l+1}(R) \\ \downarrow \\
\text{CH}^{l+1}(G)
\end{array} \longrightarrow \begin{array}{c}
\text{CH}^{l+1}(R \otimes N) \\ \downarrow \\
\text{CH}^{l+1}(M(G) \otimes N)
\end{array}
\]

the right vertical map (induced by \(\pi\)) is surjective by Lemma \ref{lemma:surjective}. We have \(\text{CH}^{l+1}(R) = \mathbb{Z}\) and by Lemma \ref{lemma:zero}, \(\text{CH}^{l+1}(R \otimes N) = \text{CH}^0(N) = \mathbb{Z}\), hence the top map is an isomorphism. It follows that the bottom map in the diagram is surjective. If \(D\) is split, the group \(\text{CH}^{l+1}(G)\) is trivial an we are done.

Suppose \(D\) is a division algebra. Since \(H^{3,2}(G) = \mathbb{Z}c_1\) (see Section \ref{section:3.2}), by Proposition \ref{proposition:division}, the image of \(\tau\) in the key diagram is equal to \(l\mathbb{Z}c_1\). It follows that \(\text{CH}^{l+1}(M(G) \otimes N)\) is a cyclic group of order \(l\). The group \(\text{CH}^{l+1}(G)\) is also cyclic of order \(l\) by Proposition \ref{proposition:cyclic}. The statement follows from the surjectivity of \(\sigma\).

It follows from Corollary \ref{corollary:corollary} and Example \ref{example:example} that \(\text{Hom}(M(G_{\text{sep}}), N_{\text{sep}}(2)[3])\) is naturally isomorphic to \(l\mathbb{Z}^{l-1}\). Consider the map

\[
\text{type} : \text{Hom}(M(G), N(2)[3]) \rightarrow \text{Hom}(M(G_{\text{sep}}), N_{\text{sep}}(2)[3]) \simeq \mathbb{Z}^{l-1}.
\]

Proposition 10.8. Let \(D\) be a division algebra of degree \(l\) and \(S = \text{SB}(D)\). Then the homomorphism

\[
\text{type} : \text{Hom}(M(G), N(2)[3]) \rightarrow \mathbb{Z}^{l-1}
\]

is injective. Its image consists of all tuples \((a_1, a_2, \ldots, a_{l-1})\) such that

\[
a_1 \equiv 2a_2 \equiv \cdots \equiv (l-1)a_{l-1} \pmod{l}.
\]

Proof. Let \(\beta \in \text{Hom}(M(G), N(2)[3])\) have zero type. We have \(\rho(\beta) = kc_1\) with \(k\) the first component of the type of \(\beta\). Hence \(k = 0\). It follows from Lemma \ref{lemma:zero} that the image of \(\beta\) in \(\text{CH}^{l+1}(G)\) is trivial, t.e., \(\beta = \xi(\gamma)\) for some \(\gamma \in \text{Hom}(M(G), M(S)(2)[3])\) with \(\text{type}(\gamma) = 0\). By Proposition \ref{proposition:division}, \(\gamma = 0\). This proves the injectivity of \(\text{type}\).

Take any \(\beta \in \text{Hom}(M(G), N(2)[3])\). We have

\[
\text{type}(\beta) = (a_1, a_2, a_3, \ldots, a_{l-1})
\]

for some \(a_i \in \mathbb{Z}\). Composing \(\beta\) with \(\theta : M(S)(2)[3] \rightarrow M(G)\) (see Section \ref{section:3.2}) we get a morphism \(M(S)(2)[3] \rightarrow N(2)[3]\) of type \((a_1, 2a_2, 3a_3, \ldots, (l-1)a_{l-1})\).

By Proposition \ref{proposition:division}, we have \(a_1 \equiv 2a_2 \equiv \cdots \equiv (l-1)a_{l-1} \pmod{l}\).

By Proposition \ref{proposition:division}, the image of the map \(\text{type}\) contains \(l\mathbb{Z}^{l-1}\). To finish the proof it suffices to find a \(\beta\) such that \(\text{type}(\beta)\) is not divisible by \(l\). By Lemma \ref{lemma:division} and diagram chase, the map \(\rho\) is surjective. Hence there is a morphism \(\beta : M(G) \rightarrow N(2)[3]\) such that the composition of \(\beta\) with \(N(2)[3] \rightarrow \mathbb{Z}(2)[3]\) coincides with \(c_1\), i.e., \(\text{type}(\beta) = (1, \ldots)\). \(\square\)
Remark 10.9. If $\alpha \in \text{Hom}(M(G), N(2)[3])$ is such that type$(\alpha)$ is not divisible by $l$, then $\rho(\alpha)$ is not divisible by $l$ in $H^{3,2}(G) = \mathbb{Z} c_1$, and hence by Lemma 10.7, the image of $\alpha$ in $\text{CH}^{l+1}(G)$ is not zero if $D$ is not split.

11. MAIN THEOREM

Now the coefficient ring is $\mathbb{Z} = \mathbb{Z}[\frac{1}{(l-1)!}]$. By Proposition 10.8, there is a unique morphism $\beta_1 : M(G) \to N(2)[3]$ with
type$(\beta_1) = (1^{-1}, 2^{-1}, \ldots, (l - 1)^{-1})$.
For every $i = 0, 1, \ldots, l - 1$ we have a composition
\[
\beta_i : M(G) \xrightarrow{\text{diag}} M(G^i) \xrightarrow{\beta^i} N(2)[3] \xrightarrow{\otimes i} \text{Sym}^i(N(2)[3]).
\]

Theorem 11.1. Let $D$ be a central simple algebra of prime degree $l$ over a perfect field $F$. Then the morphism
\[
\beta = (\beta_i) : M(\text{SL}_1(D)) \to \prod_{i=0}^{l-1} \text{Sym}^i(N(2)[3]) = \prod_{i=0}^{l-1} (\text{Alt}^i N)(2i)[3i]
\]
in the category $\text{DM}(F)$ of motives over $F$ with coefficients in $\mathbb{Z}[\frac{1}{(l-1)!}]$ is an isomorphism.

Proof. We first prove the theorem in the split case. The morphisms $\beta : M(G) \to N(2)[3]$ and $\varphi : M(G) \to N(2)[3]$ of type $(1, 1, \ldots, 1)$ defined in Section 10 differ by an automorphism of $N(2)[3]$ of type $(1, 2, \ldots, l - 1)$. Therefore, the statement follows from Proposition 10.4.

Assume that $D$ is a division algebra. We show next that $1_{M(S)} \otimes \beta$ is an isomorphism. By Corollary 10.4, the motive $M(S) \otimes M(G) = M(S \times G)$ is a direct sum of shifts of $M(S)$. The motive $M(S) \otimes N$ is a direct sum of shifts of $M(S)$ by Lemma 10.3, hence so is $M(S) \otimes (\text{Alt}^i N)$. By the first part of the proof, $\beta$ is an isomorphisms over a splitting field, hence so is $1_{M(S)} \otimes \beta$. By Lemma 10.2, $1_{M(S)} \otimes \beta$ is an isomorphism. It follows that
\[
1_{M(S)} \otimes \beta \quad \text{is an isomorphism for every} \quad i > 0.
\]

We embed the category $\text{DM}(F)$ into a larger triangulated category $\text{DM}^{eff}(F)$ of motivic complexes with coefficients in $\mathbb{Z}$ as a full subcategory (see [12]).

Let $\tilde{\mathcal{C}}(S)$ be the motive in $\text{DM}^{eff}(F)$ associated with the simplicial scheme given by the powers of $S$ (see [20, Appendix B]). Using the exact triangle in the proof of [21, Proposition 8.1] we see from (11.2) that $1_{\mathcal{C}(S)} \otimes \beta$ is an isomorphism.

It follows from Remark 10.8 that the composition
\[
M(G) \xrightarrow{\beta_1} N(2)[3] \xrightarrow{\cdot(2)[3]} \mathbb{Z}(l+1)[2l + 2]
\]
represents a nontrivial element $h \in \text{CH}^{l+1}(G)$. Therefore, for every $i = 0, 1, \ldots, l - 1$, the composition
\[
M(G) \xrightarrow{\beta_i} \text{Sym}^i(N(2)[3]) \xrightarrow{\delta} \text{Sym}^i(\mathbb{Z}(l+1)[2l + 2]) = \mathbb{Z}(i(l+1))[2i(l+1)],
\]
where $\delta_i = \text{Sym}^i(\varepsilon(2)[3])$, is equal to $h^i$. By Section 8, we have a commutative diagram

$$
\begin{array}{ccc}
M(G) & \xrightarrow{\beta} & \prod_{i=0}^{l-1} \text{Sym}^i(N(2)[3]) \\
\downarrow{\alpha} & & \downarrow{\delta} \\
M(X) & \xrightarrow{\gamma} & R,
\end{array}
$$

where $X$ is a smooth compactification of $G$ and $\delta = \prod \delta_i$.

Consider the motive $\tilde{C}(S)$ in $\text{DM}^{eff}_F(G)$ defined by the exact triangle

$$\begin{equation}
\text{(11.3)} \quad \tilde{C}(S) \to \tilde{C}(S) \to \mathbf{Z} \to \tilde{C}(S)[1].
\end{equation}$$

We also have an exact triangle

$$M(G) \to M(X) \to M_Y(X) \to M(G)[1],$$

where $Y = X \setminus G$. The algebra $D$ is split by the residue field $F(y)$ for every $y \in Y$ by Section 10, Lemma 7.1. Hence, by Section 17, Lemma 3.4, $\tilde{C}(S) \otimes M_Y(X) = 0$. Therefore, $1_{\tilde{C}(S)} \otimes \alpha$ is an isomorphism.

By Proposition 8.2, $M(X) \simeq R \oplus T$, where $T$ is a direct sum of shifts of $M(S)$ if $Z = Z(0)$. Since $\tilde{C}(S) \otimes T = 0$, we have $1_{\tilde{C}(S)} \otimes \gamma$ is an isomorphism when $Z = Z(0)$. As $\tilde{C}(S)$ vanishes over a splitting field of $D$ of degree $l$, $1_{\tilde{C}(S)} \otimes \gamma$ is an isomorphism when $Z = Z\left[\frac{1}{(l-1)!}\right]$.

By Proposition 6.4 applied to the exact triangle (4.1), there is a Postnikov tower connecting $\text{Sym}^i(\varepsilon(2)[3])$ and

$$\text{Sym}^i(Z(l+1)[2l+2]) = Z(i(l+1))[2i(l+1)]$$

with “factors” divisible by $M(S)$. Since $\tilde{C}(S) \otimes M(S) = 0$, the morphism

$$1_{\tilde{C}(S)} \otimes \text{Sym}^i \varepsilon : \tilde{C}(S) \otimes \text{Sym}^i(N(2)[3]) \to \tilde{C}(S)(i(l+1))[2i(l+1)]$$

is an isomorphism. Therefore, $1_{\tilde{C}(S)} \otimes \delta$ is an isomorphism.

It follows from the commutativity of the diagram that $1_{\tilde{C}(S)} \otimes \beta$ is an isomorphism. Finally, by 5-lemma applied to the exact triangle (4.3), the morphism $\beta$ is an isomorphism. \hfill $\square$

### 12. Applications

As an application of Theorem 4.1, we compute certain motivic cohomology of $G$. The Chow groups $\text{CH}^i(G) = H^{2i-i}(G)$ are given in Proposition 5.10. In Theorem 12.4 below we compute the groups $H^{2i+1+j+1}(G)$.

The following Lemma is an immediate application of the exact triangle in Corollary 6.5.

**Lemma 12.1.** If $p > 2q$, then

$$H^{p,q}(\text{Alt}^{i-1}N) \simeq H^{p+2l-1,q+l-1}(\text{Alt}^lN).$$

\hfill $\square$
We compute the Chow groups of $N$.

**Lemma 12.2.** We have

$$\text{CH}^i(N) = \begin{cases} 
\mathbb{Z}, & \text{if } i = 0; \\
\mathbb{Z}/l\mathbb{Z}, & \text{if } i = 1, 2, \ldots, l - 2; \\
F^\times / \text{Nrd}(D^\times), & \text{if } i = l; \\
0, & \text{otherwise.}
\end{cases}$$

**Proof.** We may assume that $Z = \mathbb{Z}$. Using (4.1) we get $\text{CH}^i(N) \cong \text{CH}^i(S)$ for $i \leq l - 2$ and apply Lemma 3.1. In the exact sequence

$$0 \to \text{CH}^{l-1}(N) \to \text{CH}^{l-1}(S) \to \text{CH}^0(\mathbb{Z})$$

the last map is injective again by Lemma 3.1, hence $\text{CH}^{l-1}(N) = 0$. In the exact sequence

$$H^{2l-1,i}(S) \to H^{1,1}(F) \to \text{CH}^i(N) \to 0$$

the first map is isomorphic to $A^{l-1}(S, K_1) \to K_1^M(F) = F^\times$ and its image is equal to $\text{Nrd}(D^\times)$ since the image is generated by the norms from finite field extensions that split $D$. By Lemma 4.5, $\text{CH}^i(N) = 0$ if $i > l$.

**Lemma 12.3.** We have

$$H^{2i+1,j}(\text{Alt}^2 N) = \begin{cases} 
\mathbb{Z}/l\mathbb{Z}, & \text{if } i = l - 1; \\
F^\times / \text{Nrd}(D^\times), & \text{if } i = 2l - 1; \\
0, & \text{otherwise.}
\end{cases}$$

**Proof.** Using the triangle in Corollary 6.5, we get an exact sequence

$$\text{CH}^i(\text{Alt}^2 M(S)) \to \text{CH}^{i-l+1}(N) \to H^{2i+1,j}(\text{Alt}^2 N) \to 0.$$ 

The middle group is trivial if $i < l - 1$, $l = 2l - 2$ and $l > 2l - 1$ by Lemma 12.2. The first map in the sequence is surjective in the split case since $\text{Alt}^2 N$ is pure and $2l + 1 > 2l$. As $\text{CH}^{i-l+1}(N) = l\mathbb{Z}$ for $i = l, l + 1, \ldots, 2l - 3$ by Lemma 12.2, the first map is also surjective in general for these values of $i$. If $i = 2l - 1$, the first group is trivial as $\text{Alt}^2 M(S)$ is a direct summand of $M(S \times S)$ and $\dim(S \times S) = 2l - 2$.

Finally consider the case $i = l - 1$. We may assume that $Z = \mathbb{Z}_{(l)}$. As

$$\text{Alt}^2 M(S) = M(S)(1)[2] \oplus M(S)(3)[6] \oplus \cdots \oplus M(S)(l - 2)[2l - 4],$$

we have

$$\text{CH}^{l-1}(\text{Alt}^2 M(S)) = \text{CH}^1(S) \oplus \text{CH}^3(S) \oplus \cdots \oplus \text{CH}^{2l-2}(S).$$

This is divisible by $l$ when going to the split case by Lemma 4.1. Whence the case $i = l - 1$.

Lemmas 12.1, 12.2 and 12.3 then yield
**Theorem 12.4.** Let $D$ be a central division algebra of degree $l$ over $F$. Then

$$H^{2i+1,j+1}(\text{SL}_1(D)) = \begin{cases} F^\times, & \text{if } i = 0; \\ \mathbb{Z}_C, & \text{if } i = 1; \\ \mathbb{Z}/l\mathbb{Z}, & \text{if } i = 2, 3, \ldots, l - 1; \\ F^\times / \text{Nrd}(D^\times), & \text{if } i = k(l + 1) + 1 \text{ for } k = 1, \ldots, l - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $G = \text{SL}_1(D)$. Note that the cup-product maps

$$F^\times \otimes \mathbb{Z}/l\mathbb{Z} = H^{1,1}(F) \otimes \text{CH}^{k(l+1)}(G) \to H^{2k(l+1)+1,k(l+1)+1}(G) = F^\times / \text{Nrd}(D^\times)$$

are natural surjections for $k = 1, \ldots, l - 1$.

Consider the motivic spectral sequence for $G$ when $D$ is not split (see [7]):

$$E_2^{p,q} = H^{p-q,q}(G) \Rightarrow K_{p-q}(G).$$

The $K$-groups of $G$ were computed in [18, Theorem 6.1]. In particular, $K_0(G) = \mathbb{Z}$ and $K_1(G) = K_1(F) \oplus K_0(D) \oplus K_0(D^{op}) \simeq F^\times \oplus 3\mathbb{Z} \oplus 3\mathbb{Z}$. It follows that the zero-diagonal limit terms $E_\infty^{p,-p}$ are trivial if $p \neq 0$. On the other hand, by Proposition 5.10, we have

$$E_2^{p,-p} = \text{CH}^p(G) = \begin{cases} \mathbb{Z}, & \text{if } p = 0; \\ \mathbb{Z}/l\mathbb{Z}, & \text{if } p = i(l+1) \text{ for } i = 1, \ldots, l - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that in the split case all the differentials coming to the zero diagonal are trivial. It follows that in the general case such differential are $l$-torsion. By [14, Theorem 3.4], nontrivial differentials coming to the zero diagonal can appear only on pages $E_i^{*,*}$ with $l - 1$ dividing $s - 1$. It follows that the nonzero differentials appear only on page $E_i^{*,*}$ and they are

$$(12.5) \quad d : E_i^{1+k(l+1),-2-k(l+1)} \to E_i^{(k+1)(l+1),-(k+1)(l+1)} = \mathbb{Z}/l\mathbb{Z}$$

for $k = 0, 1, \ldots, l - 2$. These maps are all surjective and are isomorphisms for $k > 0$, thus “clearing” the zero diagonal and partially the first diagonal. The other differentials on the $l$-th page coming to the first diagonal are the cup-products with $F^\times$ of the differentials (12.5). Nontrivial $E_\infty^{*,*}$-terms on the first diagonal are $F^\times$ and $l\mathbb{Z}$ ($l - 1$ times).
Below is a fragment of the third page of the spectral sequence when \( l = 3 \).

\[
\begin{array}{c}
\text{F}^\times \\
\text{Z} \\
* \quad \text{Z} \\
* \quad 3\text{Z} \\
* \quad 0 \\
* \quad \text{Z}/3\text{Z} \\
* \quad \text{F}^\times/\text{N} \\
\end{array}
\]

It follows from Theorem 12.4 that the Chern classes \( c_2, c_3, \ldots, c_{l-1} \) (which are defined in the split case) are not defined over \( F \) if \( D \) is not split. (Recall that \( c_1 \) is always defined over \( F \).) We will show that the product \( c_1 c_2 \cdots c_k \) is defined over \( F \) for all \( k = 1, 2, \ldots, l-1 \).

**Lemma 12.6.** For every \( i = 1, 2, \ldots, l-1 \), if \( q < i(i-1)/2 \), the group \( H^{p,q}(\text{Alt}^i N) \) is trivial for every \( p \).

**Proof.** Induction on \( k \). We may assume that \( \text{Z} = \text{Z}(l) \). The basic triangle (12.5) yields an exact sequence

\[
H^{p-2l+1,q-l+1}(\text{Alt}^{i-1} N) \to H^{p,q}(\text{Alt}^i N) \to H^{p,q}(\text{Alt}^i M(S)).
\]

The first term is trivial by induction as \( q-l+1 < (i-1)(i-2)/2 \). The last group is zero by (12.1).

**Theorem 12.7.** Let \( G = \text{SL}_1(D) \) for a central simple algebra \( D \) of prime degree \( l \). Then the product of Chern classes \( c_1 c_2 \cdots c_k \) is defined over \( F \) for all \( k = 1, 2, \ldots, l-1 \).
Proof. We may assume that $Z = Z_{(l)}$. The product $c_1 c_2 \cdots c_k$ belongs to $H^{(k+1)^2 - 1, (k+1)(k+2)/2 - 1}(G)$. Consider the following direct summand of this group (see Theorem 11.1):

$$H^{(k+1)^2 - 1, (k+1)(k+2)/2 - 1}(\text{Alt}^k N(2k)[3k]) = \text{CH}^{(k^2 - k)/2}(\text{Alt}^k N).$$

The basic triangle (6.5) yields an exact sequence

$$H^{k^2 - k - 2l + 1, (k^2 - k)/2 - l + 1}(\text{Alt}^{k-1} N) \rightarrow \text{CH}^{(k^2 - k)/2}(\text{Alt}^k N) \rightarrow \text{CH}^{(k^2 - k)/2}(\text{Alt}^k M(S)) \rightarrow \text{CH}^{(k^2 - k)/2 - l + 1}(\text{Alt}^{k-1} N).$$

The side terms are trivial by Lemma 12.6. The third term is isomorphic to $H^{0,0}(S) = Z$ by (6.6). Therefore, the group $\text{CH}^{(k^2 - k)/2}(\text{Alt}^k N)$ contains an element representing $c_1 c_2 \cdots c_k$ over a splitting field.

References


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