STEENROD OPERATIONS IN ALGEBRAIC GEOMETRY

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1. Introduction

Let p be a prime integer. For a pair of topological spaces $A \subset X$ we write $H^i(X, A; \mathbb{Z}/p\mathbb{Z})$ for the i-th singular cohomology group with coefficients in $\mathbb{Z}/p\mathbb{Z}$. A *cohomological operation of degree* r is a collection of group homomorphisms

$$H^i(X, A; \mathbb{Z}/p\mathbb{Z}) \to H^{i+r}(X, A; \mathbb{Z}/p\mathbb{Z})$$

satisfying certain naturality conditions. In particular, they commute with the pull-back homomorphisms.

The operations form the *Steenrod algebra* $\mathcal{A}_p$ *modulo* p. It is generated by the *reduced power operations* P^k, $k \geq 1$, of degree $2(p-1)k$ and the *Bockstein operation* of degree 1.

The idea of the definition of P^k is as follows. Modulo p, the p-th power operation $\alpha \mapsto \alpha^p$ is additive. It can be described as follows. Let $d : X \to X^p$ be the diagonal embedding. The composition

$$H^*(X; \mathbb{Z}/p\mathbb{Z}) \to H^*(X^p; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{d^*} H^*(X; \mathbb{Z}/p\mathbb{Z}),$$

where the first map is the p-th exterior power, takes a class α to the power α^p. Note that the composition is a homomorphism, although the first map is not.

Let G be the symmetric group S_p. It acts on X^p and we can define the composition

$$H^*(X; \mathbb{Z}/p\mathbb{Z}) \to H^*_G(X^p; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{d^*} H^*_G(X; \mathbb{Z}/p\mathbb{Z}) = H^*(X; \mathbb{Z}/p\mathbb{Z}) \otimes H^*_G(pt; \mathbb{Z}/p\mathbb{Z}).$$

The ring $H^*_G(pt; \mathbb{Z}/p\mathbb{Z})$ is isomorphic to the polynomial ring $(\mathbb{Z}/p\mathbb{Z})[t]$. Thus, the image of α is a polynomial with coefficient in $H^*(X; \mathbb{Z}/p\mathbb{Z})$. The coefficients are the *reduced powers* $P^k(\alpha)$ of a.

Let a space X be embedable into $\mathbb{R}^n$. The group

$$H_i^{BM}(X; \mathbb{Z}/p\mathbb{Z}) = H^{n-i}(\mathbb{R}^n, \mathbb{R}^n - X; \mathbb{Z}/p\mathbb{Z})$$

is independent on the embedding and is called the *Borel-Moore homology group* with coefficients in $\mathbb{Z}/p\mathbb{Z}$. An operation S of degree r defines an operation

$$H_i^{BM}(X; \mathbb{Z}/p\mathbb{Z}) = H^{n-i}(\mathbb{R}^n, X; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{S} H^{n-i+r}(\mathbb{R}^n, X; \mathbb{Z}/p\mathbb{Z}) = H_{i-r}^{BM}(X; \mathbb{Z}/p\mathbb{Z})$$

on the Borel-Moore homology groups. These operations commute with push-forward homomorphisms with respect to proper morphisms. We can them operations of *homological type*.

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V. Voevodsky has defined operations on the motivic cohomology groups $H^{i,j}(X, U; \mathbb{Z}/p\mathbb{Z})$ for a smooth variety X and an open subset $U \subset X$. The reduced power operation P^k has bidegree $\big(2(p-1)k, (p-1)k\big)$ and the Bockstein operation has bidegree $(1, 0)$.

If $d = \dim X$ we have

$$H^{2i,i}(X, U; \mathbb{Z}/p\mathbb{Z}) \simeq \mathrm{CH}_{d-i}(X - U)/p.$$

In particular, the reduced power operations (but not the Bockstein operation) act on the Chow groups modulo p. Our aim is to give an "elementary" definition of these operations. We follows P. Brosnan's approach.

Notation:
A *scheme* is a quasi-projective scheme over a field.
A *variety* is an integral scheme.

2. Chow groups

2.1. **Chow groups of homological type.** Let X be a scheme over a field F and let i be an integer. The *Chow groups* of i-dimensional cycles $\mathrm{CH}_i(X)$ is the factor group of the free group generated by dimension i closed subvarieties of X modulo rational equivalence. In particular, $\mathrm{CH}_i(X)$ is trivial if $i < 0$ or $i > \dim X$.

Properties:

1. For a projective morphism $f : X \to Y$ there is the *push-forward* homomorphism

$$f_* : \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y).$$

2. For a flat morphism $f : Y \to X$ of relative dimension d there is the *pull-back* homomorphism

$$f^* : \mathrm{CH}_i(X) \to \mathrm{CH}_{i+d}(Y).$$

3. (Homotopy invariance) For a vector bundle $f : E \to X$ of rank r the pull-back homomorphism $f_* : \mathrm{CH}_i(X) \to \mathrm{CH}_{i+r}(E)$ is an isomorphism.

4. For two schemes X and Y over F there is an *exterior product* homomorphism

$$\mathrm{CH}_i(X) \otimes \mathrm{CH}_j(Y) \to \mathrm{CH}_{i+j}(X \times Y), \quad \alpha \otimes \beta \mapsto \alpha \times \beta.$$

Remark 2.1. We can view Chow groups $\mathrm{CH}_*(X)$ as a Borel-Moore type theory associated to the motivic cohomology groups of bidegree $(2i, i)$.

2.2. **Chern classes.** Let $E \to X$ be a vector bundle over a scheme X. For every $i \geq 0$ there is a group homomorphism

$$c_i(E) : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-i}(X), \quad \alpha \mapsto c_i(E) \cap \alpha$$

called the *i-th Chern class of E*. We can view $c_i(E)$ as an endomorphism of $\mathrm{CH}_*(X)$. The class $c_\bullet = c_0 + c_1 + c_2 + \ldots$ is called the *total Chern class.*

Properties:

1. $c_0(E)$ is the identity.

2. $c_i(E) = 0$ for $i > \operatorname{rank} E$.

3. (Cartan formula) For an exact sequence $0 \to E' \to E \to E'' \to 0$,
$$c_\bullet(E) = c_\bullet(E') \circ c_\bullet(E'').$$

4. The operation $c_1(L)$ for a line bundle $L \to X$ is the intersection with the Cartier divisor associated with L.

The Chern classes uniquely determined by the properties 1-4.

If $E \to X$ is a vector bundle of rank r, the class $e(E) = c_r(E)$ is called the *Euler class* of E. For an exact sequence as above we have
$$e(E) = e(E') \circ e(E'').$$

Let A be a commutative ring. A *characteristic class over* A is a power series w over A in the Chern classes c_i. For a vector bundle $E \to X$ the operation $w(E)$ on $\mathrm{CH}_*(X) \otimes A$ is well defined.

Let $f(x) = 1 + a_1x + a_2x^2 + \dots$ be a power series over A. A characteristic class w_f over A is called a *multiplicative class associated with* f if the following holds:

1. For a line bundle $L \to X$,
$$w_f(L) = f\big(c_1(L)\big) = \mathrm{id} + a_1c_1(L) + a_2c_1(L)^2 + \dots;$$

2. For an exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles,
$$w_f(E) = w_f(E') \circ w_f(E'').$$

The class w_f exists and unique for every f. If E has a filtration by subbundles with line factors L_i. Then
$$w_f(E) = \prod_i f\big(c_1(L_i)\big).$$

Example 2.2. If $f = 1 + x$, the class w_f coincides with the total Chern class $c_\bullet$.

Note that the operation $w_f(E)$ is invertible. We write $w_f(-E)$ for $w_f(E)^{-1}$.

2.3. **Chow groups of cohomological type.** If X is a smooth variety of dimension d we write
$$\mathrm{CH}^i(X) = \mathrm{CH}_{d-i}(X).$$

Properties:

1. For a morphism $f : Y \to X$ of smooth varieties there is the *pull-back* homomorphism
$$f^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(Y).$$

2. For a smooth variety X, the graded group $\mathrm{CH}^*(X)$ has a structure of a graded ring with identity $1_X = [X]$ via the pull-back with respect to the diagonal morphism $d : X \to X \times X$: $\alpha \cdot \beta = d^*(\alpha \times \beta)$.

Remark 2.3. The group $\mathrm{CH}^i(X)$ for a smooth X coincides with the motivic cohomology group $H^{2i,i}(X) = H^{2i,i}(X, \emptyset)$ and the latter has obvious contravariant properties. We can view the assignment $X \mapsto \mathrm{CH}^*(X)$ as a cohomology theory.

Example 2.4. Let $h \in \mathrm{CH}^1(\mathbb{P}^n)$ be the class of a hyperplane section of the projective space $\mathbb{P}^n$. Then

$$\mathrm{CH}^i(\mathbb{P}^n) = \begin{cases} \mathbb{Z}h^i & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\mathrm{CH}^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1}).$$

2.4. **Refined Gysin homomorphisms.** Let $i : M \hookrightarrow P$ be a closed embedding of smooth varieties of codimension d. Let X be a closed subscheme of P and set $Y = M \cap X$. Thus we have a pull-back diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow \\ M & \xrightarrow{i} & P. \end{array}$$

The pull-back homomorphism i^* (called also a *Gysin homomorphism*) has a refinement, called the *refined Gysin homomorphism*

$$i^! : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-d}(Y).$$

In the case $X = P$ the homomorphism $i^!$ coincides with i^*.

Remark 2.5. The refined Gysin homomorphism coincide with the pull-back homomorphism

$$H^{2i.i}(P, P - X) \to H^{2i.i}(M, M - Y).$$

for $i = \dim P - k$.

Suppose we are given two diagrams as above:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle f_l}\downarrow & & \downarrow \\ M_l & \xrightarrow{i_l} & P_l \end{array}$$

$(l = 1, 2)$ with $i_l : M_l \to P_l$ closed embeddings of smooth varieties of codimension d_l. Let N_l be the normal bundle of the closed embedding i_l over M_l. It is a vector bundle on M_l.

The following proposition is a variant of so-called excess formula.

Proposition 2.6. *For an element $\alpha \in \mathrm{CH}_k(X)$,*

$$e(f_2^* N_2) \cap i_1^!(\alpha) = e(f_1^* N_1) \cap i_2^!(\alpha)$$

in $\mathrm{CH}_{k-d_1-d_2}(Y)$.

3. Equivariant Chow groups

In topology: Let G be a group and X be a G-space. For a (co)homology theory H we can define an equivariant (co)homology group

$$H^G(X) = H\big((X \times EG)/G\big)$$

where EG is a contractible G-space with free G-action.

In algebra we don't have a scheme representing EG. Instead, we consider certain approximations of EG.

Let G be an algebraic group of dimension d_G over F and let X be a G-scheme. Let V be a G-vector space (a representation of G) of dimension d_V such that G acts regularly on a nonempty open subscheme $U \subset V$. Then for every $i \in \mathbb{Z}$, the group

$$\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+d_V-d_G}\big((X \times U)/G\big)$$

is independent of V as long as $V - U$ has sufficiently large codimension in V. This group is the *i-th equivariant Chow group* of X.

A closed G-equivariant subvariety $Z \subset X$ has its class $[Z]^G$ in $\mathrm{CH}_i^G(X)$ defined as

$$[(Z \times U)/G] \in \mathrm{CH}_{i+d_V-d_G}\big((X \times U)/G\big) = \mathrm{CH}_i^G(X).$$

If G is the trivial group we have

$$\mathrm{CH}_i^G(X) = \mathrm{CH}_i(X).$$

If H is a subgroup of G, for a G-scheme X there is a natural *restriction* homomorphism

$$\mathrm{res}_{G/H} : \mathrm{CH}_i^G(X) \to \mathrm{CH}_i^H(X).$$

For a smooth G-variety X of dimension d we write

$$\mathrm{CH}^i_G(X) = \mathrm{CH}^G_{d-i}(X).$$

The equivariant Chow groups have similar functorial properties as the ordinary ones (projective push-forwards, flat pull-backs, refined Gysin homomorphisms, homotopy invariance). In particular, the graded group $\mathrm{CH}^*_G(X)$ has a structure of a commutative ring.

Example 3.1. Let $G = \mathbb{G}_m$ the multiplicative group. Consider the natural G-action on the affine space $\mathbb{A}^{r+1}$. Clearly G acts regularly on $\mathbb{A}^{r+1} - \{0\}$ with the factor variety $\mathbb{P}^r$. Hence

$$\mathrm{CH}^i_G(pt) = \mathrm{CH}^G_{-i}(pt) = \mathrm{CH}_{r-i}(\mathbb{P}^r) = \mathbb{Z}h^i$$

and therefore, $\mathrm{CH}^*_G(pt) = \mathbb{Z}[h]$ is a polynomial ring over $\mathbb{Z}$. Note that this ring is a domain contrary to the ordinary Chow rings having only nilpotent elements in positive degrees.

Let p be a prime integer. We introduce a new notation:

$$\mathrm{Ch}_*(X) = \mathrm{CH}_*(X)/p, \quad \mathrm{Ch}^G_*(X) = \mathrm{CH}^G_*(X)/p.$$

Example 3.2. Let $H = \boldsymbol{\mu}_p$. We consider H as a subgroup of $G = \mathbb{G}_m$. The restriction ring homomorphism $\mathrm{Ch}^*_G(pt) \to \mathrm{Ch}^*_H(pt)$ is an isomorphism, so that

$$\mathrm{Ch}^*_H(pt) = (\mathbb{Z}/p\mathbb{Z})[h].$$

More generally, if H acts trivially on a scheme X, we have a canonical isomorphism

$$\mathrm{Ch}^H_*(X) = \mathrm{Ch}_*(X) \otimes \mathbb{Z}/p\mathbb{Z}[h] = \mathrm{Ch}_*(X)[h].$$

Example 3.3. Let G be the semidirect product of $\boldsymbol{\mu}_p$ and $\mathrm{Aut}(\boldsymbol{\mu}_p) = (\mathbb{Z}/p\mathbb{Z})^\times$. Since the order of $(\mathbb{Z}/p\mathbb{Z})^\times$ is prime to p, we have for a G-scheme X:

$$\mathrm{Ch}^G_*(X) = [\mathrm{Ch}^{\mu_p}_*(X)]^{(\mathbb{Z}/p\mathbb{Z})^\times}.$$

A class $i+p\mathbb{Z} \in (\mathbb{Z}/p\mathbb{Z})^\times$ acts on h by $h \mapsto h^i$. The restriction homomorphism

$$\mathrm{Ch}^*_G(pt) \to \mathrm{Ch}^*_{\boldsymbol{\mu}_p}(pt) = \mathbb{Z}/p\mathbb{Z}[h]$$

is injective and identifies the ring $\mathrm{Ch}^*_G(pt)$ with $\mathbb{Z}/p\mathbb{Z}[h^{p-1}]$. More generally, if G acts trivially on a scheme X, we have a canonical isomorphism

$$\mathrm{Ch}^G_*(X) = [\mathrm{Ch}^{\mu_p}_*(X)]^{\mathrm{Aut}(\boldsymbol{\mu}_p)} = \mathrm{Ch}_*(X)[h^{p-1}].$$

3.1. **Equivariant Chern classes.** Let X be G-scheme and let $E \to X$ be a G-vector bundle over X. For every $i \geq 0$ there is a group homomorphism

$$c^G_i(E) : \mathrm{CH}^G_k(X) \to \mathrm{CH}^G_{k-i}(X), \quad \alpha \mapsto c^G_i(E) \cap \alpha$$

called the *equivariant i-th Chern class of E*. The equivariant Chern classes satisfy similar properties as the ordinary ones.

Example 3.4. If $X = pt$, a G-vector bundle over X is a representation V of G. Let $G = \mathbb{G}_m$ or $\boldsymbol{\mu}_n$ and let V be the canonical 1-dimensional representation of G. Then $c^G_1(V)$ is the multiplication by h in $\mathrm{Ch}^*_G(pt)$. We will simply write $c^G_1(V) = h$.

Example 3.5. Let p be a prime integer. Let G be the semidirect product of $\boldsymbol{\mu}_p$ and $\mathrm{Aut}(\boldsymbol{\mu}_p) = (\mathbb{Z}/p\mathbb{Z})^\times$. Consider the algebra $F_p = F[x]/(x^p - 1)$. It is an étale F-algebra of degree p. If F contains all p-th roots of unity, F_p is a product of p copies of F. The group G acts naturally on F_p by algebra automorphisms: the group μ_p acts on x by multiplication and an automorphism $i+p\mathbb{Z} \in (\mathbb{Z}/p\mathbb{Z})^\times$ by $x \mapsto x^i$. In particular, F_p is a G-module having a (trivial) submodule F. We claim that the mod p equivariant Euler class $e^G(\widetilde{F})$, where $\widetilde{F} = F_p/F$, is the multiplication by $-h^{p-1}$.

Indeed, since $\mathrm{Ch}^*_G(pt)$ is contained in $\mathrm{Ch}^*_{\boldsymbol{\mu}_p}(pt)$ it is sufficient to replace G by $\boldsymbol{\mu}_p$. Then $\widetilde{F}$ is a direct sum of 1-dimensional modules $Fx^i \simeq V^{\otimes i}$ for $i = 1, 2, \dots, p-1$. Hence

$$e^G(\widetilde{F}) = \prod_{i=1}^{p-1} e^G(V^{\otimes i}) = \prod_{i=1}^{p-1}(ih) = (p-1)! \cdot h^{p-1} = -h^{p-1}.$$

We set $t = -h^{p-1}$ so that $\mathrm{CH}^*_G(pt)/p = \mathbb{Z}/p\mathbb{Z}[t]$ and $t = e^G(\widetilde{F})$ modulo p.

4. Definition of reduced powers

We fix a prime integer p. We assume that $\operatorname{char} F \neq p$.

4.1. **Scheme $R^p(X)$.** Consider again the algebra $F_p = F[x]/(x^p - 1)$. Let X be a scheme over a field F. Write $R^p(X)$ for the scheme $R_{F_p/F}(X \otimes_F F_p)$ representing the functor $S \mapsto X(S \otimes_F F_p)$. If F contains all p-th roots of unity, X is a product of p copies of X. In particular, $\dim R^p(X) = p \dim X$.

Consider the semidirect product G of $\boldsymbol{\mu}_p$ with $\operatorname{Aut}(\boldsymbol{\mu}_p) = (\mathbb{Z}/p\mathbb{Z})^\times$. The action of G on F_p extends to an action of G on $R^p(X)$.

The embedding $X(S) \hookrightarrow X(S \otimes_F F_p)$ gives rise to the closed embedding of X into $R^p(X)$. In fact, X is identified with the subscheme of G-invariant elements of $R^p(X)$. We have a well defined map (not a homomorphism!)

$$P_X : \mathrm{CH}_k(X) \to \mathrm{CH}^G_{pk}\big(R^p(X)\big), \quad [Z] \mapsto [R^p(Z)]^G.$$

Now assume that X is a smooth variety. We would like to compute the normal cone N of the closed embedding $X \to R^p(X)$. One checks that

$$T_{R^p(X)}|_X = T_X \otimes_F F_p,$$

hence

$$N = T_X \otimes_F \widetilde{F},$$

where as above $\widetilde{F} = F_p/F$. The group G acts on N via $\widetilde{F}$. Let us compute the Euler class $e^G(N)$.

Consider the multiplicative characteristic class $w = w_f$ corresponding to the power series $f(x) = 1 + x^{p-1}$. Recall that for every line bundle L, we have $w(L) = 1 + c_1(L)^{p-1}$. We write

$$w(y) = 1 + w_{p-1}y + w_{2(p-1)}y^2 + \dots,$$

so that for a line bundle L,

$$w(L, y) = 1 + c_1(L)^{p-1} \cdot y.$$

Proposition 4.1. *For a vector bundle $E \to X$ of rank r we have*

$$e^G(E \otimes_F \widetilde{F}) = t^r w(E, 1/t) \in \mathrm{Ch}^G_*(X) = \mathrm{Ch}_*(X)[t].$$

Proof. Since $\mathrm{Ch}^G_*(X)$ injects into $\mathrm{Ch}^{\boldsymbol{\mu}_p}_*(X)$ we may replace G by $\boldsymbol{\mu}_p$. We also may assume that $E = L$ is a line bundle. Then

$$\begin{aligned}
e^G(L\otimes_F \widetilde{F}) &= \prod_{i=1}^{p-1} c_1^G(L\otimes V^{\otimes i})\\
&= \prod_{i=1}^{p-1}\big(c_1^G(L)+c_1^G(V^{\otimes i})\big)\\
&= \prod_{i=1}^{p-1}\big(c_1^G(L)+ih\big)\\
&= c_1^G(L)^{p-1}-h^{p-1}\\
&= t\big(1+c_1^G(L)^{p-1}/t\big)\\
&= t\cdot w(L,1/t).
\end{aligned}$$

□

Remark 4.2. It follows from the proposition that the Euler class $e^G(E\otimes_F \widetilde{F})$ is invertible if we invert t.

4.2. **Reduced power operations.** Let X be a scheme over F. We embed X as a closed subscheme into a smooth variety M of dimension d. The commutative diagram

$$\begin{array}{ccc}
X & \longrightarrow & R^p(X)\\
\downarrow & & \downarrow\\
M & \xrightarrow{\ i\ } & R^p(M)
\end{array}$$

gives rise to a refined Gysin homomorphism

$$i^! : \mathrm{CH}^G_r\big(R^p(X)\big)\to \mathrm{CH}^G_{r-(p-1)d}(X),$$

where G be the semidirect product of $\boldsymbol{\mu}_p$ and $(\mathbb{Z}/p\mathbb{Z})^\times$.

Note that the composition

$$D^M_X : \mathrm{Ch}_k(X)\xrightarrow{P_X}\mathrm{Ch}^G_{pk}\big(R^p(X)\big)\xrightarrow{i^!}\mathrm{Ch}^G_{pk-(p-1)d}(X)$$

is a group homomorphism.

Since G acts trivially on X, we have

$$\mathrm{Ch}^G_*(X)=\mathrm{Ch}_*(X)[t]=\coprod_{i\ge 0}\mathrm{Ch}_*(X)t^i.$$

The map D^M_X changes the degree by a multiple of $p-1$. Therefore, for an element $\alpha\in\mathrm{Ch}_k(X)$ we can then write

$$D^M_X(\alpha)(t)=\sum_i S^M_i(\alpha)t^{d-k-i}$$

for some elements

$$S^M_i(\alpha)\in\mathrm{Ch}_{k-(p-1)i}(X).$$

In other words,

$$S^M_\bullet(\alpha)(t) \stackrel{def}{=} \sum_i S^M_i(\alpha)t^i = t^{d-k} D^M_X(\alpha)(1/t).$$

Thus, for any scheme X over F and a smooth variety M containing X as a closed subscheme we get the *reduced power operations* for the pair (M, X):

$$S^M_i : \mathrm{Ch}_k(X) \to \mathrm{Ch}_{k-(p-1)i}(X).$$

Note that S^M_i lowers the dimension by $(p-1)i$.

We also have the *total reduced power operation*

$$S^M_\bullet = S^M_\bullet(t)|_{t=1} = \sum S^M_i : \mathrm{Ch}_*(X) \to \mathrm{Ch}_*(X).$$

Remark 4.3. On the motivic cohomology groups the operation S^M_i coincides with the reduced power

$$S^i : H^{2k,k}(M, M-X) \to H^{2k-2(p-1)i, k-(p-1)i}(M, M-X).$$

Properties:

1. Let $Y \xrightarrow{i} X \to M$ be closed embeddings with M smooth. Then

$$S^M_\bullet(i_*\alpha) = i_*(S^M_\bullet \alpha)$$

for every $\alpha \in \mathrm{Ch}_*(Y)$.

2. (Cartan formula) Let $X \to M$ and $Y \to N$ be closed embeddings with M and N smooth. Then

$$S^{M\times N}_\bullet(\alpha \times \beta) = S^M_\bullet \alpha \times S^N_\bullet \beta$$

for $\alpha \in \mathrm{Ch}_*(X)$ and $\beta \in \mathrm{Ch}_*(Y)$.

4.3. **Steenrod operations of cohomological type.** Let X be a smooth variety over F. Set $S^\bullet = S^X_\bullet$. Thus, S^i is an operation

$$S^i : \mathrm{Ch}^k(X) \to \mathrm{Ch}^{k+(p-1)i}(X),$$

called the *reduced power of cohomological type*.

Properties:

1. Let $f : Y \to X$ be a morphism of smooth varieties over F. Then

$$f^* S^\bullet(\alpha) = S^\bullet f^*(\alpha)$$

for every $\alpha \in \mathrm{Ch}^*(X)$.

2. (Cartan formula) $S^\bullet(\alpha\beta) = S^\bullet(\alpha)\cdot S^\bullet(\beta)$ for $\alpha, \beta \in \mathrm{Ch}^*(X)$, $S^\bullet(1_X) = 1_X$, i.e., $S^\bullet$ is a ring endomorphism of $\mathrm{Ch}^*(X)$.

3. For every $\alpha \in \mathrm{Ch}^k(X)$,

$$S^i\alpha = \begin{cases} \alpha^p, & \text{if } i = k, \\ 0, & \text{if } i > k \text{ or } i < 0. \end{cases}$$

4.4. **Steenrod operations of homological type.**

Proposition 4.4. *Let X be a scheme and let M_1 and M_2 be two smooth varieties containing X. Then*

$$w(T_{M_2}|_X) \cap S_\bullet^{M_1} = w(T_{M_1}|_X) \cap S_\bullet^{M_2}.$$

Proof. The normal bundle N_l of the closed embedding $M_l \to R^p(M_l)$ is equal to $T_{M_l}|_X \otimes_F \widetilde{F}$, $(l = 1, 2)$. Let $d_l = \dim M_l$. For an element $\alpha \in \mathrm{Ch}_k(X)$, we have

$$\begin{aligned} w(T_{M_2}|_X, 1/t) \cap S_\bullet^{M_1}(1/t) &= t^{-d_2} \cdot e^G(N_2) \cap t^{k-d_1} D_X^{M_1}(\alpha)(t) \quad \text{(Prop. 4.1)} \\ &= t^{k-d_1-d_2} \cdot e^G(N_2) \cap i_1^!\big(P_X(\alpha)\big). \end{aligned}$$

It follows from Proposition 2.6 that the latter is equal to

$$t^{k-d_1-d_2} \cdot e^G(N_1) \cap i_2^!\big(P_X(\alpha)\big).$$

The result follows. □

Let X be a subscheme of a smooth variety M. For every $\alpha \in \mathrm{Ch}_k(X)$ set

$$S_\bullet(\alpha) = w(T_M|_X)^{-1} \cap S_\bullet^M(\alpha).$$

By Proposition 4.4, $S_\bullet(\alpha)$ is independent of M. The operations

$$S_i : \mathrm{Ch}_k(X) \to \mathrm{Ch}_{k-(p-1)i}(X)$$

are called the *reduced powers of homological type.*

Properties:

1. If $i : Y \to X$ is a projective morphism of schemes over F, then

$$S_\bullet(i_*\alpha) = i_*(S_\bullet\alpha)$$

for every $\alpha \in \mathrm{Ch}_*(Y)$. In fact, this is a variant of a Riemann-Roch theorem by I. Panin and A. Smirnov.

2. S_0 is the identity.

3. Let a scheme X be embedable in a smooth scheme of dimension d and let $\alpha \in \mathrm{Ch}_k(X)$. Then $S_i(\alpha) = 0$ for $i \notin [0, d-k]$.

If X is smooth, we have the operations $S^\bullet$ and $S_\bullet$ on the Chow groups of X modulo p. For every $\alpha \in \mathrm{Ch}_k(X)$ we have

$$S_\bullet(\alpha) = w(T_X)^{-1} \cap S^\bullet(\alpha) = w(-T_X) \cap S^\bullet(\alpha).$$

In particular,

$$S_\bullet(1_X) = w(-T_X) \cap 1_X, \quad \text{i.e.,} \quad S_i(1_X) = w_{(p-1)i}(-T_X) \cap 1_X.$$

Remark 4.5. The homological operation S_i is a special case of the cohomological operation S_i^M if X is affine. Let $i : X \hookrightarrow \mathbb{A}^n$ be a closed embedding. Since the tangent bundle of $\mathbb{A}^n$ is trivial, we have

$$S_i = S_i^{\mathbb{A}^n}.$$

5. Reduced Steenrod algebra

We fix a prime integer p in this section. Consider the polynomial ring

$$\mathcal{H} = (\mathbb{Z}/p\mathbb{Z})[\mathbf{b}] = (\mathbb{Z}/p\mathbb{Z})[b_1, b_2, \dots]$$

in infinitely many variables $b_1, b_2, \dots$ as a graded ring with $\deg b_i = p^i - 1$. The monomials

$$b^R = b_1^{r_1} b_2^{r_2} \dots,$$

where R ranges over all sequences $(r_1, r_2, \dots)$ of non-negative integers such that almost all of the r_i's are zero, form a basis of $\mathcal{H}$ over $\mathbb{Z}/p\mathbb{Z}$. We set

$$|R| = \sum_{i \geq 1} r_i(p^i - 1).$$

Clearly, $\deg b^R = |R|$.

Denote the scheme $\operatorname{Spec} \mathcal{H}$ by $\mathcal{G}$. For a commutative $\mathbb{Z}/p\mathbb{Z}$-algebra A the set of A-points

$$\mathcal{G}(A) = \operatorname{Hom}_{rings}\big(\mathcal{H}, A\big)$$

can be identified with the set of sequences $(a_1, a_2, \dots)$ of the elements of A and, therefore, with the set of power series of the form

$$x + a_1 x^p + a_2 x^{p^2} + a_3 x^{p^3} + \dots \in A[[x]].$$

The composition law $(f_1 * f_2)(x) = f_2(f_1(x))$ makes $\mathcal{G}$ a group scheme over $\mathbb{Z}/p\mathbb{Z}$ and $\mathcal{H}$ a graded Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra. The co-multiplication morphism $c : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is given by the rule

$$c(b_k) = \sum_{i+j=k} b_i^{p^j} \otimes b_j.$$

We write $\overline{\mathcal{A}}_p$ for the graded Hopf $\mathbb{Z}/p\mathbb{Z}$-algebra dual to $\mathcal{H}$. Thus, $\overline{\mathcal{A}}_p$ is the graded Hopf algebra such that the d-component of $\overline{\mathcal{A}}_p$ is the $\mathbb{Z}/p\mathbb{Z}$-space dual to the d-component of $\mathcal{H}$. The algebra $\overline{\mathcal{A}}_p$ is the *reduced Steenrod $\mathbb{Z}/p\mathbb{Z}$-algebra*, i.e., the Steenrod $\mathbb{Z}/p\mathbb{Z}$-algebra modulo the ideal generated by the Bockstein element. It can be also considered as the subalgebra of the Steenrod algebra $\mathcal{A}_p$ generated by the reduced power operations.

Let $\{P^R\}$ be the basis of $\overline{\mathcal{A}}_p$ dual to the basis $\{b^R\}$ of $\mathcal{H}$.

We write P^i for the basis element P^R with $R = (i, 0, \dots)$. The element P^0 is the identity of $\overline{\mathcal{A}}_p$ and the algebra $\overline{\mathcal{A}}_p$ is generated by the P^i, $i \geq 1$. These generators are subject to the set of defining *Adem relations*:

$$P^b \cdot P^c = \sum_{i=0}^{[b/p]} (-1)^{b+i} \binom{(p-1)(c-i)-1}{b-pi} \cdot P^{b+c-i} \cdot P^i.$$

P. Brosnan proved that the power operations S^i and S_i satisfy the Adem relations respectively. Hence, for any smooth variety X, the group $\mathrm{Ch}^*(X)$ has a structure of a left $\overline{\mathcal{A}}_p$-module (the generator P^i acts as S^i)), and for every scheme X, the group $\mathrm{Ch}_*(X)$ has a structure of a left $\overline{\mathcal{A}}_p$-module (the generator P^i acts as S_i)). Denote by S^R (resp. S_R) the operation induced by P^R on $\mathrm{CH}_*(X)$ if X is smooth (resp. arbitrary scheme). The operations S^R commute with the pull-back homomorphisms and the operations S_R commute with the push-forward homomorphisms.

Consider the "generic" power series $f_{gen}(x) = 1 + b_1x^{p-1} + b_2x^{p^2-1} + \dots$ over the ring $\mathbb{Z}[\mathbf{b}]$. Write the corresponding multiplicative class $w_{gen} = w_{f_{gen}}$ in the form

$$w_{gen} = \sum_R w_R b^R$$

for unique characteristic classes w_R over $\mathbb{Z}$. In particular, if $R = (i, 0, \dots)$, we have $w_R = w_{(p-1)i}$ in the old notation.

If X is a smooth variety, the equality $S_i(1_X) = w_{(p-1)i}(-T_X) \cap 1_X$ generalizes as follows: $S_R(1_X) = w_R(-T_X) \cap 1_X$ for every sequence R.

6. Degree formulas

Let X be a scheme over F. For a closed point $x \in X$ we define its *degree* as the integer $\deg(x) = [F(x) : F]$ and set

$$n_X = \gcd \deg(x),$$

where the gcd is taken over all closed points of X.

Fix a prime integer $p \neq \operatorname{char} F$. Let X be a projective variety of dimension $d > 0$ and let $q : X \to \operatorname{Spec} F$ be the structure morphism. For every sequence R with $|R| = d$, the group $\mathrm{Ch}_d(\operatorname{Spec} F)$ in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ch}_d(X) & \xrightarrow{S_R} & \mathrm{Ch}_0(X) & & \\
{\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle q_*=\deg} & & \\
\mathrm{Ch}_d(\operatorname{Spec} F) & \xrightarrow{S_R} & \mathrm{Ch}_0(\operatorname{Spec} F) & = & \mathbb{Z}/p
\end{array}$$

is trivial. Hence the degree of a cycle $\alpha_R^X \in \mathrm{CH}_0(X)$ representing the element $S_R([X]) \in \mathrm{Ch}_0(X)$ is divisible by p. The class of the integer $\deg(\alpha_R^X)/p$ modulo n_X is independent on the choice of α_R^X; we denote it by

$$u_R(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

Clearly, $p \cdot u_R(X) = 0$.

If X is a smooth variety, $S_R(1_X) = w_R(-T_X) \cap 1_X$, hence we can take $\alpha_R^X = w_R(-T_X) \cap 1_X$. Thus, $\deg\big(w_R(-T_X) \cap 1_X\big)$ is divisible by p and

$$u_R(X) = \frac{\deg\big(w_R(-T_X) \cap 1_X\big)}{p} + n_X\mathbb{Z}.$$

Remark 6.1. The degree of the 0-cycle $w_R(-T_X) \cap 1_X$ does not change under field extensions. In particular, it can be computed over an algebraic closure of F.

Let $f : X \to Y$ be a morphism of varieties over F of the same dimension. We set

$$\deg(f) = \begin{cases} [F(X) : F(Y)], & \text{if } f \text{ is dominant;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if f is projective, then $f_*(1_X) = \deg(f) \cdot 1_Y$.

Theorem 6.2. (Degree formula) *Let $f : X \to Y$ be a morphism of projective varieties over F of dimension $d > 0$. Then n_Y divides n_X and for every sequence R with $|R| = d$, and for any prime integer $p \neq \operatorname{char} F$, we have*

$$u_R(X) = \deg(f) \cdot u_R(Y) \in \mathbb{Z}/n_Y\mathbb{Z}.$$

In particular, if X and Y are smooth, then

$$\frac{\deg\big(w_R(-T_X) \cap 1_X\big)}{p} \equiv \deg(f) \cdot \frac{\deg\big(w_R(-T_Y) \cap 1_X\big)}{p} \quad \big(mod\ n_Y\big).$$

Proof. It follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Ch}_d(X) & \xrightarrow{S_R} & \mathrm{Ch}_0(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Ch}_d(Y) & \xrightarrow{S_R} & \mathrm{Ch}_0(Y) \end{array}$$

and the equality $f_*(1_X) = \deg(f) \cdot 1_Y$ that

$$f_* S_R(1_X) = \deg(f) \cdot S_R(1_Y) \in \mathrm{Ch}_0(Y)$$

and therefore,

$$f_* \alpha_R^X \equiv \deg(f) \cdot \alpha_R^Y \quad \big(mod\ p\,\mathrm{CH}_0(Y)\big).$$

Applying the degree homomorphism, we get

$$\deg(\alpha_R^X) = \deg(f_* \alpha_R^X) \equiv \deg(f) \cdot \deg(\alpha_R^Y) \quad (mod\ pn_Y),$$

whence the result. $\square$

Remark 6.3. The case of sequences $R = (i, 0, 0, \dots)$ was considered by M. Rost.

7. Applications

Let X and Y be varieties over a field F, $d = \dim(X)$. A *correspondence from X to Y*, denoted $\alpha\colon X \rightsquigarrow Y$, is a dimension d algebraic cycle on $X \times Y$. A correspondence α is called *prime* if α is given by a prime (irreducible) cycle. Any correspondence is a linear combination with integer coefficients of prime correspondences.

Let $\alpha : X \rightsquigarrow Y$ be a prime correspondence. Suppose that α is given by a closed subvariety $Z \subset X \times Y$. We define *multiplicity* of α as the degree of the projection $Z \to X$. We extend the notion of multiplicity to arbitrary correspondences by linearity.

A rational morphism $X \to Y$ defines a multiplicity 1 prime correspondence $X \rightsquigarrow Y$ given by the closure of its graph. One can think of a correspondence of multiplicity m as a "generically m-valued morphism".

We fix a prime integer p and a field F such that $\operatorname{char} F \neq p$. Let R be a nonzero sequence and let X be a projective variety over F of dimension $|R|$. The variety X is called *R^p-rigid* if $u_R(X) \neq 0 \in \mathbb{Z}/n_X\mathbb{Z}$.

We write v_p for the p-adic valuation of $\mathbb{Q}$.

Theorem 7.1. *Let X and Y be projective varieties over F and let R be a sequence such that $\dim(X) = |R| > 0$. Suppose that*

(1) *There is a correspondence $\alpha : X \rightsquigarrow Y$ of multiplicity not divisible by p;*
(2) *X is R^p-rigid;*
(3) *$v_p(n_X) \leq v_p(n_Y)$.*

Then

(1) $\dim(X) \leq \dim(Y)$*;*
(2) *If* $\dim(X) = \dim(Y)$*,*
   (2a) *There is a correspondence $\beta : Y \rightsquigarrow X$ of multiplicity not divisible by p;*
   (2b) *Y is R^p-rigid;*
   (2c) *$v_p(n_X) = v_p(n_Y)$.*

Proof. Suppose that $m = \dim(X) - \dim(Y) \geq 0$ and set $Y' = Y \times \mathbb{P}^m_F$. Clearly, $n_{Y'} = n_Y$. We embed Y into Y' as $Y \times z$ where z is a rational point of $\mathbb{P}^m_F$.

We may assume that α is a prime correspondence, replacing if necessary, α by one of its prime components. Let $Z \subset X \times Y$ be the closed subvariety representing α. We have two natural morphisms $f : Z \to X$ and $g : Z \to Y \hookrightarrow Y'$. By assumption, $\deg(f)$ is not divisible by p.

We write the degree formulas of Theorem 6.2 for the morphisms f and g:

$$u_R(Z) = \deg(f) \cdot u_R(X) \in \mathbb{Z}/n_X\mathbb{Z}, \tag{1}$$

$$u_R(Z) = \deg(g) \cdot u_R(Y') \in \mathbb{Z}/n_Y\mathbb{Z}. \tag{2}$$

The variety X is R^p-rigid and the degree $\deg(f)$ is not divisible by p, hence it follows from (1) that $u_R(Z) \neq 0$ in $\mathbb{Z}/n_X\mathbb{Z}$. Since $v_p(n_X) \leq v_p(n_Y)$ and $p \cdot u_R(Z) = 0$, we have $u_R(Z) \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$ and it follows from the degree formula (2) that $\deg(g)$ is not divisible by p, so that g is surjective, and $u_R(Y') \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$. The image of g is contained in Y, therefore $Y = Y'$, i.e., $m = 0$ and $\dim(X) = \dim(Y)$.

The variety Z defines a correspondence $\beta : Y \rightsquigarrow X$ of multiplicity $\deg(g)$ not divisible by p. Since $u_R(Y) = u_R(Y') \neq 0$ in $\mathbb{Z}/n_Y\mathbb{Z}$, it follows that Y is R^p-rigid. Finally, $p \cdot u_R(Z) = 0$ and $u_R(Z)$ is nonzero in both $\mathbb{Z}/n_X\mathbb{Z}$ and $\mathbb{Z}/n_Y\mathbb{Z}$, therefore, we must have $v_p(n_X) = v_p(n_Y)$. □

Corollary 7.2. *The classes $u_R(X) \in \mathbb{Z}/n_X\mathbb{Z}$ are birational invariants of a smooth projective variety X.*

8. Examples

Let X be a complete intersection in $\mathbb{P}^n$ of m smooth hypersurfaces of degrees $d_1, d_2, \dots, d_m$. Let $L = L_{can}$ be the restriction on X of the canonical line bundle on $\mathbb{P}^n$. Then

$$[T_X] = [T_{\mathbb{P}^n}|_X] - \sum_{i=1}^{m} [L^{\otimes d_i}] \in K_0(X).$$

We have

$$[T_{\mathbb{P}^n}|_X] = (n+1)[L] - 1_X \in K_0(X).$$

and

$$w_{gen}(L) = 1 + h^{p-1}b_1 + h^{p^2-1}b_2 + \dots,$$

where $h = c_1(L) \in \mathrm{Ch}^1(X)$ is the class of a hyperplane section. Therefore,

$$w_{gen}(-T_X) = \frac{\prod w_{gen}(L^{\otimes d_i})}{w_{gen}(L)^{n+1}} = \frac{\prod (1 + (d_i h)^{p-1} b_1 + (d_i h)^{p^2-1} b_2 + \dots)}{(1 + h^{p-1} b_1 + h^{p^2-1} b_2 + \dots)^{n+1}}.$$

Let R be a sequence such that $|R| = \dim X = n - m$. Since

$$\deg h^{n-m} = \prod d_i,$$

we have

$$\deg w_R(-T_X) = \prod d_i \cdot \left[\frac{\prod (1 + d_i^{p-1} b_1 + d_i^{p^2-1} b_2 + \dots)}{(1 + b_1 + b_2 + \dots)^{n+1}} \right]_R .$$

Example 8.1. Let X be the Severi-Brauer variety of a central simple algebra A of degree p^n. Then X is a twisted form of the projective space $\mathbb{P}^{p^n-1}$. For $R = (0, \dots, 0, \overset{n}{1}, 0, \dots)$ we have

$$\deg w_R(-T_X) = -p^n.$$

Thus, X is R^p-rigid if $n_X = p^n$, i.e., A is a division algebra.

Example 8.2. Let X be a smooth hypersurface of degree p in $\mathbb{P}^{p^n}$. For a sequence R as above we have

$$\deg w_R(-T_X) = p(p^{p^n-1} - p^n - 1).$$

Thus, X is R^p-rigid if $n_X = p$.

Example 8.3. Let X be a smooth intersection of two quadrics in $\mathbb{P}^4$. Then for $R = (2, 0, 0, \dots)$ and $p = 2$ we have

$$\deg w_R(-T_X) = -4.$$

Thus, X is R^2-rigid if $n_X = 4$.

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