DEVELOPMENTS IN ALGEBRAIC K-THEORY AND QUADRATIC FORMS AFTER THE WORK OF MILNOR

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In the book [21] Milnor introduced the K_2-groups for arbitrary rings. Milnor's discovery of K_2 using partly Steinberg's ideas of universal central extensions turned out to be a truly revolutionary step. The definition of the higher algebraic K-theory was in the air, and soon after that, Quillen showed the world how to define higher K-groups in [25].

On the other hand, Matsumoto's theorem on the presentation of the group K_2 of a field led Milnor in [19] to the definition the Milnor's K-groups $K_n^M(F)$ of a field F. Milnor also proposed two conjectures that connect the groups $k_n^M(F) := K_n^M(F)/2K_n^M(F)$ with certain Galois cohomology groups $H^n(F)$ and the components $GW_n(F)$ of the graded Witt ring of F via the two homomorphisms

$$\begin{array}{ccc} & k_n^M(F) & \\ {}^{h_n^F}\swarrow & & \searrow^{s_n^F} \\ H^n(F) & & GW_n(F) \end{array}$$

Milnor conjectured that both h_n^F and s_n^F were isomorphisms. The aim of this paper is to survey the results influenced by [19] that culminate in the solution of Milnor's conjectures.

Milnor assumed that the characteristic of the field F is different from 2. In the case $\mathrm{char}(F) = 2$, one can still define the maps h_n^F and s_n^F and prove the analogs of Milnor's conjectures. The Witt ring of quadratic forms should be replaced by the Witt ring of bilinear forms. We include the case of fields of characteristic 2 in the present paper.

In Section 3 we review Voevodsky's proof of Milnor's conjecture on the bijectivity of h_n^F. In the proof Voevodsky introduces a number of revolutionary ideas and tools, the main one being the use of motivic cohomology. Another tool is the motivic Steenrod operations defined by Voevodsky in analogy with the classical topological operations. This part of the work was influenced by the paper [18], where Milnor determined the structure of the dual of the Steenrod algebra. In particular, Milnor introduced a basis for the Steenrod algebra. The motivic analogs of some of the basis operations, the *Milnor operations* Q_i, played an essential role in Voevodsky's proof.

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1. K_0, K_1, K_2 and Milnor's K-theory of fields

Let R be an associative ring with unit. The group $K_0(R)$ is defined as the Grothendieck group of the category $\mathbf{P}(R)$ of finitely generated projective left R-modules. If R is a connected commutative ring, the rank of a module provides a group homomorphism $K_0(R) \to \mathbb{Z}$. This map is an isomorphism if R is a field.

Identifying each $n \times n$ matrix a with the $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives an embedding of $\mathrm{GL}_n(R)$ into $\mathrm{GL}_{n+1}(R)$. The union of the groups of the resulting sequence

$$\mathrm{GL}_1(R) \subset \mathrm{GL}_2(R) \subset \cdots \subset \mathrm{GL}_n(R) \subset \cdots$$

is called the *infinite general linear group* $\mathrm{GL}(R)$. The subgroup $\mathrm{E}(R)$ of $\mathrm{GL}(R)$ generated by elementary matrices coincides with the commutator subgroup of $\mathrm{GL}(R)$. The group $K_1(R)$ is defined as the factor group $\mathrm{GL}(R)/E(R)$ (see [4]).

If R is a commutative ring, the determinant of a matrix provides a group homomorphism $K_1(R) \to R^\times$. This map is an isomorphism if R is a field.

The group K_2 of a ring R was defined by Milnor in [21]. For $n \geq 3$ the *Steinberg group* $\mathrm{St}_n(R)$ is the group defined by generators $x_{ij}(r)$, with $i, j = 1, \ldots, n$, $i \neq j$ and $r \in R$, subject to the following relations:

- $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$,
- $[x_{ij}(r), x_{jl}(s)] = x_{il}(rs)$ for $i \neq l$,
- $[x_{ij}(r), x_{kl}(s)] = 1$ for $j \neq k$ and $i \neq l$.

The Steinberg relations are satisfied by the elementary matrices $e_{ij}(r)$ in the group $\mathrm{GL}_n(R)$ and can be viewed as "elementary relations" between the elementary matrices. There is a group homomorphism $\varphi_n : \mathrm{St}_n(R) \to \mathrm{GL}_n(R)$ taking $x_{ij}(r)$ to $e_{ij}(r)$.

There is an obvious homomorphism $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ for any $n \geq 3$. We write $\mathrm{St}(R)$ for the colimit of the groups $\mathrm{St}_n(R)$. The homomorphisms φ_n give rise to a group homomorphism

$$\varphi : \mathrm{St}(R) \to \mathrm{GL}(R).$$

The image of φ is the subgroup $\mathrm{E}(R)$ of $\mathrm{GL}(R)$. The group $K_2(R)$ is defined as the kernel of φ. Thus there is an exact sequence of groups

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 1.$$

The group $K_2(R)$ is abelian and it coincides with the center of $\mathrm{St}(R)$.

The Steinberg group $\mathrm{St}(R)$ can be described in terms of universal central extensions as follows. A *central extension* of a group G is a surjective group homomorphism $\alpha : H \to G$ such that the kernel of α is a central subgroup of H. For example, the canonical homomorphism $\mathrm{St}(R) \to \mathrm{E}(R)$ is a central extension of $E(R)$.

A central extension $\alpha : H \to G$ of a group G is called *universal* if for any other central extension $\alpha' : H' \to G$ of G there is a unique homomorphism $H \to H'$ over G. A universal central extension of G is unique up to a canonical isomorphism. A group G admits a universal central extension $\alpha : H \to G$ if and only if G is perfect. In this case the kernel of α is canonically isomorphic to $H_2(G, \mathbb{Z})$.

Theorem 1.1. ([21, Th. 5.10]) *For any ring R, the canonical homomorphism $\mathrm{St}(R) \to \mathrm{E}(R)$ is a universal central extension of $\mathrm{E}(R)$. In particular, $K_2(R) \simeq H_2(\mathrm{E}(R), \mathbb{Z})$.*

Let $A, B \in \mathrm{E}(R)$ be two commuting matrices. Choose elements $a, b \in \mathrm{St}(R)$ such that $\varphi(a) = A$ and $\varphi(b) = B$. As $K_2(R)$ is central in $\mathrm{St}(R)$, the element

$$A \star B := [a, b] \in K_2(R)$$

is well defined. If r and s are commuting invertible elements in R, we define the *Steinberg symbol* $\{r, s\} \in K_2(R)$ as follows:

$$\{r, s\} = \mathrm{diag}(r, r^{-1}, 1) \star \mathrm{diag}(s, 1, s^{-1}),$$

where $\mathrm{diag}(r, s, t)$ is the diagonal 3×3 matrix with the diagonal terms r, s and t. The Steinberg symbol satisfies the following relations:

- $\{r, s_1 s_2\} = \{r, s_1\} + \{r, s_2\}$,
- $\{r_1 r_2, s\} = \{r_1, s\} + \{r_2, s\}$,
- (*Steinberg Relation*) $\{r, 1 - r\} = 0$ if $r, 1 - r \in R^\times$.

Let R be a commutative ring. Write $K_2^M(R)$ for the abelian group defined by generators $\{r, s\}$ for all $r, s \in R^\times$ subject to the relations above. There is an obvious group homomorphism $K_2^M(R) \to K_2(R)$. The following important result was proven by Matsumoto in [14]:

Theorem 1.2. *If R is a field, then the homomorphism $K_2^M(R) \to K_2(R)$ is an isomorphism.*

1.A. **Milnor ring of a field.** Inspired by Matsumoto's theorem, Milnor defined in [19] the higher Milnor K-groups for a field as follows. Let F be a field and let T_* denote the tensor ring of the multiplicative group $F^\times$ of the field F. That is a graded ring with

$$T_n = F^\times \otimes_{\mathbb{Z}} F^\times \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F^\times$$

the nth tensor power of $F^\times$ over $\mathbb{Z}$. For instance, $T_0 = \mathbb{Z}$, and $T_1 = F^\times$. The *Milnor ring $K_*(F)$ of F* is the factor ring of T_* by the ideal generated by tensors of the form $a \otimes b$ with $a + b = 1$.

The class of a tensor $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ in $K_*^M(F)$ is denoted by $\{a_1, a_2, \ldots, a_n\}$ and is called a *symbol*. We have $K_n^M(F) = 0$ if $n < 0$, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$. For $n \geq 2$, $K_n^M(F)$ is generated (as an abelian group) by the symbols $\{a_1, a_2, \ldots, a_n\}$ with $a_i \in F^\times$ subject to the following defining relations:

- $\{a_1, \ldots, a_i a_i', \ldots, a_n\} = \{a_1, \ldots, a_i, \ldots, a_n\} + \{a_1, \ldots, a_i', \ldots, a_n\}$;
- (*Steinberg Relation*) $\{a_1, a_2, \ldots, a_n\} = 0$ if $a_i + a_j = 1$ for some $i \neq j$.

The canonical homomorphism

$$K_n^M(F) \to K_n(F)$$

is an isomorphism for $n = 0, 1, 2$.

Note that the operation in the group $K_n^M(F)$ is written additively. In particular, $\{ab\} = \{a\} + \{b\}$ in $K_1^M(F)$ where $a, b \in F^\times$.

The product in the ring $K_*^M(F)$ is given by the rule

$$\{a_1, a_2, \ldots, a_n\} \cdot \{b_1, b_2, \ldots, b_m\} = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}.$$

Suppose that a field E has a discrete valuation v with residue field F_v. For an element $u \in E^\times$ with $v(u) = 0$ write $\bar{u}$ for the residue of u in F_v. There is the *residue homomorphism*

$$\partial_v : K_n^M(E) \to K_{n-1}^M(F_v)$$

satisfying

$$\partial_v\big(\{x_1, x_2, \dots, x_n\}\big) = v(x_1)\{\bar{x}_2, \dots, \bar{x}_n\}$$

for arbitrary $x_1, x_2, \dots, x_n \in E^\times$ such that $v(x_2) = \cdots = v(x_n) = 0$.

Each monic irreducible polynomial $p \in F[t]$ gives rise to (p)-adic discrete valuation on the rational function field $F(t)$ with residue field $F_p := F[t]/(p)$. Hence there is the associated residue homomorphism

$$\partial_p : K_n^M\big(F(t)\big) \to K_{n-1}^M(F_p).$$

In [19, Th. 2.3] Milnor computed the K^M-groups of the field $F(t)$:

Theorem 1.3. *The sequence*

$$0 \to K_n^M(F) \xrightarrow{\operatorname{res}_{F(t)/F}} K_n^M\big(F(t)\big) \xrightarrow{(\partial_p)} \coprod K_{n-1}^M(F_p) \to 0,$$

where the direct sum extends over all monic irreducible polynomial $p \in F[t]$, is exact.

In [25] Quillen defined the higher K_n-groups of a small exact category. Quillen's higher K-groups $K^Q(R)$ of a ring R are defined as the groups $K_n(\mathbf{P}(R))$. There are canonical isomorphisms $K_n(R) \to K_n^Q(R)$ for $n = 0, 1$ and 2. Hence for a field F, the canonical homomorphism $K_n^M(F) \to K_n^Q(F)$ is an isomorphism for $n \leq 2$.

Let X be an algebraic variety over F and let $X^{(i)}$ denote the set of points in X of codimension i. For any i, we write $A^i(X, K_{n+i}^M)$ for the homology group of the complex

$$\coprod_{x \in X^{(i-1)}} K_{n+1}^M\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{(i)}} K_n^M\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{(i+1)}} K_{n-1}^M\big(F(x)\big),$$

where the differentials are induced by residue homomorphisms (see [29]). For example, Theorem 1.3 asserts that $A^0(\mathbb{A}^1, K_n^M) \simeq K_n^M(F)$ and $A^1(\mathbb{A}^1, K_n^M) = 0$.

In a similar fashion one defines the groups $A^i(X, K_{n+i}^Q)$.

If X is a projective variety of dimension d, then the corestriction homomorphisms $K_n^M(F(x)) \to K_n^M(F)$ for every $x \in X^{(d)}$ yield the *norm homomorphism*

$$N_X : A^d(X, K_{n+d}^M) \to K_n^M(F).$$

1.B. **The norm residue homomorphism.** Let F be a field, F_{sep} a separable closure of F and $\Gamma_F = \operatorname{Gal}(F_{\text{sep}}/F)$. For a discrete Γ_F-module M we write $H^n(F, M)$ for the nth cohomology group $H^n(\Gamma_F, M)$.

We write $k_*^M(F)$ for the factor ring $K_*^M(F)/2K_*^M(F)$ and $\{a_1, a_2, \dots, a_n\}$ for the class of the symbol $\{a_1, a_2, \dots, a_n\}$ in $k_n^M(F)$.

Suppose first that the characteristic of F is different from 2. We view the group $\mathbb{Z}/2\mathbb{Z}$ as a Γ_F-module with trivial action. The graded group $H^*(F, \mathbb{Z}/2\mathbb{Z})$ has the structure of a commutative ring with respect to the cup-product. We simply write $H^*(F)$ for this ring.

The group $\mathbb{Z}/2\mathbb{Z}$ can be identified with the subgroup $\{\pm 1\}$ of the multiplicative group $F^\times$. The exact sequence of Γ_F-modules

$$0 \to \mathbb{Z}/2\mathbb{Z} \to F_{\text{sep}}^\times \xrightarrow{2} F_{\text{sep}}^\times \to 1$$

yields an exact sequence

$$H^0(F, F_{\text{sep}}^\times) \xrightarrow{2} H^0(F, F_{\text{sep}}^\times) \to H^1(F) \to H^1(F, F_{\text{sep}}^\times)$$

of cohomology groups. The group $H^1(F, F_{\text{sep}}^\times)$ is zero by Hilbert's Theorem 90. Identifying the first two groups with $F^\times$, we get an isomorphism

$$h_1^F : F^\times/F^{\times 2} \xrightarrow{\sim} H^1(F).$$

Lemma 1.4. (Bass-Tate) *$h_1^F(a) \cup h_1^F(b) = 0$ in $H^2(F)$ for all $a, b \in F^\times$ with $a + b = 1$.*

It follows from Lemma 1.4 that the ring homomorphism $T^*(F^\times/F^{\times 2}) \to H^*(F)$ induced by h_1^F factors through $k_*^M(F)$. The resulting homomorphisms

$$h_n^F : k_n^M(F) \to H^n(F)$$

are called the *norm residue homomorphisms modulo* 2.

Suppose now that F is a field of characteristic 2. Let Ω_F^n be the nth exterior power of the absolute differential module Ω_F^1. Let

$$d : \Omega_F^{n-1} \to \Omega_F^n$$

be the exterior derivation defined by

$$d(xdy_1 \wedge dy_2 \wedge \cdots \wedge dy_{n-1}) = dx \wedge dy_1 \wedge dy_2 \wedge \cdots \wedge dy_{n-1}.$$

Write $H^n(F)$ for the kernel of the homomorphism

$$\wp : \Omega_F^n \to \Omega_F^n/d(\Omega_F^{n-1}),$$

given by

$$\wp(x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_n}{y_n}) = (x^2 + x)\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_n}{y_n}.$$

Note that $H^0(F) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^1(F) \simeq F^\times/F^{\times 2}$, so the groups $H^n(F)$ in the case $\operatorname{char}(F) = 2$ can be viewed as the analogs of the cohomology groups $H^n(F)$ for fields of characteristic different from 2.

If $x + y = 1$ in F, then $dx \wedge dy = 0$ in Ω_F^2. It follows that the homomorphism

$$h_n^F : k_n^M(F) \to H^n(F),$$

$$h_n^F(\{x_1, x_2, \ldots, x_n\}) = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

is well defined.

Thus we have the homomorphisms h_n^F defined for any $n \geq 0$ and every field F.

In [19, p. 340] Milnor wrote: "I don't know of any examples for which the homomorphism h_n^F fails to be injective." The following statement is known as the *Milnor's Conjecture 1* for a field F:

MC1(n): The homomorphism $h_n^F : k_n^M(F) \to H^n(F)$ is an isomorphism.

If $\operatorname{char}(F) = 2$, the conjecture **MC1**(n) was proved by Kato in [12].

Suppose that $\operatorname{char}(F) \neq 2$. The conjectures **MC1**(0) and **MC1**(1) obviously hold for all fields.

In [19, Lemma 4.5] Milnor proved:

Proposition 1.5. **MC1**(n) *holds for F a finite field, a local field or a global field.*

Using Milnor's Theorem 1.3 and similar statement for Galois cohomology (see [9, Th. 9.3]), one deduces the following:

Proposition 1.6. *If* $\mathbf{MC1}(n)$ *holds for a field* F *and* $\mathbf{MC1}(n-1)$ *holds for all finite extensions* L/F*, then* $\mathbf{MC1}(n)$ *holds for the rational function field* $F(t)$*.*

We shall review the proof of $\mathbf{MC1}(n)$ in Section 3.

2. Bilinear and quadratic forms

Let F be an arbitrary field. Let $b : V \times V \to F$ be a symmetric bilinear form on a finite dimensional vector space V over F. For a subspace $W \subset V$ write $W^\perp$ for the *orthogonal complement* of W in V with respect to b.

A subspace $W \subset V$ is called *totally isotropic* if $W \subset W^\perp$. If b is nondegenerate and W is a totally isotropic subspace, then $\dim W \leq \frac{1}{2}\dim V$. We say that W is a *lagrangian* for b if $\dim W = \frac{1}{2}\dim V$, equivalently $W^\perp = W$. A nondegenerate symmetric bilinear form is called *metabolic* if it has a lagrangian (see [7, Ch. I]).

The isometry classes of nondegenerate symmetric bilinear forms over F is a semi-ring under orthogonal sum and tensor product. The Grothendieck ring of this semi-ring is called the *Witt-Grothendieck* ring of F and denoted by $\widehat{W}(F)$.

The quotient ring $W(F)$ of $\widehat{W}(F)$ by the ideal generated the classes of metabolic forms is called the *Witt ring* of nondegenerate symmetric bilinear forms over F. We write $[b]$ for the class of a form b. Two nondegenerate anisotropic symmetric bilinear forms b_1 and b_2 are isomorphic if and only if $[b_1] = [b_2]$ in $W(F)$. A nondegenerate symmetric bilinear form b is metabolic if and only if $[b] = 0$ in $W(F)$.

We write $\langle a_1, a_2, \dots, a_n\rangle$ for the bilinear form on the space F^n with the diagonal matrix $\operatorname{diag}(a_1, a_2, \dots, a_n)$ and write $\langle\langle a\rangle\rangle$ for the binary symmetric bilinear form $\langle 1, -a\rangle$. Let $I(F)$ be the ideal in $W(F)$ consisting of the classes of even-dimensional forms. It is called the *fundamental ideal* of $W(F)$ and is generated by the classes $\langle\langle a\rangle\rangle$ with $a \in F^\times$.

The powers $I^n(F) := I(F)^n$ of the fundamental ideal $I(F)$ in $W(F)$ define the filtration

$$W(F) \supset I(F) \supset I^2(F) \supset \cdots \supset I^n(F) \supset \cdots .$$

In [19, Question 4.4] Milnor conjectured that the intersection of the ideals $I^n(F)$ is zero. This conjecture was proved by Arason and Pfister in [2]. Moreover, they proved the following

Theorem 2.1. *Let* b *be a binary symmetric bilinear form over* F *such that* $[b] \in I^n(F)$*. If* $[b] \neq 0$ *then* $\dim(b) \geq 2^n$*.*

We write $GW_*(F)$ for the associated *graded Witt ring*

$$\coprod_{n\geq 0} I^n(F)/I^{n+1}(F).$$

The homomorphism $W(F) \to \mathbb{Z}/2\mathbb{Z}$ taking the class of a form b to $\dim(b)$ modulo 2 yields an isomorphism $GW_0(F) \simeq \mathbb{Z}/2\mathbb{Z}$. The *signed discriminant* homomorphism

$$I(F) \to F^\times/F^{\times 2}, \qquad [b] \mapsto (-1)^{\dim(b)/2}\det(b)$$

gives rise to an isomorphism $GW_1(F) \simeq F^\times/F^{\times 2}$.

Let $a_1, a_2, \dots, a_n \in F^\times$. We denote the tensor product $\langle\langle a_1\rangle\rangle \otimes \langle\langle a_2\rangle\rangle \otimes \cdots \otimes \langle\langle a_n\rangle\rangle$ by $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle$ and call it a *bilinear n-fold Pfister form.* The isometry classes of bilinear n-fold Pfister forms generate $I^n(F)$ as an abelian group.

The map $F^\times \to GW_1(F) = I(F)/I^2(F)$ defined by $a \mapsto \langle\langle a\rangle\rangle + I^2(F)$ is a homomorphism. If $a, b \in F^\times$ with $a + b = 1$, then the form $\langle\langle a, b\rangle\rangle$ is metabolic, hence this map gives rise to a graded ring homomorphism

$$s_*^F : k_*^M(F) \to GW_*(F) \tag{2.2}$$

taking the symbol $\{a_1, a_2, \dots, a_n\}$ to $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle + I^{n+1}(F)$. Since the graded ring $GW_*(F)$ is generated by the degree one component $I(F)/I^2(F)$, the map s_*^F is surjective.

In [19, Question 4.3] Milnor raised the following problem that we call *Milnor's Conjecture 2* for a field F:

MC2(n): The homomorphism $s_n^F : k_n^M(F) \to GW_n(F)$ is an isomorphism.

We saw above that s_n^F is an isomorphism for $n = 0$ and 1. In [19, Th. 4.1] Milnor proved that s_2^F is an isomorphism. If $\operatorname{char}(F) = 2$, the conjecture **MC2**$(n)$ was proved by Kato in [12]. The proof of **MC2**(n) in the case $\operatorname{char}(F) \neq 2$ is due to Orlov, Vishik and Voevodsky (see [23] and Section 4).

Let b be a symmetric bilinear form on a vector space V over F. The map $q : V \to F$ given by $q(v) = b(v, v)$ is the *quadratic form associated to* b. If $\operatorname{char}(F) \neq 2$, the bilinear form b can be reconstructed from q via

$$b(v, w) = [q(v + w) - q(v) - q(w)]/2.$$

In the case $\operatorname{char}(F) = 2$ the theory of quadratic forms deviates from that of bilinear forms. The latter theory was studied by Milnor in [20].

Let F be a field and

$$\alpha = (a_1, a_2, \dots, a_n) \in (F^\times)^n.$$

Write $\{\alpha\}$ for the symbol $\{a_1, a_2, \dots, a_n\}$ in $k_n^M(F)$ and $\langle\langle\alpha\rangle\rangle$ for the n-fold Pfister form $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle$.

Consider the bilinear form

$$b_\alpha = \langle\langle a_1, a_2, \dots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$$

and the smooth projective quadric hypersurface X_α of dimension $2^{n-1} - 1$ given by the equation $q_\alpha = 0$, where q_α is the quadratic form associated to b_α.

The following statement was proved by Elman and Lam in [8, Th. 3.2] in the case $\operatorname{char}(F) \neq 2$ and by Aravire and Baeza in [3] if $\operatorname{char}(F) = 2$.

Proposition 2.3. *Let $\alpha \in (F^\times)^n$. Then the following conditions are equivalent:*

(1) *The form b_α is isotropic.*
(2) *The n-fold Pfister form $\langle\langle\alpha\rangle\rangle$ is metabolic.*
(3) *The symbol $\{\alpha\}$ is trivial in $k_n^M(F)$.*
(4) *The quadric X_α has a point over F.*

3. Milnor's Conjecture 1

In this Section we review the proof of Milnor's Conjecture $\mathbf{MC1}(n)$ for fields of characteristic different from 2 given by Voevodsky in [35]. A detailed account on the history of Milnor's Conjectures can be found in [24].

Recall that the homomorphisms h_0^F and h_1^F are isomorphisms, i.e., $\mathbf{MC1}(0)$ and $\mathbf{MC1}(1)$ hold.

3.A. **The homomorphism h_2^F.** Below we review the proof of Milnor's Conjecture $\mathbf{MC1}(2)$ in the case $\operatorname{char}(F) \neq 2$ given in [15].

One of the approaches to the study of h_n^F is the comparison of h_n^F with h_n^L for various field extensions L/F. The case of a quadratic extension was considered by Arason in [1]. He proved that there is an infinite exact sequence of cohomology groups for the quadratic field extension $L = F(\sqrt{a})$ of F:

$$\cdots \to H^n(F) \xrightarrow{\operatorname{res}_{L/F}} H^n(L) \xrightarrow{\operatorname{cor}_{L/F}} H^n(F) \xrightarrow{\cup(a)} H^{n+1}(F) \xrightarrow{\operatorname{res}_{L/F}} \dots \tag{3.1}$$

A similar sequence (a complex) can be written for the k_*^M-groups:

$$\cdots \to k_n^M(F) \xrightarrow{\operatorname{res}_{L/F}} k_n^M(L) \xrightarrow{\operatorname{cor}_{L/F}} k_n^M(F) \xrightarrow{\cup\{a\}} k_{n+1}^M(F) \xrightarrow{\operatorname{res}_{L/F}} \dots \tag{3.2}$$

The two sequences are connected by the norm residue homomorphisms h_*^F and h_*^L. If one could prove that the second sequence is exact then Milnor's Conjecture $\mathbf{MC1}(n)$ would follow by induction on the number of symbols (injectivity) and on the degree of a splitting field (surjectivity).

One can attempt to prove the exactness of (3.2) at the term $k_n^M(L)$ as follows. Let $u \in k_n^M(L)$ be in the kernel of $\operatorname{cor}_{L/F}$. There is an irreducible algebraic variety X (depending on u) over the field $F_1 = F_0(a)$, where F_0 is the prime subfield of F, and a "universal" element $U \in k_n^M(L_1(X))$, where $L_1 = F_1(\sqrt{a})$ and $L_1(X)$ is the function field of X_{L_1}, such that

- The corestriction of U for the quadratic extension $L_1(X)/F_1(X)$ is trivial in $k_n^M(F_1(X))$.
- There is a place $F_1(X) \dashrightarrow F$ specializing U to u.

If $\mathbf{MC1}(n)$ holds for the fields $F_1(X)$ and $L_1(X)$, it would follow from the exactness of (3.1) that the sequence (3.2) for the quadratic field extension $L_1(X)/F_1(X)$ is exact at the term $k_n^M(L_1(X))$, therefore U belongs to the image of the restriction map $k_n^M(F_1(X)) \to k_n^M(L_1(X))$ and hence, by specialization, u belongs to the image of the restriction map $k_n^M(F) \to k_n^M(L)$, i.e., the sequence (3.2) is exact at the term $k_n^M(L)$.

The field F_1 is either finite or global, or rational function field over a finite or global field. By Propositions 1.5 and 1.6, $\mathbf{MC1}(n)$ holds for the fields F_1 and L_1. The variety X over F_1 appears to be a product of projective quadrics of the form X_α for $\alpha = (a_1, \dots, a_n)$ with $a_n = a$ (see Section 2). Thus, to complete the proof one needs to prove the following "going-up" statement:

Let $\alpha = (a_1, \dots, a_n) \in (F^\times)^n$ and $L = F(\sqrt{a_n})$. If $\mathbf{MC1}(n)$ holds for the fields F and L, then $\mathbf{MC1}(n)$ holds for the fields $F(X_\alpha)$ and $L(X_\alpha)$.

In [30] Suslin proved this statement in the case $n = 2$. In the proof he used a computation of Quillen's K^Q-groups of the projective conic curve X_α given in [25, §8]. This completes the proof of $\mathbf{MC1}(2)$.

3.B. **Hilbert's Theorem 90.** Consider the following statement $\mathbf{H90}(n)$ for a Galois quadratic extension L/F:

Let σ the generator of $\mathrm{Gal}(L/F)$. *Then the following sequence*

$$K_n^M(L) \xrightarrow{1-\sigma} K_n^M(L) \xrightarrow{cor_{L/F}} K_n^M(F) \tag{3.3}$$

is exact.

The statement $\mathbf{H90}(0)$ is obvious and $\mathbf{H90}(1)$ is the classical Hilbert's Theorem 90 for quadratic extensions.

Another approach to the proof of $\mathbf{MC1}(n)$ is to prove $\mathbf{H90}(n)$ first for Galois quadratic field extensions which then implies that the sequence (3.2) is exact at the first and the second terms. This exactness implies $\mathbf{MC1}(n)$ via the specialization argument as described in Section 3.A.

In order to prove $\mathbf{H90}(n)$ for a quadratic field extension $L = F(\sqrt{a})/F$ by induction on n, Suslin proposed the following "going-down" method. Suppose first that the corestriction homomorphism $K_{n-1}^M(L) \to K_{n-1}^M(F)$ is surjective. We can construct the homomorphism

$$s : K_n^M(F) \to K_n^M(L)/(1-\sigma)K_n^M(L)$$

as follows. Let $u = \{a_1, \dots, a_n\} \in K_n^M(F)$. Write $\{a_1, \dots, a_{n-1}\} = cor_{L/F}(v)$ for some $v \in K_{n-1}^M(L)$ and set $s(u) = v \cdot \{a_n\} + (1-\sigma)K_n^M(L)$. One checks that s extends to a well defined homomorphism that is the inverse to the homomorphism $K_n^M(L)/(1-\sigma)K_n^M(L) \to K_n^M(F)$ induced by the corestriction, whence $\mathbf{H90}(n)$ for L/F.

Write $V(n, L/F)$ for the homology of the complex (3.3). It is sufficient to construct a field extension F'/F such that the corestriction homomorphism

$$K_{n-1}^M(F'L) \to K_{n-1}^M(F')$$

is surjective and the natural map $V(n, L/F) \to V(n, F'L/F')$ is injective.

Let $A_n(F)$ be the set of all tuples $(a_1, \dots, a_n)$ with $a_i \in F^\times$ for all i and $a_n = a$. For any $\alpha = (a_1, \dots, a_n) \in A_n(F)$ consider the projective quadric X_α as in Section 2. By Proposition 2.3, the symbol $\{a_1, \dots, a_n\}$ is trivial in k_n^M over any field extension of F over which X_α has a point. In particular, $\{a_1, \dots, a_{n-1}\}$ belongs to the image of the corestriction $K_{n-1}^M(L(X_\alpha)) \to K_{n-1}^M(F(X_\alpha))$.

For any finite subset $S \subset A_n(F)$, let X_S be the product of X_α over all $\alpha \in S$. If S' is a subset of S, we have a natural projection $X_S \to X_{S'}$ inducing the field extension $F(X_S)/F(X_{S'})$. Let F_1 be the colimit of the fields $F(X_S)$ over all finite subsets $S \subset A_n(F)$. By construction, the image of $K_{n-1}^M(F) \to K_{n-1}^M(F_1)$ belongs to the image of the corestriction $K_{n-1}^M(F_1 L) \to K_{n-1}^M(F_1)$.

Iterating the construction, we build a tower of field extensions $F \subset F_1 \subset F_2 \subset \dots$. Let F' be the union of all F_i. Clearly, the corestriction $K_{n-1}^M(F'L) \to K_{n-1}^M(F')$ is surjective.

Suppose that for any field extension K/F and any $\alpha \in A_n(K)$, the natural homomorphism

$$V(n, KL/K) \to V(n, KL(X_\alpha)/K(X_\alpha)) \tag{3.4}$$

is injective. It follows that the map $V(n, L/F) \to V(n, LF'/F')$ is injective and hence $\mathbf{H90}(n)$ holds for L/F.

The injectivity of (3.4) in the case $n = 2$ follows from the injectivity of the restriction homomorphism $A^1(X_\alpha, K_2^M) \to A^1((X_\alpha)_L, K_2^M)$. The latter is equivalent to the injectivity of the norm map

$$N_{X_\alpha} : A^1(X_\alpha, K_2^M) \to K_1(F).$$

The injectivity can be deduced from Quillen's computation of the K^Q-groups of the conic curve X_α.

There are also "elementary" proofs of the injectivity of the norm map N_{X_α} in the case $n = 2$ given in [37] and [7] that don't use Quillen's K-theory.

3.C. **The homomorphism h_3^F.** The conjecture **MC1**(3) was proved in [16] and by Rost in [26]. One deduces **MC1**(3) from **H90**(3) as explained in Section 3.B and applies the "going-down" method described in the previous section in order to prove **H90**(3). We need to prove that for a quadratic field extension L/F and an appropriate quadric X_α, the restriction homomorphism $A^1(X_\alpha, K_3^M) \to A^1((X_\alpha)_L, K_3^M)$ is injective.

How to compute $A^1(X_\alpha, K_3^M)$? Although the groups K_3^M and K_3^Q for fields are different, one still can prove that $A^1(X_\alpha, K_3^M) = A^1(X_\alpha, K_3^Q)$. In order to compute $A^1(X_\alpha, K_3^Q)$, we use the Brown-Gersten-Quillen spectral sequence [25, §8]

$$A^p(X_\alpha, K_{-q}^Q) \Rightarrow K_{-p-q}^Q(X_\alpha)$$

and Swan's computation of the K^Q-theory of quadrics [32]. One has to prove that the differential $A^1(X_\alpha, K_3^Q) \to A^3(X_\alpha, K_4^Q) = A^3(X_\alpha, K_4^M)$ in the spectral sequence is zero. This follows from the injectivity of the norm homomorphism

$$N_{X_\alpha} : A^3(X_\alpha, K_4^M) \to K_1(F).$$

Note that the injectivity of the norm homomorphism for the quadric X_α played an essential role in the proof of Milnor's Conjecture in the cases $n = 2$ and 3. It is also used in Voevodsky's proof of the general case that will be considered in the next section.

The drawback of the use of Quillen's K-groups is obvious: in higher degrees the group K_n^Q for fields deviates from K_n^M, so we cannot use the powerful machinery of the higher K^Q-theory. On the other hand, we are lacking tools for the calculation of Milnor's K-groups of the function fields of algebraic varieties.

3.D. **Proof of Milnor's Conjecture MC1(n).** Milnor's Conjecture **MC1**(n) was proved by Voevodsky in [35]. He has introduced a number of breakthrough ideas in the proof. The main one is the use of motivic cohomology in the place of Quillen's K-theory. The motivic cohomology fits better in the context of Milnor's Conjecture. In particular, Milnor's K-groups of a field are certain motivic cohomology groups.

Most of Voevodsky's tools have analogs in algebraic topology. In particular, the motivic cohomology in algebraic geometry plays the role of singular cohomology for CW-complexes.

Let F be a field. The existence of motivic complexes over F was predicted by Beilinson in [5]. Suslin and Voevodsky introduced in [31] the *motivic complexes* $\mathbb{Z}(q)$, for $q \geq 0$, of sheaves of abelian groups in the étale topology on the category of smooth schemes over F such that:

(i) $\mathbb{Z}(0) = \mathbb{Z}$ placed in degree 0, (*ii*) $\mathbb{Z}(1)$ is quasi-isomorphic to the sheaf $\mathbb{G}_m$ of invertible functions placed in degree 1, i.e., $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$,

(*iii*) $\mathbb{Z}(q)$ is acyclic in degree greater than q.

For an abelian group A, let $A(n)$ denote the derived tensor product $\mathbb{Z}(n) \overset{L}{\otimes} A$. For a smooth scheme X over F define the *motivic cohomology groups*

$$H^{p,q}(X, A) := \mathbb{H}^p_{\mathrm{Zar}}(X, A(q)),$$

and the *étale motivic cohomology groups*

$$H^{p,q}_{\text{ét}}(X, A) := \mathbb{H}^p_{\text{ét}}(X, A(q)).$$

If $X = \mathrm{Spec}(F)$, we write $H^{p,q}(F, A)$ and $H^{p,q}_{\text{ét}}(F, A)$ for the motivic and the étale motivic cohomology groups of X respectively.

By Property (*iii*),

$$H^{p,q}(F, A) = 0 \quad \text{if} \quad p > q. \tag{3.5}$$

More generally, if X is a smooth scheme over F, then

$$H^{p,q}(X, A) = 0 \quad \text{if} \quad p > q + \dim(X). \tag{3.6}$$

Milnor's K-groups are the motivic cohomology: in the case $p = q$ we have a canonical isomorphism

$$H^{p,p}(F, \mathbb{Z}) \simeq K^M_p(F).$$

This can be generalized as follows:

Proposition 3.7. ([35, Lemma 4.11]) *Let X be smooth scheme over F of dimension d. Then*

$$H^{p+2d,p+d}(X, \mathbb{Z}) \simeq A^d(X, K^M_{p+d})$$

for all p.

If $\mathrm{char}(F) \neq 2$, then the complex $\mathbb{Z}(q)/2\mathbb{Z}(q)$ is quasi-isomorphic to $\mathbb{Z}/2\mathbb{Z}$ in the étale topology, hence

$$H^{p,q}_{\text{ét}}(F, \mathbb{Z}/2\mathbb{Z}) = H^p(F, \mathbb{Z}/2\mathbb{Z}) = H^p(F). \tag{3.8}$$

In [34] Voevodsky introduced the triangulated category of *motivic complexes* $\mathbf{DM}(F)$ over F containing the motivic complexes $\mathbb{Z}(q)$ as objects. There is the functor M from the category of smooth schemes over F to $\mathbf{DM}(F)$ that takes a scheme X to the *motive* $M(X)$ *of* X. This functor is analogous to the singular chain complex functor in topology.

The motivic cohomology groups are represented by the motivic complexes:

$$H^{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{\mathbf{DM}(F)}(M(X), \mathbb{Z}(q)[p]).$$

Therefore, the motivic cohomology can be defined for all objects in $\mathbf{DM}(F)$.

Let π be the canonical morphism from the étale cite to the Zariski cite. We have the following relation between motivic and étale motivic cohomology groups for a smooth scheme X:

$$H^{p,q}_{\text{ét}}(X, \mathbb{Z}) \simeq H^p_{Zar}(X, R\pi_*(\pi^*(\mathbb{Z}(q)))).$$

Let $L(q)$ be the canonical truncation of the complex $R\pi_*(\pi^*(\mathbb{Z}(q)))$ at level $q+1$, i.e., $L(q)$ is a subcomplex of $R\pi_*(\pi^*(\mathbb{Z}(q)))$ whose ith cohomology sheaf is the same as for $R\pi_*(\pi^*(\mathbb{Z}(q)))$ if $i \leq q+1$ and zero if $i > q+1$. The canonical morphism

$\mathbb{Z}(q) \to R\pi_*(\pi^*(\mathbb{Z}(q)))$ factors through $L(q)$. Let $K(q)$ be the complex defined by the distinguished triangle in $\mathbf{DM}(F)$:

$$\mathbb{Z}(q) \to L(q) \to K(q) \to \mathbb{Z}(q)[1]. \tag{3.9}$$

Lichtenbaum has conjectured in [13] that the complex $K(q)$ is acyclic, i.e., the morphism of complexes $\mathbb{Z}(q) \to L(q)$ is a quasi-isomorphism, in particular, the group $H^{n+1,n}_{\text{ét}}(F, \mathbb{Z})$ is trivial. If $n = 0$ we have $\mathbb{Z}(0) = \mathbb{Z}$ and

$$H^{1,0}_{\text{ét}}(F, \mathbb{Z}) = H^1_{\text{ét}}(F, \mathbb{Z}) = \operatorname{Hom}_{cont}(\operatorname{Gal}(F_{\text{sep}}/F), \mathbb{Z}) = 0.$$

If $n = 1$ we have $\mathbb{Z}(1) = \mathbb{G}_m[-1]$ and Lichtenbaum's conjecture is equivalent to the classical Hilbert's Theorem 90: $H^1_{\text{ét}}(F, \mathbb{G}_m) = 0$.

We consider the 2-part of Lichtenbaum's Conjecture. Let $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal $2\mathbb{Z}$. We say that the *2-part of motivic Hilbert's Theorem 90* holds for a field F if the following condition holds:

MH90(n)*: The group* $H^{n+1,n}_{\text{ét}}(F, \mathbb{Z}_{(2)})$ *is trivial.*

The condition **MH90**(n) for fields of characteristic 2 was proved by Geisser and Levine in [10, Th 8.6].

Note that by [35, Lemma 6.8] and (3.5),

$$H^{n+1,n}_{\text{ét}}(F, \mathbb{Q}) \simeq H^{n+1,n}(F, \mathbb{Q}) = 0,$$

hence $H^{n+1,n}_{\text{ét}}(F, \mathbb{Z}_{(2)})$ is a 2-primary torsion group.

In fact, **MH90**(n) is equivalent to the 2-part of Lichtenbaum's Conjecture:

Theorem 3.10. ([35, Th. 6.6]) *The condition* **MH90**(n) *holds for all fields if and only if the complex* $K(n) \otimes \mathbb{Z}_{(2)}$ *is quasi-isomorphic to zero for all fields.*

Corollary 3.11. *Assume that* **MH90**(n) *holds for all fields. Then for any smooth simplicial scheme* $\mathcal{X}$ *one has:*

(1) *The homomorphisms*

$$H^{p,q}(\mathcal{X}, \mathbb{Z}_{(2)}) \to H^{p,q}_{\text{ét}}(\mathcal{X}, \mathbb{Z}_{(2)})$$

are isomorphisms for $p - 1 \leq q \leq n$ *and monomorphisms for* $p = q + 2$ *and* $q \leq n$.

(2) *The homomorphisms*

$$H^{p,q}(\mathcal{X}, \mathbb{Z}/2\mathbb{Z}) \to H^{p,q}_{\text{ét}}(\mathcal{X}, \mathbb{Z}/2\mathbb{Z})$$

are isomorphisms for $p \leq q \leq n$ *and monomorphisms for* $p = q + 1$ *and* $q \leq n$.

In the special case $p = q = n$ and $\mathcal{X} = \operatorname{Spec}(F)$, the isomorphism in Corollary 3.11(2) coincides with the norm residue homomorphism h^F_n. Thus, **MH90**(n) implies **MC1**(n).

The motivic Hilbert's Theorem 90 implies the classical one:

Proposition 3.12. ([35, Lemma 6.11]) **MH90**(n) *for all fields implies* **H90**(n) *for all Galois quadratic field extensions.*

Thus, in order to prove **MC1**(n) it suffices to show **MH90**(n). By induction, we may assume **MH90**(q) for all $q < n$ and therefore, **MC1**(q) and **H90**(q) hold by Proposition 3.12.

The proof of **MH90**(n) for fields of characteristic not 2 splits into several steps.

A field F is called 2-*special* if F has no nontrivial odd degree extensions.

Step 1: **MH90**(n) *holds for* 2*-special fields* F *with* $k_n^M(F)=0$.

Indeed, under the assumption and induction hypothesis, the corestriction homomorphism $K_{n-1}^M(L)\to K_{n-1}^M(F)$ is surjective for any quadratic field extension L/F. As in Section 3.B, one deduces **H90**(n) for any quadratic field extension L/F. It follows that the sequence (3.2) is exact at the second term. Hence the two assumptions on F are preserved under quadratic extensions and hence hold for all finite extensions of F as F is 2-special.

We claim that $H^n(F)=0$. Let $u\in H^n(F)$. By induction on the degree of a splitting field, we may assume that u is split over a quadratic field extension $L=F(\sqrt{a})/F$. From the commutativity of the diagram

$$\begin{array}{ccccc} k_{n-1}^M(F) & \xrightarrow{\{a\}} & k_n^M(F) & & \\ {\scriptstyle h_{n-1}^F}\downarrow\wr & & {\scriptstyle h_n^F}\downarrow & & \\ H^{n-1}(F) & \xrightarrow{(a)} & H^n(F) & \xrightarrow{\mathrm{res}_{L/F}} & H^n(L) \end{array}$$

with the exact bottom row, the triviality of $k_n^M(F)$ and **MC1**$(n-1)$, it follows that $u=0$. This proves the claim.

The exactness of the sequence (see (3.8))

$$H^n(F)\to H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})\xrightarrow{2}H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)}),$$

induced by $0\to\mathbb{Z}_{(2)}(n)\xrightarrow{2}\mathbb{Z}_{(2)}(n)\to(\mathbb{Z}/2\mathbb{Z})(n)\to 0$, and the claim imply that $H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})$ has no 2-torsion and hence is trivial as $H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})$ is a 2-primary torsion group. Thus **MH90**(n) holds for F.

Recall that for an $\alpha=(a_1,a_2,\dots,a_n)\in(F^\times)^n$ we write X_α for the projective quadric hypersurface X_α of dimension $2^{n-1}-1$ given by the equation $q_\alpha=0$, where $q_\alpha=\langle\langle a_1,a_2,\dots,a_{n-1}\rangle\rangle\perp\langle -a_n\rangle$.

Step 2: *Reduction to the proof of the injectivity of*

$$H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})\to H^{n+1,n}_{\text{ét}}(F(X_\alpha),\mathbb{Z}_{(2)}). \tag{3.13}$$

One employs the "going-down" method to prove **MH90**(n). In order to reduce to the special case in Step 1, we need to prove the injectivity of (3.13) and the injectivity of the restriction $H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})\to H^{n+1,n}_{\text{ét}}(E,\mathbb{Z}_{(2)})$ for an odd degree field extension E/F. The latter follows by the corestriction argument and the fact that $H^{n+1,n}_{\text{ét}}(F,\mathbb{Z}_{(2)})$ is a 2-torsion group.

Another breakthrough idea of Voevodsky is the introduction and application of the simplicial scheme associated to a usual scheme over F. For a smooth scheme X over F write $\check{C}(X)$ for the *Cech simplicial scheme* with the terms $\check{C}(X)_n=X^{n+1}$ and face and degeneracy morphisms given by partial projections and diagonals respectively (see [35, Appendix B]). If X has a point over F, then $\check{C}(X)$ is contractible.

For an $\alpha\in(F^\times)^n$ set $\mathcal{X}_\alpha:=\check{C}(X_\alpha)$. Denote by $M(\mathcal{X}_\alpha)$ the motive of $\mathcal{X}_\alpha$ in the triangulated category $\mathbf{DM}(F)$.

If X_α has a point over F, then $M(\mathcal{X}_\alpha)=\mathbb{Z}$, so in general, $M(\mathcal{X}_\alpha)$ is a "twisted form" of $\mathbb{Z}$. Write $H^{p,q}(\mathcal{X}_\alpha,\mathbb{Z})$ for the motivic cohomology group $H^{p,q}\big(M(\mathcal{X}_\alpha),\mathbb{Z}\big)$.

In the étale topology X_α "has a point", hence (see [35, Lemma 7.3])

$$H^{p,q}_{\text{ét}}(\mathcal{X}_\alpha, \mathbb{Z}) \simeq H^{p,q}_{\text{ét}}(F, \mathbb{Z}). \tag{3.14}$$

Step 3: *Reduction to the proof of the triviality of* $H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z})$.

The exact triangle of complexes (3.9) yields a commutative diagram

$$\begin{array}{ccccc}
H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) & \longrightarrow & H^{n+1,n}_{\text{ét}}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) & \longrightarrow & \mathbb{H}^{n+1}_{\text{ét}}(\mathcal{X}_\alpha, K(n)_{(2)}) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n+1,n}(F(X_\alpha), \mathbb{Z}_{(2)}) & \longrightarrow & H^{n+1,n}_{\text{ét}}(F(X_\alpha), \mathbb{Z}_{(2)}) & \longrightarrow & \mathbb{H}^{n+1}_{\text{ét}}(F(X_\alpha), K(n)_{(2)})
\end{array}$$

The middle term of the top row reduces to $H^{n+1,n}_{\text{ét}}(F, \mathbb{Z}_{(2)})$ by (3.14).

By the induction hypothesis, the complexes $K(q) \otimes \mathbb{Z}_{(2)}$ are quasi-isomorphic to zero if $q < n$. Then the exactness of the Gysin sequence shows that the right vertical homomorphism in the diagram is an isomorphism (see [11, Lemme 33]).

As $H^{n+1,n}(F(X_\alpha), \mathbb{Z}_{(2)}) = 0$ by (3.5), the diagram yields an exact sequence

$$H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) \to H^{n+1,n}_{\text{ét}}(F, \mathbb{Z}_{(2)}) \to H^{n+1,n}_{\text{ét}}(F(X_\alpha), \mathbb{Z}_{(2)}).$$

Thus, by Step 3, it suffices to prove that $H^{n+1,n}(\mathcal{X}_\alpha, \mathbb{Z}) = 0$.

Step 4: *Reduction to the proof of the triviality of* $H^{2^n-1,2^{n-1}}(X_\alpha, \mathbb{Z})$.

In this step Voevodsky introduced another powerful novelty: the Steenrod operations in motivic cohomology.

In topology, the operations on the cohomology groups with coefficients in $\mathbb{Z}/l\mathbb{Z}$ for a prime integer l form the (Hopf) *Steenrod algebra*. In the famous paper [18] Milnor determined the structure of the dual of the Steenrod algebra. Milnor proved that there are elements τ_i, $i \geq 0$, of degree $2l^i - 1$ and elements ξ_i, $i \geq 1$, of degree $2l^i - 2$ in the dual Steenrod algebra satisfying the following. Let $r = (r_1, r_2, \dots)$ be a sequence of non-negative integers that are almost all zero and let $e = (e_0, e_1, \dots)$ be a sequence of zeros and ones that are almost all zero. Then the elements

$$\tau^e \xi^r := \prod_{i \geq 0} \tau_i^{e_i} \prod_{j \geq 1} \xi_i^{r_i}$$

form a basis for the dual algebra.

Voevodsky defined the *motivic Steenrod algebra* and proved in [36] that there are elements τ_i, $i \geq 0$, of bidegree $(2l^i - 1, l^i - 1)$ and ξ_i, $i \geq 1$, of bidegree $(2l^i - 2, l^i - 1)$ in the dual motivic Steenrod algebra such that the elements $\tau^e \xi^r$ form a basis of the dual algebra.

Let Q_i, $i \geq 0$, be the operation of bidegree $(2l^i - 1, l^i - 1)$ dual to τ_i. The operations Q_i are called the *Milnor operations*. The operation $B := Q_0$ is the *Bockstein homomorphism*, i.e., the connecting homomorphism of bidegree $(1, 0)$ for the exact sequence

$$0 \to \mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/l^2\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0.$$

The operations Q_i anti-commute: $Q_k Q_j = -Q_j Q_k$ for $j \neq k$ and $Q_i^2 = 0$ for all i.

Let q_i, $i \geq 1$, be the operation of bidegree $(2l^i - 2, l^i - 1)$ dual to ξ_i. We have ([36, Prop. 13.6])

$$Q_i = [B, q_i]. \tag{3.15}$$

Let X be a smooth scheme of dimension d and $E \to X$ a vector bundle. Write $\mathrm{Th}(E)$ for the *Thom space of* E in the *pointed motivic homotopy category* $\mathrm{H}_\bullet(F)$ introduced in [22] and let $t_E \in \widetilde{H}^{2d,d}(\mathrm{Th}(E), \mathbb{Z})$ be the *Thom class of* E (see [36, §4]).

The Steenrod operations act on the Thom classes modulo l by multiplication by certain Chern classes. For any integer $k \geq 0$, let $s_k(E) \in H^{2k,k}(X, \mathbb{Z})$ be the additive in E Chern class uniquely determined by the property $s_k(L) = e(L)^k$, where L is a line bundle over X and $e(L) \in H^{2,1}(X, \mathbb{Z}) = \mathrm{Pic}(X)$ is the *Euler class of* L.

Proposition 3.16. ([36, Cor. 14.3]) *For any vector bundle E over a smooth scheme X of dimension d we have*

$$q_i(t_E) = s_{l^i-1}(E)t_E$$

in $\widetilde{H}^{2(d+l^i-1),d+l^i-1}\big(\mathrm{Th}(E), \mathbb{Z}/l\mathbb{Z}\big)$ *for any* $i \geq 1$.

For a smooth scheme X over F, let $\widetilde{C}(X)$ be the cone of the canonical morphism $\check{C}(X)_+ \to \mathrm{Spec}(F)_+$ in $\mathrm{H}_\bullet(F)$, where $\mathcal{X}_+$ denotes $\mathcal{X} \coprod \mathrm{Spec}(F)$. Using (3.15) and Proposition 3.16, Voevodsky proved the following

Theorem 3.17. ([35, Th. 3.2]) *Let Y be a smooth projective variety over F such that there is a morphism $X \to Y$, where X is a smooth projective variety over F of dimension $l^i - 1$ satisfying*

$$\deg s_{l^i-1}(T_X) \neq 0 \pmod{l^2},$$

where T_X is the tangent bundle of X. Then the sequence

$$\widetilde{H}^{p,q}(\widetilde{C}(Y), \mathbb{Z}/l\mathbb{Z}) \xrightarrow{Q_i} \widetilde{H}^{p+2l^i-1,q+l^i-1}(\widetilde{C}(Y), \mathbb{Z}/l\mathbb{Z}) \xrightarrow{Q_i} \widetilde{H}^{p+4l^i-2,q+2l^i-2}(\widetilde{C}(Y), \mathbb{Z}/l\mathbb{Z})$$

is exact for all p and q.

Corollary 3.18. ([35, Prop. 3.6]) *Let Y be a smooth projective quadric of dimension $2^m - 1$. Then the sequence in Theorem 3.17 is exact for $l = 2$ and any $i \leq m$.*

Indeed, if X is a smooth quadric of dimension $d = 2^i - 1$, then the integer

$$\deg s_{2^i-1}(T_X) = 2(2^i + 1 - 2^d)$$

is not divisible by 4. The quadric Y has smooth subquadrics of every dimension $2^i - 1 \leq 2^m - 1$.

Recall that $\mathcal{X}_\alpha := \check{C}(X_\alpha)$ for $\alpha \in (F^\times)^n$.

Lemma 3.19. *The group $\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z})$ is 2-torsion. If $p > q$ then the natural homomorphism $H^{p,q}(\mathcal{X}_\alpha, \mathbb{Z}) \to \widetilde{H}^{p+1,q}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z})$ is an isomorphism.*

Proof. If $\alpha = 0$, then $\widetilde{\mathcal{X}}_\alpha$ is contractible. As in general, α is split by a quadratic extension of F, the group $\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z})$ is 2-torsion. The last statement follows from (3.5). □

Lemma 3.20. *The group $\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z})$ is trivial if $p - 1 \leq q < n$.*

Proof. As we assume **MH90**(q) for all $q < n$, by Corollary 3.11 and 5-lemma applied to the diagram

$$\begin{array}{ccccccccc} H^{p-1,q}(F,\mathbb{Z}_{(2)}) & \to & H^{p-1,q}(\mathcal{X}_\alpha,\mathbb{Z}_{(2)}) & \to & \widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) & \to & H^{p,q}(\mathcal{X}_\alpha,\mathbb{Z}_{(2)}) & \to & H^{p,q}(F,\mathbb{Z}_{(2)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{p-1,q}_{\text{ét}}(F,\mathbb{Z}_{(2)}) & \to & H^{p-1,q}_{\text{ét}}(\mathcal{X}_\alpha,\mathbb{Z}_{(2)}) & \to & \widetilde{H}^{p,q}_{\text{ét}}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) & \to & H^{p,q}_{\text{ét}}(\mathcal{X}_\alpha,\mathbb{Z}_{(2)}) & \to & H^{p,q}_{\text{ét}}(F,\mathbb{Z}_{(2)}) \end{array}$$

the vertical homomorphism in the middle is an isomorphism if $p-1 \leq q < n$. It follows from (3.14) that $\widetilde{H}^{p,q}_{\text{ét}}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) = 0$. □

We say that an element $\alpha \in \widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\theta,\mathbb{Z}/2\mathbb{Z})$ is *integral* if α belongs to the image of the natural homomorphism

$$\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) \to \widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}/2\mathbb{Z}).$$

It follows from the fact that the group $\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z})$ is 2-torsion (Lemma 3.19) and from the equality $Q_iB = -BQ_i$ that Q_i takes integral elements to the integral ones. The restriction of Q_i on the subgroup of integral elements $H^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z})$ is still denoted by Q_i. Corollary 3.18 yields

Proposition 3.21. *For every $i = 1,\dots,n-1$, the sequence*

$$\widetilde{H}^{p,q}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) \xrightarrow{Q_i} \widetilde{H}^{p+2^{i+1}-1,q+2^i-1}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) \xrightarrow{Q_i} \widetilde{H}^{p+2^{i+2}-2,q+2^{i+1}-2}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z})$$

is exact for all p and q.

Lemma 3.20 then implies

Corollary 3.22. *If $p-1 \leq q < n$, then the operation*

$$Q_i : \widetilde{H}^{p+2^{i+1}-1,q+2^i-1}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z}) \to \widetilde{H}^{p+2^{i+2}-2,q+2^{i+1}-2}(\widetilde{\mathcal{X}}_\alpha,\mathbb{Z})$$

is injective for any $i = 1,2,\dots,n-1$.

It follows from Lemma 3.19 and Corollary 3.22 that the operation

$$Q_i : H^{n+2^{i+1}-i-2,n+2^i-i-1}(\mathcal{X}_\alpha,\mathbb{Z}) \to H^{n+2^{i+2}-i-3,n+2^{i+1}-i-2}(\mathcal{X}_\alpha,\mathbb{Z})$$

is injective for any $i = 1,2,\dots,n-1$. Hence the composition $Q_{n-2} \circ \cdots \circ Q_2 \circ Q_1$ is an injective homomorphism

$$H^{n+1,n}(\mathcal{X}_\alpha,\mathbb{Z}) \to H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha,\mathbb{Z}).$$

Remark 3.23. Note that in the case $n = 2$ this map is the identity, i.e., the Steenrod operations are not used in the proof in this case. In the case $n = 3$ the only Steenrod operation Q_1 used in the proof is closely related to the differential in the spectral sequence (see Section 3.C).

Step 5: *Proof of the triviality of $H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha,\mathbb{Z})$.*

If the quadric X_α has a rational point, i.e., the quadratic form q_α is isotropic and $q_\alpha \simeq q'_\alpha \perp \mathbb{H}$ for some form q'_α and the hyperbolic plane $\mathbb{H}$, the motive of X_α decomposes into direct sum of motives as follows:

$$M(X_\alpha) \simeq \mathbb{Z} \oplus \mathbb{Z}(d)[2d] \oplus M(X'_\alpha),$$

where $d = \dim(X_\alpha) = 2^{n-1}-1$ and X'_α is the quadric of the form q'_α. In general, when X_α may not have a rational point, Rost proved in [28] that $M(X_\alpha)$ still splits off a canonical motive M_α, called the *Rost motive of* α. Over a field extension over which X_α has a point, the Rost motive M_α is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(d)[2d]$.

The simplicial motive $\mathcal{X}_\alpha$ and the Rost motive fit into an exact triangle in $\mathbf{DM}(F)$ [35, Th. 4.4]:

$$M(\mathcal{X}_\alpha)(2^{n-1}-1)[2^n-2] \to M_\alpha \to M(\mathcal{X}_\alpha) \to M(\mathcal{X}_\alpha)(2^{n-1}-1)[2^n-1]. \tag{3.24}$$

The exact triangle yields an exact sequence

$$H^{0,1}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2^n-1,2^{n-1}}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2^n-1,2^{n-1}}(M_\alpha, \mathbb{Z}) \to H^{1,1}(\mathcal{X}_\alpha, \mathbb{Z}). \tag{3.25}$$

As $\mathbb{Z}(1) = \mathbb{G}_m[-1]$, we have

$$H^{p,1}(\mathcal{X}_\alpha, \mathbb{Z}) = H^{p-1}(\mathcal{X}_\alpha, \mathbb{G}_m).$$

It follows that the first group in the exact sequence (3.25) is trivial and the last one is equal to $F^\times$. The last map in the exact sequence is the restriction to the direct summand M_α of $M(X_\alpha)$ of the norm homomorphism (see Proposition 3.7):

$$N_{X_\alpha} : H^{2^n-1,2^{n-1}}(X_\alpha, \mathbb{Z}) = A^{2^{n-1}-1}(X_\alpha, K^M_{2^{n-1}}) \to F^\times.$$

Thus, it suffices to show that the norm homomorphism is injective. This was proved by Rost in [27]. This is also a consequence of the following general result [6, Th. 6.2]:

Theorem 3.26. *Let q be a non-degenerate quadratic form over a field F, X projective quadric hypersurface of dimension d given by $q = 0$. Then the kernel of the norm homomorphism $N_X : A^d(X, K_{d+1}) \to F^\times$ is naturally isomorphic to the group of R-equivalence classes* $\mathrm{Spin}(q)/R$ *in the spinor group of q.*

When q is a Pfister neighbor, i.e., q is a subform of an n-fold Pfister form of dimension at least $2^{n-1}+1$, the variety of the algebraic group $\mathrm{Spin}(q)$ is rational by [17, Th. 6.4], hence $\mathrm{Spin}(q)/R = 1$.

4. Milnor's Conjecture 2

We review the proof of $\mathbf{MC2}(n)$ by Orlov, Vishik and Voevodsky given in [23]. Assume that $\mathrm{char}(F) \neq 2$.

Proposition 4.1. *Let $\alpha \in (F^\times)^n$. Then the kernel of the natural homomorphism $k_n^M(F) \to k_n^M(F(X_\alpha))$ coincides with $\{0, \{\alpha\}\}$.*

Proof. We may assume that X_α has no rational points. Since by Lemma 3.19, $H^{n+1,n-1}(\mathcal{X}_\alpha, \mathbb{Z}) = \widetilde{H}^{n+1,n-1}(\widetilde{\mathcal{X}}_\alpha, \mathbb{Z})$ is 2-torsion, we have an exact sequence

$$H^{n,n-1}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) \to H^{n,n-1}(\mathcal{X}_\alpha, \mathbb{Z}/2\mathbb{Z}) \to H^{n+1,n-1}(\mathcal{X}_\alpha, \mathbb{Z}) \to 0.$$

It follows from Corollary 3.11, (3.14) and $\mathbf{MH90}(n-1)$ that

$$H^{n,n-1}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) \simeq H^{n,n-1}_{\text{ét}}(\mathcal{X}_\alpha, \mathbb{Z}_{(2)}) \simeq H^{n,n-1}_{\text{ét}}(F, \mathbb{Z}_{(2)}) = 0.$$

Hence the group $H^{n+1,n-1}(\mathcal{X}_\alpha, \mathbb{Z})$ is canonically isomorphic to $H^{n,n-1}(\mathcal{X}_\alpha, \mathbb{Z}/2\mathbb{Z})$. By the proof of [33, proof of Lemma 6.5], the latter group is canonically isomorphic to the kernel of the homomorphism $H^n(F) \to H^n(F(X_\alpha))$. In view of $\mathbf{MC1}(n)$, we have

$$H^{n+1,n-1}(\mathcal{X}_\alpha, \mathbb{Z}) \simeq \mathrm{Ker}\big(k_n^M(F) \to k_n^M(F(X_\alpha))\big). \tag{4.2}$$

It follows from Lemma 3.19 and Corollary 3.22 that the operation

$$Q_i : H^{n+2^{i+1}-i-2,n+2^i-i-2}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{n+2^{i+2}-i-3,n+2^{i+1}-i-3}(\mathcal{X}_\alpha, \mathbb{Z})$$

is injective for any $i = 1, 2, \dots, n-1$. The composition $Q_{n-2} \circ \cdots \circ Q_2 \circ Q_1$ is then an injective homomorphism

$$H^{n+1,n-1}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2^n-1,2^{n-1}-1}(\mathcal{X}_\alpha, \mathbb{Z}). \tag{4.3}$$

The exact triangle (3.24) yields an exact sequence

$$H^{0,0}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2^n-1,2^{n-1}-1}(\mathcal{X}_\alpha, \mathbb{Z}) \to H^{2^n-1,2^{n-1}-1}(M_\alpha, \mathbb{Z}).$$

The third group in the sequence is a subgroup of $H^{2^n-1,2^{n-1}-1}(X_\alpha, \mathbb{Z})$ and hence is trivial by (3.6) as $(2^n - 1) - (2^{n-1} - 1) = 2^{n-1} > \dim(X_\alpha)$. The first group in the sequence is isomorphic to $\mathbb{Z}$, hence the group $H^{2^n-1,2^{n-1}-1}(\mathcal{X}_\alpha, \mathbb{Z})$ is cyclic of order at most 2 as it is a group of exponent 2. It follows from (4.2) and the injectivity of (4.3) that the kernel of $k_n^M(F) \to k_n^M(F(X_\alpha))$ has at most two elements. But $\{\alpha\}$ is a nontrivial element in the kernel, hence the kernel coincides with $\{0, \{\alpha\}\}$. □

Now we can prove **MC2**(n) for all fields of characteristic not 2. It suffices to prove that the map s_n^F in injective. Let $u \in \operatorname{Ker}(s_n^F)$ be a sum of m symbols. We prove by induction on m that $u = 0$. Write $u = \{\alpha\} + w$, where $\alpha \in (F^\times)^n$ and w is a sum of $m-1$ symbols. Over the field $L = F(X_\alpha)$, the element $u_L = w_L$ is a sum of $m-1$ symbols. As u_L belongs to the kernel of s_n^L, it is trivial by induction. By Proposition 4.1, $u = 0$ or $u = \{\alpha\}$. We show that in the latter case $\{\alpha\} = 0$. Indeed, as $s_n^F(\{\alpha\}) = 0$, we have $[\langle\langle\alpha\rangle\rangle] \in I^{n+1}(F)$. It follows from Theorem 2.1 that $[\langle\langle\alpha\rangle\rangle] = 0$ in $W(F)$, i.e., the form $\langle\langle\alpha\rangle\rangle$ is metabolic. By Proposition 2.3, $\{\alpha\} = 0$.

As a result of positive solution of both Milnor's conjectures we have the isomorphisms

$$e_n^F : I^n(F)/I^{n+1}(F) \xrightarrow{(s_n^F)^{-1}} k_n^M(F) \xrightarrow{s_n^F} H^n(F).$$

The maps e_n^F provide *cohomological invariants*

$$\bar{e}_n^F : I^n(F) \to H^n(F)$$

of bilinear forms so that the invariant $\bar{e}_n^F(b)$ for a form b is defined if the previous one $\bar{e}_{n-1}^F(b)$ was defined and vanished. As the intersection of the ideals $I^n(F)$ is zero, these invariants determine bilinear forms up to isomorphism.

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