LIFTING PROPERTY FOR FINITE GROUPS

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Abstract. We classify all finite groups that have lifting property of mod p representations to mod p^2 representations for all prime p.

1. Introduction

Let R be a commutative local ring with maximal ideal M satisfying $M^2 = 0$ and residue field k of positive characteristic p. For example, one can take $R = \mathbb{Z}/p^2\mathbb{Z}$ with $k = \mathbb{F}_p$, or more generally, for a field k of characteristic $p > 0$, $R = W_2(k)$ the ring of the p-typical length 2 Witt vectors of k.

The group homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$ is surjective for every n and its kernel $A_n(R)$ is an abelian group of exponent p.

Let G be a finite group. The following two statements are equivalent:

1. Every group homomorphism $G \to \mathrm{GL}_n(k)$ for any $n > 0$ extends to a group homomorphism $G \to \mathrm{GL}_n(R)$.

2. Every continuous $k[G]$-module M of finite k-dimension lifts to a $R[G]$-module N, that is N is free as an R-module, and $M \simeq N \otimes_R k$ as $k[G]$-modules.

If these two conditions hold for all R (respectively for $R = \mathbb{Z}/p^2\mathbb{Z}$, where p is a prime integer) we say that G is *liftable* (respectively, *p-liftable*). The theorem below shows it to be a very restrictive property for finite groups.

Theorem 1.1. *Let G be a finite group. The following are equivalent:*
(1) *G is liftable.*
(2) *G is p-liftable for every prime p.*
(3) *G is isomorphic to one of the following groups: C_{2^n}, $C_3 \times C_{2^n}$ or $C_3 \rtimes C_{2^n}$. (The semidirect product is taken with respect the only nontrivial action of C_{2^n} on C_3.)*

We say a few words about a context for our result, and earlier related work. The modularity conjecture of Serre [5], proved in [3], asserts that odd, irreducible, 2-dimensional mod p representations ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k),$$

with k a finite field, arise from newforms, and in particular lift to characteristic 0. This has led to a lot of work about lifting mod p representations of absolute Galois groups of local and global fields. For absolute Galois groups of fields, lifting questions have been studied in [4]. The lifting property for finite groups was first studied in [2]: this is almost complementary to the case of absolute Galois groups as, by a theorem of Artin and Schreier, the only finite groups that are absolute Galois groups are 1 and C_2.

More precisely, in [2], G is said to be *liftable* (respectively, p-*liftable*), if any homomorphism $f : G \to \mathrm{GL}_n(k)$ with k a finite field (respectively, a finite extension of $\mathbb{F}_p$) lifts to $\mathrm{GL}_n(W_2(k))$. The definition of liftable in [2] is *a priori* weaker than our definition, and of p-liftable in [2] is *a priori* stronger than our definition. Under either of the definitions, it is easy to see that a finite group G is p-liftable if and only if a Sylow p-subgroup of G is p-liftable. From this and the equivalence of (1) and (2) of the above theorem, it follows that the definitions of liftable and p-liftable here and in loc. cit. are equivalent. Thus the theorem above sharpens Propositions 1.3 and 1.4 of [2], which left open whether the quaternion group Q_8 is liftable: we show in Proposition 4.2 below that Q_8 is not 2-liftable.

2. Preliminaries

Let R be a commutative local ring with maximal ideal M satisfying $M^2 = 0$ and residue field k of characteristic $p > 0$. We have an exact sequence

$$1 \to A_n(R) \to \mathrm{GL}_n(R) \to \mathrm{GL}_n(k) \to 1, \tag{2.1}$$

where $A_n(R)$ is an abelian group of exponent p with a natural structure of a $\mathrm{GL}_n(k)$-module.

If $R = \mathbb{Z}/p^2\mathbb{Z}$ for a prime p, the group $A_n(R)$ is identified with the (additive) group $M_n(\mathbb{F}_p)$ of all $n \times n$ matrices over $\mathbb{F}_p$ via $1 + px \mapsto \bar{x}$, the residue of x modulo p. The group $\mathrm{GL}_n(\mathbb{F}_p)$ acts on $M_n(\mathbb{F}_p)$ by conjugation.

Lemma 2.2. *Let H be a finite group and X and Y two finite dimensional $k[H]$-modules. Then $X \oplus Y$ is p-liftable if and only if X and Y are p-liftable.*

Proof. Let $n = \dim(X)$, $m = \dim(Y)$, $R = \mathbb{Z}/p^2\mathbb{Z}$ and $k = \mathbb{F}_p$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{n+m}(k) & \longrightarrow & \mathrm{GL}_{n+m}(R) & \xrightarrow{\gamma} & \mathrm{GL}_{n+m}(k) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & M_{n+m}(k) & \longrightarrow & E & \longrightarrow & \mathrm{GL}_n(k) \times \mathrm{GL}_m(k) & \longrightarrow & 1 \\
 & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\delta}\downarrow & & \| & & \\
0 & \longrightarrow & M_n(k) \oplus M_m(k) & \longrightarrow & \mathrm{GL}_n(R) \times \mathrm{GL}_m(R) & \longrightarrow & \mathrm{GL}_n(k) \times \mathrm{GL}_m(k) & \longrightarrow & 1,
\end{array}$$

where

$$\alpha(S,T) = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \quad \beta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (A, D),$$

E is the inverse image under γ of the image of α. It consists of all invertible matrices of the form $\begin{pmatrix} A & pB \\ pC & D \end{pmatrix}$ such that A, B, C, D are matrices over R. The map δ is a homomorphism taking $\begin{pmatrix} A & pB \\ pC & D \end{pmatrix}$ to (A, D).

Let $\varepsilon : H \to \mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$ be a homomorphism afforded by X and Y. If $X \oplus Y$ is p-liftable, then there is a homomorphism $\eta : H \to \mathrm{GL}_{n+m}(R)$ such that $\gamma\eta = \alpha\varepsilon$. It follows that the image of η is contained in E. Composing η with δ, we get a lifting $H \to \mathrm{GL}_n(R) \times \mathrm{GL}_m(R)$ of ε, hence X and Y are p-liftable. $\square$

Proposition 2.3. [2, Lemma 2.3] *Let G be a finite group and $H \subset G$ a subgroup. If G is p-liftable, then so is H.*

Proof. Let X be an $k[H]$-module. Consider the induced $k[G]$-module $Z := k[G] \otimes_H X$. As a $k[H]$-module, Z contains X as a direct summand: $Z = X \oplus Y$ for a $k[H]$-module Y. Since Z is p-liftable as a $k[G]$-module, it is p-liftable as a $k[H]$-module. By Lemma 2.2, the $k[H]$-module X is p-liftable, hence H is p-liftable. □

Proposition 2.4. [2, Lemma 2.2] *If every Sylow subgroup of a finite group G is liftable, then G is liftable.*

Proof. Let R be a commutative local ring with residue field k of characteristic $p > 0$ as above. Let $f : G \to \mathrm{GL}_n(k)$ be a group homomorphism. The restriction h of f to G_p lifts to a homomorphism $G_p \to \mathrm{GL}_n(R)$, that is the pullback of the sequence (2.1) with respect to h is splits. Equivalently, the image of the class of (2.1) under the pullback map $h^* : H^2(\mathrm{GL}_n(k), A_n(R)) \to H^2(G_p, A_n(R))$ is zero. Since $A_n(R)$ is p-torsion and $[G : G_p]$ is prime to p, the restriction homomorphism $H^2(G, A_n(R)) \to H^2(G_p, A_n(R))$ is injective. Hence the image of the class of (2.1) under the pullback map $f^* : H^2(\mathrm{GL}_n(k), A_n(R)) \to H^2(G, A_n(R))$ is also zero, that is f lifts to a homomorphism $G \to \mathrm{GL}_n(R)$. □

3. Cyclic groups

Set $R = \mathbb{Z}/p^2\mathbb{Z}$ and $k = \mathbb{F}_p$ for a prime p. Let G be a finite group and let f and h be two elements in $k[G]$ such that $fh = 0$. Choose their lifts $\hat{f}$ and $\hat{h}$ in $R[G]$. Then $\hat{f}\hat{h} = pu$ for some $u \in R[G]$ that is unique modulo $pR[G]$. If $\hat{f} + pv$ and $\hat{h} + pw$ are two other lifts, then

$$(\hat{f} + pv)(\hat{h} + pw) = p(u + \hat{f}w + v\hat{h}).$$

Thus, the residue of u is uniquely determined in the quotient group $k[G]/(fk[G] + k[G]h)$. We denote this class by $\theta(f, h)$.

Proposition 3.1. *Let G be a p-liftable group. Then for every pair f and h of elements in $k[G]$ such that $fh = 0$ we have $\theta(f, h) = 0$.*

Proof. By assumption, the left $k[G]$-module $X := k[G]/I$, where $I = k[G]h$, admits an $R[G]$-lifting $\widehat{X}$. The surjective composition $R[G] \to k[G] \to X$ lifts to a homomorphism of left $R[G]$-modules $R[G] \to \widehat{X}$ that is surjective by Nakayama. Let $\widehat{I}$ be its kernel.

We claim that the left ideal $\widehat{I} \subset R[G]$ is a lifting of I. Indeed, since $\widehat{X}$ is free as an R-module, the exact sequence $0 \to \widehat{I} \to R[G] \to \widehat{X} \to 0$ is split as a sequence of R-modules. It follows that $\widehat{I}$ is free as an R-module and $I \simeq \widehat{I}/p\widehat{I}$. This proves the claim.

Let $\hat{h} \in \widehat{I}$ be a lift of h. By Nakayama, $\hat{h}$ generates $\widehat{I}$, that is $\widehat{I} = R[G]\hat{h}$.

Now the map $k[G] \xrightarrow{r(h)} I$ of right multiplication by h has a lifting $R[G] \xrightarrow{r(\hat{h})} \widehat{I}$. Therefore, the kernel $\widehat{J}$ of $r(\hat{h})$ is a lifting of the kernel J of $r(h)$. In particular, the map $\widehat{J} \to J$ is surjective and we can choose a lift $\hat{f} \in \widehat{J}$ of f. Since $\hat{f}\hat{h} = 0$, we have $\theta(f, h) = 0$. □

Corollary 3.2. [2, Proposition 2.8] *Let $p > 2$ be a prime and n a positive integer such that $n \geqslant 2$ if $p = 3$. Then a cyclic group of order p^n is not p-liftable.*

Proof. Let σ be a generator of a cyclic group G order p^n. Let $m := p^{n-1}+1$. Consider the elements $f = (1-\sigma)^m$ and $h = (1-\sigma)^{p^n-m}$ in $k[G]$. Since $(1-\sigma)^{p^n} = 0$, we have $fh = 0$. We will prove that $\theta(f,h) \neq 0$ and hence G is not p-liftable.

Write $k[G]$ as the quotient of the polynomial ring $k[t]$ by the ideal generated by $(1-t)^{p^n}$. Since $m \leqslant p^n - m$ by assumption, the class $\theta(f,h)$ is represented by $Q(\sigma)$ in

$$k[G]/(1-\sigma)^m = k[t]/(1-t)^m,$$

where Q is the integer polynomial $[(1-t)^{p^n} - (1-t^{p^n})]/p$. Setting $s := 1-t$ we have $Q = [(1-s)^{p^n} - (1-s^{p^n})]/p$. The s^{m-1}-coefficient of Q is equal to $\binom{p^n}{p^{n-1}}/p$ and hence is not divisible by p. It follows that the class of $Q(\sigma)$ in $k[G]/(1-\sigma)^m = k[s]/(s^m)$ is nontrivial, hence $\theta(f,h) \neq 0$. $\square$

Proposition 3.3. *Let G be a cyclic group of order p^n. Suppose that for every $i = 1, 2, \dots, p^n$ there is a monic divisor P_i of the polynomial $t^{p^n}-1$ in $\mathbb{Z}[t]$ such that $P_i \equiv (t-1)^i$ modulo p. Then G is liftable.*

Proof. Let R be a commutative local ring with maximal ideal M satisfying $M^2 = 0$ and residue field k of positive characteristic p. Choose a generator σ of G. Every finite dimensional $k[G]$-module is a direct sum of modules of the form $M_i = k[G]/(\sigma-1)^i$ for $i = 1, 2, \dots, p^n$, so it suffices to show that every M_i lifts to an $R[G]$-module. Indeed, M_i lifts to the $R[G]$-module $R[G]/(P_i(\sigma))$. $\square$

Corollary 3.4. [2, Propositions 2.6 and 2.7] *Every cyclic group of order* 3 *and* 2^n *is liftable.*

Proof. If $p = 3$, the polynomial $(t-1)^2$ lifts to the divisor t^2+t+1 of t^3-1 in $\mathbb{Z}[t]$. In the case $p = 2$, let $i = 1, 2, \dots, 2^n$. Write i is base 2: $i = 2^{s_0} + 2^{s_1} + \cdots + 2^{s_m}$, where $0 \leqslant s_0 < s_1 < \cdots < s_m$. The polynomial $t^{2^n} - 1$ is the product $(t-1)Q_0Q_1\cdots Q_{n-1}$ in $\mathbb{Z}[t]$, where $Q_j = t^{2^j}+1$. Therefore, we can take $P_i = Q_{s_0}Q_{s_1}\cdots Q_{s_m}$ for the lift of $(t-1)^i$ modulo 2 in $\mathbb{Z}[t]$. $\square$

4. $C_2 \times C_2$ and Q_8 are not 2-liftable

Set $S = M_2(\mathbb{F}_2)$, $T = M_2(\mathbb{Z}/4\mathbb{Z})$, and

$$x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in S, \quad y = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in T.$$

Proposition 4.1. *The Klein group $C_2 \times C_2$ is not 2-liftable.*

Proof. Consider the following 4-dimensional representation of $C_2 \times C_2 = \langle \sigma, \tau \rangle$:

$$C_2 \times C_2 \to \mathrm{GL}_2(S) = \mathrm{GL}_4(\mathbb{F}_2), \quad \sigma \mapsto \begin{bmatrix} 1_S & 1_S \\ 0_S & 1_S \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} 1_S & x \\ 0_S & 1_S \end{bmatrix}.$$

We claim that this representation is not liftable modulo 4. Suppose it is liftable:

$$\sigma \mapsto A \cdot \begin{bmatrix} 1_T & 1_T \\ 0_T & 1_T \end{bmatrix}, \quad \tau \mapsto B \cdot \begin{bmatrix} 1_T & y \\ 0_T & 1_T \end{bmatrix},$$

where A and B are in the kernel of $\mathrm{GL}_2(T) \to \mathrm{GL}_2(S)$. We identify this kernel with $M_2(S)$ (written additively). The group $\mathrm{GL}_2(S)$ acts on $M_2(S)$ by conjugation. Let a and b be two matrices in $M_2(S)$ corresponding to A and B, respectively, under this identification. The relations $\sigma^2 = 1 = \tau^2$ and $\sigma\tau = \tau\sigma$ yield

$$a + \sigma(a) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b + \tau(b) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad a + \tau(a) = b + \sigma(b).$$

in $M_2(S)$. Writing $a = (a_{ij})$ and $b = (b_{ij})$ with a_{ij} and b_{ij} in S, we have the following relations in S:

$$\begin{aligned} a_{21} = b_{21} &= 0, \\ a_{11} + a_{22} &= 1, \\ b_{11}x + xb_{22} &= x, \\ a_{11}x + xa_{22} &= b_{11} + b_{22}. \end{aligned}$$

Consider the subspace

$$V := [x, S] = \{xs + sx, \ s \in S\} \subset S.$$

Note that $xV = V = x^{-1}V$ since $x(xs + sx) = x(xs) + (xs)x$. We then have the following equalities in S/V:

$$1 = x^{-1}(b_{11}x + xb_{22}) = b_{11} + b_{22} = a_{11}x + xa_{22} = x(a_{11} + a_{22}) = x,$$

hence $x + 1 \in V$, a contradiction since $\mathrm{Trace}(x + 1) \neq 0$. □

Proposition 4.2. *The quaternion group Q_8 is not 2-liftable.*

Proof. Note the relation $x^2 + x + 1 = 0$ in S.

Consider the quaternion group $Q_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^2, \sigma^4 = 1, \tau\sigma = \sigma^3\tau \rangle$ and its 6-dimensional representation over $\mathbb{F}_2$ given by

$$Q_8 \to \mathrm{GL}_3(S) = \mathrm{GL}_6(\mathbb{F}_2), \quad \sigma \mapsto j = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \tau \mapsto k = \begin{bmatrix} 0 & x & 1 \\ x & x^2 & x \\ x^2 & 0 & x \end{bmatrix}.$$

We claim that this representation is not liftable modulo 4. Suppose it is liftable:

$$\sigma \mapsto J + 2A, \quad \tau \mapsto K + 2B,$$

where A and B are two 3×3 matrices over T and

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & y & 1 \\ y & y^2 & y \\ y^2 & 0 & y \end{bmatrix}$$

in $M_3(T)$. The equality $\sigma^2 = \tau^2$ yields $(J + 2A)^2 = (K + 2B)^2$ in $M_3(T)$. Since

$$J^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad K^2 = \begin{bmatrix} 2y^2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2y^2 \end{bmatrix},$$

we get an equality

$$2(JA + AJ) = \begin{bmatrix} 2y^2 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & 2y \end{bmatrix} + 2(KB + BK)$$

in $M_3(T)$, hence

$$ja + aj = \begin{bmatrix} x^2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & x \end{bmatrix} + (kb + bk) \tag{4.3}$$

in $M_3(S)$, where a and $b = (b_{ij})$ in $M_3(S)$ are the residues modulo 2 of A and B, respectively. A computation shows that the trace in S of the 3×3 matrix $ja + aj$ is zero and the trace of $kb + bk$ is equal to

$$[x, b_{12}] + [x, b_{21}] + [x, b_{32}] + [x, b_{33}] + [x^2, b_{13}] + [x^2, b_{22}].$$

This is contained in $V = [x, S]$ since $[x^2, c] = [x + 1, c] = [x, c] \in V$. It follows from (4.3) that $1 = x^2 + x \in V$, hence $x \in xV = V$, a contradiction since $\mathrm{Trace}(x) \neq 0$. □

5. $C_3 \times C_3$ is not 3-liftable

Proposition 5.1. [4, Claim 5.4] *The group $C_3 \times C_3$ is not 3-liftable.*

Proof. For every $1 \leqslant i < j \leqslant 3$, let e_{ij} be the matrix with 1 on the (ij)-entry, and 0 everywhere else and set $\sigma_{ij} := 1 + e_{ij}$. Let $C_3 \times C_3 \to \mathrm{GL}_3(\mathbb{F}_3)$ be a 3-dimensional representation taking generators to σ_{12} and σ_{13}, respectively. If this homomorphism lifts to $\mathrm{GL}_3(\mathbb{Z}/9\mathbb{Z})$, there should exist two elements $\rho, \tau \in \mathrm{GL}_3(\mathbb{Z}/9\mathbb{Z})$ such that $\rho^3 = \tau^3 = [\rho, \tau] = 1$, ρ reduces to σ_{12} modulo 3, and τ reduces to σ_{13} modulo 3. We now prove that such ρ and τ do not exist. We have $\rho = A\sigma_{12}$ and $\tau = B\sigma_{13}$ in $\mathrm{GL}_3(\mathbb{Z}/9\mathbb{Z})$, for some $A, B \in A_3(\mathbb{Z}/9\mathbb{Z})$. Identifying $A_3(\mathbb{Z}/9\mathbb{Z})$ with the additively written group $M_3(\mathbb{F}_3)$, we rewrite the relations $\rho^3 = \tau^3 = [\rho, \tau] = 1$ in the form

$$N_{12}(a) = -e_{12}, \quad N_{13}(b) = -e_{13} \quad \text{and} \quad (\sigma_{13} - 1)a = (\sigma_{12} - 1)b$$

in $M_3(\mathbb{F}_3)$, where $a, b \in M_3(\mathbb{F}_3)$ and $N_{ij} := 1 + \sigma_{ij} + \sigma_{ij}^2$. Letting $a = (a_{ij})$, a matrix computation shows

$$N_{12}(a) = \begin{pmatrix} 0 & a_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad -(\sigma_{13} - 1)a + (\sigma_{12} - 1)b = \begin{pmatrix} * & * & * \\ * & * & a_{21} \\ * & * & * \end{pmatrix}.$$

Thus $-1 = a_{21} = 0$, a contradiction. □

6. Proof of Theorem 1.1

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Let G be a p-liftable finite group for every prime p. By Proposition 2.3, a Sylow p-subgroup G_p of G is p-liftable for all p. If $p > 3$ then $G_p = 1$ since otherwise G_p would contain a cyclic group of order p which is not p-liftable by Corollary 3.2. This contradicts Proposition 2.3.

The subgroup G_3 has order at most 3. Indeed, if $|G_3| > 3$, then G_3 contains a subgroup of order 9, hence G_3 contains either $C_3 \times C_3$ or C_9. The latter two groups are not 3-liftable by Proposition 5.1 and Corollary 3.2. Again, we get a contradiction by Proposition 2.3.

We claim that every abelian subgroup $H \subset G_2$ (which is 2-liftable by Proposition 2.3) is cyclic. Indeed, if H were not cyclic, it would contain a subgroup $C_2 \times C_2$. This contradicts Propositions 4.1 and 2.3. The claim is proved.

By [1, Chapter IV, Corollary 6.6], a 2-group with all abelian subgroups cyclic is either cyclic or generalized quaternion group Q_{2^n}, $n \geqslant 3$, generated by σ and τ subject to the relations $\sigma^{2^{n-2}} = \tau^2$, $\tau^4 = 1$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. The subgroup of Q_{2^n} generated by $\sigma^{2^{n-3}}$ and τ is isomorphic to Q_8 which is not 2-liftable by Proposition 4.2. It follows from Proposition 2.3 that G_2 is cyclic. Thus, we proved that all Sylow subgroups of G are cyclic and the order of G divides $3 \cdot 2^n$. It follows from the proof of [2, Proposition 2.4] that G is one of the groups in the list (3).

(3) $\Rightarrow$ (1): By Proposition 2.4 it suffices to prove that all Sylow subgroups G_p of G are liftable. The group G_p is either trivial or cyclic of order 3 or 2^n. By Corollary 3.4, G_p is liftable. □

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