ALGEBRAIC COBORDISM THEORY

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1. Oriented Borel-Moore functors

We will follow the work of M. Levine and F. Morel.

Notation:
Scheme = quasi-projective scheme over a field F;
Variety = integral scheme;
$\mathbf{Sch}(F)$ = category of schemes over F;
$\mathbf{Sm}(F)$ = category of schemes over F;
$\mathcal{V}$ is either $\mathbf{Sm}(F)$ or $\mathbf{Sch}(F)$;
$\mathcal{V}'$ has the same objects as $\mathcal{V}$ and projective morphisms;
$pt = \operatorname{Spec} F$.

We will be considering pairs of functors (theories):

- A Borel-Moore homology functor

$$H_* : \mathcal{V}'(F) \to \mathbf{Ab}_*,$$

- A cohomology functor

$$H^* : \mathbf{Sm}(F)^{op} \to \mathbf{Rings}^*$$

together with the canonical isomorphism

$$H^*(X) \simeq H_{d-*}(X)$$

for every smooth variety X of dimension d (Poincaré duality).

1.1. **Chow groups.** Let X be a scheme over a field F and let $k \geq 0$ be an integer. A *cycle of dimension k on X* is a formal finite sum with integer coefficients

$$\alpha = \sum n_V [V]$$

taken over closed subvarieties $V \subset X$ of dimension k. All cycles of dimension i on X form a free abelian group $Z_k(X)$. Let $R_k(X)$ be the subgroup of $Z_k(X)$ of rationally trivial cycles. The factor group

$$\mathrm{CH}_k(X) = Z_k(X)/R_k(X)$$

is called the *Chow group of X of dimension k*.

Clearly, $\mathrm{CH}_k(X) = 0$ for $k > \dim X$ or $k < 0$.

Example 1.1. $\mathrm{CH}_*(pt) = \mathbb{Z} \cdot [pt]$.

1.1.1. *Push-forward.* Let $f : Y \to X$ be a *projective* morphism of schemes over F. Let $V \subset X$ be a closed subvariety of dimension k. The image $f(V)$ is a closed subvariety of X. Clearly, $\dim W \leq k$ and $\dim W = k$ if and only if the function field $F(V)$ is a finite field extension of $F(W)$. We set

$$\deg_V(f) = \begin{cases} [F(V) : F(W)], & \text{if } \dim W = k; \\ 0, & \text{otherwise,} \end{cases}$$

For every $k \geq 0$ we can define a homomorphism

$$f_* : Z_k(Y) \to Z_k(X)$$

by the rule

$$f_*([V]) = \deg_V(f) \cdot [W].$$

We have

$$f_*(R_k(Y)) \subset R_k(X)$$

and therefore f_* induces the *push-forward homomorphism*

$$f_* : \mathrm{CH}_k(Y) \to \mathrm{CH}_k(X).$$

If $g : Z \to Y$ be another projective morphism, then $f \circ g$ is projective and $(f \circ g)_* = f_* \circ g_*$. Thus, the correspondence $X \mapsto \mathrm{CH}_*(X)$ gives rise to a functor

$$\mathrm{CH}_* : \mathbf{Sch}(F)' \to \mathbf{Ab}_*$$

that is an example of what will be called a *Borel-Moore functor* in what follows.

1.1.2. *Flat pull-back.* Let $f : Y \to X$ be a flat morphism of schemes of relative dimension d. Then for a closed subvariety $V \subset X$ of dimension k the closed subscheme $f^{-1}(V)$ is equidimensional of dimension $k+d$. The homomorphism

$$f^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+d}(Y), \qquad [V] = [f^{-1}(V)]$$

is well-defined and is called the *pull-back homomorphism.*

If $g : Z \to Y$ is another flat morphism of relative dimension e, then the composite $f \circ g$ is flat of relative dimension $d+e$ and $(f \circ g)^* = g^* \circ f^*$. Thus, the correspondence $X \mapsto \mathrm{CH}_*(X)$ gives rise to a co-functor from the category of schemes and flat morphisms of constant relative dimension to the category of graded groups.

Let

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{f} & X \end{array}$$

be a pull-back diagram of schemes such that the morphism f is flat of relative dimension d and g is proper. Then f' is flat of relative dimension d and g is proper and we have

$$g_* \circ f^* = f'^* \circ h_*.$$

1.1.3. *Homotopy Invariance* **(HI)**. Let $p : E \to X$ be a vector bundle of rank r. Then p is flat of relative dimension r and therefore we have a pull-back homomorphism

$$p^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E).$$

The *homotopy invariance* property asserts that map p^* is an isomorphism. In fact the homotopy invariance holds if p is just an affine bundle.

1.1.4. *Chern classes.* Let $L \to X$ be a line bundle with the zero section $z : X \to L$. We define the *first Chern class of* L as the homomorphism

$$c_1(L) : \mathrm{CH}_k(X) \xrightarrow{z_*} \mathrm{CH}_k(L) \xrightarrow{(p^*)^{-1}} \mathrm{CH}_{k-1}(X).$$

Let L and L' be two line bundles over X. We have

$$c_1(L \otimes L') = c_1(L) + c_1(L') \qquad \text{and} \qquad c_1(L) \circ c_1(L') = c_1(L') \circ c_1(L).$$

1.1.5. *Projective bundle theorem* **(PBT)**. Let $E \to X$ be a vector bundle of rank r and let $q : \mathbb{P}(E) \to X$ be the associated projective bundle. Denote by $L_t \to \mathbb{P}(E)$ the *tautological line bundle* (with the sheaf of sections $\mathcal{O}(-1)$). Let ξ be the operator $c_1(L_t)$ in $\mathrm{CH}_*\big(\mathbb{P}(E)\big)$. Then every element $a \in \mathrm{CH}_{k-1}\big(\mathbb{P}(E)\big)$ can be written in the form

$$a = \sum_{i=1}^{r} \big(\xi^{r-i} \circ q^*\big)(a_i)$$

for unique elements $a_i \in \mathrm{CH}_{k-i}(X)$.

The **PBT** allows to define the *higher Chern operations*

$$c_i(E) : \mathrm{CH}_*(X) \to \mathrm{CH}_{*-i}(X), \quad i \geq 0$$

by the rules $c_0(E) = \mathrm{id}$ and

$$\sum_{i=0}^{r} (-1)^i \xi^{r-i} \circ q^* \circ c_i(E) = 0.$$

In the case $\mathrm{rank}(E) = 1$ we have $\mathbb{P}(E) = X$ and $L_t = E$ so that the new definition of $c_1(E)$ agrees with the old one.

1.1.6. *Localization exact sequence* **(LOC)**. Let Z be a closed subscheme of a scheme X and set $U = X - Z$. Denote by $i : Z \to X$ and $j : U \to X$ the closed and open embeddings respectively. Then the *localization sequence*

$$\mathrm{CH}_k(Z) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \to 0$$

is exact.

1.1.7. *External products.* Let X and Y be two schemes. The assignment $([Z], [T]) \mapsto [Z \times T]$ gives rise to the *external product*

$$\mathrm{CH}_k(X) \otimes \mathrm{CH}_l(Y) \to \mathrm{CH}_{k+l}(X \times Y).$$

1.1.8. *Pull-back homomorphisms with respect to regular closed embeddings.* A sequence of elements $a_1, a_2, \dots a_d$ of a commutative ring A is called a *regular sequence of length* d is a_i is not a zero divisor in $A/(a_1, \dots, a_{i-1})$ for all $i = 1, \dots, d$. A closed embedding $i : Y \to X$ is called *regular of codimension* d if the ideal of Y in X in a neighborhood of every point of Y is generated by a regular sequence of length d.

Example 1.2. A regular closed embedding of codimension 1 is a locally principal divisor.

If $i : Y \to X$ is a regular closed embedding of codimension d then there are *pull-back* homomorphisms

$$i^* : \mathrm{CH}_k(X) \to \mathrm{CH}_{k-d}(Y).$$

Let $i : Y \to X$ be a regular closed embedding of codimension d and let $f : X' \to X$ be a projective morphism. Suppose that in the fiber square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$

the morphism i' is also a regular closed embedding of codimension d. Then

$$i^* \circ f_* = i'^* \circ g_*.$$

1.1.9. *Smooth schemes.* Let X be a smooth variety of dimension d. We write $\mathrm{CH}^i(X)$ for $\mathrm{CH}_{d-i}(X)$. The diagonal embedding $\delta : X \to X \times X$ is then regular of codimension d. The composition

$$\mathrm{CH}^i(X) \otimes \mathrm{CH}^j(X) \xrightarrow{product} \mathrm{CH}^{i+j}(X \times X) \xrightarrow{\delta^*} \mathrm{CH}^{i+j}(X)$$

makes the graded group $\mathrm{CH}^*(X)$ a commutative ring, called the *Chow ring of* X.

In general, a smooth scheme X is a disjoint union of the components X_i. We define the ring $\mathrm{CH}^*(X)$ as the product of the rings $\mathrm{CH}^*(X_i)$.

Every morphism $f : Y \to X$ of smooth schemes can be factored as

$$Y \xrightarrow{g} Y \times X \xrightarrow{p} X$$

where $g = (1_Y, f)$ and p is the projection. We have g a regular closed embedding and therefore we have the *pull-back* ring homomorphism

$$f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

well defined. We get a functor

$$\mathbf{Sm}(F)^{op} \to \mathbf{Rings}^*, \qquad X \mapsto \mathrm{CH}^*(X).$$

This functor can be viewed as a *cohomology* functor.

1.2. G**-groups.** Let X be a scheme. The group $G(X)$ is defined by generators and relations. The generators are the isomorphism classes $[\mathcal{M}]$ of coherent $\mathcal{O}_X$-module $\mathcal{M}$. Relations are of the forms $[\mathcal{M}'] + [\mathcal{M}''] = [\mathcal{M}]$ for every exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

of coherent $\mathcal{O}_X$-modules.

The theory $X \mapsto G(X)$ is not graded. We force it to be graded by tensoring with $\mathbb{Z}[\beta, \beta^{-1}]$:

$$G_*(X) = G(X) \otimes \mathbb{Z}[\beta, \beta^{-1}].$$

where we consider the *Bott element* β of degree 1. In other words,

$$G_i(X) = G(X) \cdot \beta^i, \quad i \in \mathbb{Z}.$$

Example 1.3. If $X = \operatorname{Spec} F$ is the point, then $G_*(X) = \mathbb{Z}[\beta, \beta^{-1}]$.

1.2.1. *Push-forward.* Let $f : Y \to X$ be a *projective* morphism of schemes over F. We define the *push-forward homomorphisms*

$$f_* : G_k(Y) \to G_k(X)$$

by

$$f_*([\mathcal{M}] \cdot \beta^k) = \sum_{i \geq 0} (-1)^i [R^i f_*(\mathcal{M})] \cdot \beta^k$$

for a coherent $\mathcal{O}_Y$-module $\mathcal{M}$. The we get a Borel-Moore functor

$$G_* : \mathbf{Sch}(F)' \to \mathbf{Ab}_*.$$

1.2.2. *The Grothendieck ring.* The Grothendieck ring $K(X)$ is defined similarly to $G(X)$ using locally free $\mathcal{O}_X$-modules instead of coherent $\mathcal{O}_X$-modules. Set

$$G^*(X) = K_0(X) \otimes \mathbb{Z}[\beta, \beta^{-1}].$$

The functor

$$G^* : \mathbf{Sm}(F)^{op} \to \mathbf{Rings}^*$$

can be viewed as a cohomology functor.

Note that $G_*(X)$ is a module over the ring $G^*(X)$ and the canonical homomorphism $G^*(X) \to G_{d-*}(X)$ is an isomorphism if X is a smooth variety of dimension d. This can be regarded as the *Poincaré duality.*

1.2.3. *Flat pull-back.* Let $f : Y \to X$ be a flat morphism of schemes of relative dimension d. We define the *pull-back homomorphisms*

$$f^* : G_k(Y) \to G_{k+d}(X)$$

by

$$f^*([\mathcal{M}] \cdot \beta^k) = [f^*(\mathcal{M})] \cdot \beta^{k+d}.$$

for a coherent $\mathcal{O}_X$-module $\mathcal{M}$.

1.2.4. *Chern classes.* The functor G_* satisfies **HI**, hence the chern classes are well defined. Let $p : L \to X$ be a line bundle. Recall that

$$L = \operatorname{Spec} S^\bullet(\mathcal{L}^\vee)$$

where $S^\bullet$ is the symmetric algebra and $\mathcal{L}$ is the sheaf of sections of L. Then we have an exact sequence of coherent $\mathcal{O}_L$-modules

$$0 \to p^*\mathcal{L}^\vee \to \mathcal{O}_L \to z_*\mathcal{O}_X \to 0$$

where z is the zero section. Tensoring with $p^*\mathcal{M}$ over $\mathcal{O}_L$ for an $\mathcal{O}_X$-module $\mathcal{M}$ we get an exact sequence

$$0 \to p^*(\mathcal{L}^\vee \otimes \mathcal{M}) \to p^*\mathcal{M} \to z_*\mathcal{M} \to 0.$$

We have

$$\begin{aligned} c_1\big([\mathcal{M}] \cdot \beta^k\big) &= (p^*)^{-1} z_*\big([\mathcal{M}] \cdot \beta^k\big) \\ &= (p^*)^{-1}\big(p^*[\mathcal{M}] \cdot \beta^k - p^*[\mathcal{L}^\vee \otimes \mathcal{M}] \cdot \beta^k\big) \\ &= (1 - [\mathcal{L}^\vee])[\mathcal{M}] \cdot \beta^{k-1}. \end{aligned}$$

In other words, $c_1(L)$ is the multiplication by $(1 - [\mathcal{L}^\vee]) \cdot \beta^{-1}$.

Let L and L' be two line bundles over X. We have

$$1 - [L \otimes L'] = \big(1 - [L]\big) + \big(1 - [L']\big) - \big(1 - [L]\big)\big(1 - [L']\big)$$

hence

$$c_1(L \otimes L') = c_1(L) + c_1(L') - \beta c_1(L) c_1(L').$$

1.2.5. *External products.* The external product

$$G_k(X) \otimes G_l(Y) \to G_{k+l}(X \times Y)$$

is given by

$$([\mathcal{M}] \cdot \beta^k, [\mathcal{N}] \cdot \beta^l) \mapsto [p_1^*(\mathcal{M}) \otimes p_2^*(\mathcal{N})] \cdot \beta^{k+l}$$

where p_1 and p_2 are two projections of $X \times Y$ to X and Y respectively.

Note that the theory G_* satisfies **PBT**, **LOC** a has pull-backs with respect to regular closed embeddings.

1.3. **Borel-Moore functors.** The idea is weaken the assumption: we will assume existence of the pull-back homomorphisms for *smooth morphisms* only, not all flat ones.

Consider the category $\mathcal{V}'(F)$ where $\mathcal{V}(F) = \mathbf{Sch}(F)$ or $\mathbf{Sm}(F)$. A *Borel-Moore functor* is a functor

$$H_* : \mathcal{V}'(F) \to \mathbf{Ab}_* \,.$$

Thus for every projective morphism $f : Y \to X$ a (*push-forward*) homomorphism

$$f_* : H_*(Y) \to H_*(X)$$

is given.

A Borel-Moore functor is called *additive* if for any $X_1, X_2, \ldots, X_n$ in $\mathcal{V}'(F)$ the natural homomorphism

$$\coprod_{i=1}^{n} H_*(X_i) \to H_*(\coprod_{i=1}^{n} X_i)$$

is an isomorphism. In particular, $H_*(\emptyset) = 0$.

An additive Borel-Moore functor H_* is called *oriented* (**OBMF**) if there are given:

(1) For each smooth morphism $f : Y \to X$ of relative dimension d a homomorphism of graded groups (*pull-backs*)

$$f^* : H_*(X) \to H_{*+d}(Y).$$

(2) For each line bundle $L \to X$ a homomorphism of graded groups (*first Chern class*)

$$c_1(L) : H_*(X) \to H_{*-1}(X).$$

All these should satisfy the following axioms:

(A1) For any pair $Z \xrightarrow{g} Y \xrightarrow{f} X$ of equidimensional smooth morphisms one has $(f \circ g)^* = g^* \circ f^*$. Also $\mathrm{Id}_X^* = \mathrm{Id}_{H_*(X)}$.

(A2) For a fiber square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

where f is projective and g is smooth equidimensional one has

$$g^* \circ f_* = f'_* \circ g'^*.$$

(A3) For a projective morphism $f : Y \to X$ and a line bundle $L \to X$, one has

$$f_* \circ c_1(f^* L) = c_1(L) \circ f_*.$$

(A4) For a smooth equidimensional morphism $f : Y \to X$ and a line bundle $L \to X$, one has

$$c_1(f^* L) \circ f^* = f^* \circ c_1(L).$$

(A5) For any two line bundles L and L' on X one has

$$c_1(L) \circ c_1(L') = c_1(L') \circ c_1(L).$$

Moreover, if L and L' are isomorphic then $c_1(L) = c_1(L')$.

A morphism $H_* \to H'_*$ of oriented Borel-Moore functors is a natural transformation of functors which commutes with the smooth-pull-backs and the first Chern classes.

Remark 1.4. Consider the category $\mathcal{A}$ with the same objects as in $\mathcal{V}(F)$. A morphism $X \rightsquigarrow Y$ is a "roof" $X \xleftarrow{f} X' \xrightarrow{g} Y$ where f is smooth of relative dimension d and g is projective. The composition of $X \rightsquigarrow Y$ and another morphism $Y \rightsquigarrow Z$, represented by the roof $Y \xleftarrow{h} Y' \xrightarrow{k} Z$, is given by the roof $X \xleftarrow{fh'} X'' \xrightarrow{kg'} Z$ defined by the fiber square

$$\begin{array}{ccc} X'' & \xrightarrow{g'} & Y' \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g} & Y. \end{array}$$

Every BM-functor H defines a functor $\mathcal{A} \to \mathbf{Ab}$ taking an object X to $H(X)$ and a morphism $X \rightsquigarrow Y$ given by a roof $X \xleftarrow{f} X' \xrightarrow{g} Y$ to

$$g_* f^* : H_*(X) \to H_{*+d}(Y).$$

The property (A2) shows that the functor is well defined.

1.3.1. *Oriented Borel-Moore functor with product.* An *oriented Borel-Moore functor with product* is a **OBMF** together with the following data:

An element $1 \in H_0(pt)$ and for every two schemes X and Y, a bilinear graded pairing (called the *external product*)

$$H_*(X) \otimes H_*(Y) \to H_*(X \times Y), \qquad \alpha \otimes \beta \mapsto \alpha \times \beta$$

which is commutative, associative and admits 1 as unit. The product satisfies:

(A6) For every projective morphisms f and g,

$$f_* \times g_* = (f \times g)_*.$$

(A7) For every smooth equidimensional morphisms f and g,

$$f^* \times g^* = (f \times g)^*.$$

(A8) For every two schemes X and Y and a line bundle $L \to X$ one has

$$(c_1(L)(\alpha) \times \beta = c_1\big(p_1^*(L)\big)(\alpha \times \beta)$$

for every $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$.

If H_* is a **OBMF** with product then the axioms give $H_*(pt)$ a structure of a commutative graded ring, and to each $H_*(X)$ a structure of a $H_*(pt)$-module so that the operations f_*, f^* and $c_1(L)$ preserve the $H_*(pt)$-module structure.

For $X \in \mathbf{Sm}(F)$ the distinguished element $p^*(1)$ is denoted by $1_X \in H^0(X)$ (here $p : X \to pt$ is the structure morphisms).

Remark 1.5. If X is not smooth, the group $H_*(X)$ may have no distinguished element.

1.4. **Formal group law.** Let H_* be a **OBMF** with products. Assume that **PBT** holds. Let L_n be the canonical line bundle on the projective space $\mathbb{P}^n$. For simplicity we write L_n and L_m for the pull-backs of L_n and L_m on $\mathbb{P}^n \times \mathbb{P}^m$ with respect to two projections. By **PBT**, applied twice, we have

$$c_1(L_n \otimes L_m) = \sum_{\substack{0 \le i \le n \\ 0 \le j \le m}} a_{ij}\ c_1(L_n)^i\ c_1(L_m)^j$$

for uniquely determined elements $a_{ij} \in H_{i+j-1}(pt)$. One shows that the elements a_{ij} do not depend on $n \ge i$ and $m \ge j$.

Suppose now that H_* has the pull-backs with respect to morphisms of smooth schemes. Let L and M be line bundles over $X \in \mathbf{Sm}(F)$ generated by global sections. Then there are morphism $f : X \to \mathbb{P}^n$ and $g : X \to \mathbb{P}^m$ such that $L \simeq f^*L_n$ and $M \simeq g^*L_m$. Pulling back the formula above we get

$$c_1(L \otimes M) = \sum_{i,j \ge 0} a_{ij}\ c_1(L)^i\ c_1(M)^j \tag{1}$$

on X.

Suppose that L and M are arbitrary line bundles on X. By Jouanolou's trick, there is an affine bundle $h : Y \to X$ with an affine scheme Y. Since every vector bundle over an affine scheme is generated by sections, the formula (1) holds for h^*L and h^*M over Y. Assume that H_* satisfies **HI**. Then (1) holds for L and M over X.

Consider the power series

$$\Phi_H(u, v) = \sum_{i,j \ge 0} a_{ij}\ u^i\ v^j \in H_*(pt)[[u, v]].$$

One shows that $\Phi = \Phi_H$ satisfies:

(1) $\Phi\big(u, \Phi(v, w)\big) = \Phi\big(\Phi(u, v), w\big)$;
(2) $\Phi(u, v) = \Phi(v, u)$;
(3) $\Phi(u, v) = u + v + \sum_{i,j \ge 1} a_{ij}\ u^i\ v^j$.

Thus Φ_H is a (commutative) *formal group law* over $H_*(pt)$. It is called the *formal group law of* H_*.

Consider the "universal" power series

$$\Phi(u, v) = u + v + \sum_{i,j \ge 1} A_{ij}\ u^i\ v^j \in H_*(pt)[[u, v]]$$

over the polynomial ring $\mathbb{Z}[A_{ij}]$ where A_{ij} are variables. Let I be the ideal in $\mathbb{Z}[A_{ij}]$ given by the conditions (1) and (2) above. The factor ring

$$\mathbb{L} = \mathbb{Z}[A_{ij}]/I$$

is called the *Lazard ring* and the image $\Phi_{\mathbb{L}}$ of F in $\mathbb{L}$ is called the *universal formal group law*. The coefficients a_{ij} of $\Phi_{\mathbb{L}}$ generate the Lazard ring $\mathbb{L}$. For every commutative ring R we have a canonical bijection

$$\boxed{\begin{array}{c}\text{Formal group}\\ \text{laws over } R\end{array}} \simeq \operatorname{Hom}_{Rings}(\mathbb{L}, R).$$

We consider $\mathbb{L}$ as a graded ring by $\deg a_{ij} = i + j - 1$. For any **OBMF** considered in this section there is a canonical graded ring homomorphism $\mathbb{L}_* \to H_*(pt)$ giving the formal group law Φ_H.

Example 1.6. If $H_* = \mathrm{CH}_*$ then $\Phi_H(u, v) = u + v$ the *additive* groups law. The ring homomorphism $\mathbb{L}_* \to H_*(pt) = \mathbb{Z}$ takes all a_{ij} to 0.

Example 1.7. If $H_* = G_*$ then $\Phi_H(u, v) = u + v - \beta uv$ the *multiplicative periodic* groups law. The ring homomorphism $\mathbb{L}_* \to H_*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$ takes a_{11} to $-\beta$ and all other a_{ij} to 0.

Let R_* be a commutative graded ring. An **OBMF** *over* R_* is **OBMF** H_* with product together with a graded ring homomorphism $R_* \to H_*(pt)$. For such a functor, one gets the structure of an R_*-module on $H_*(X)$ for every X. All the operations in H_* are R_*-linear.

Example 1.8. The functors CH_* and G_* are **OBMF** over L_*.

Given an **OBMF** over R_* and a homomorphism of commutative graded rings $R_* \to S_*$, one can construct an **OBMF** $H_* \otimes_{R_*} S_*$ over S_* defined by $X \mapsto H_*(X) \otimes_{R_*} S_*$.

1.5. **OBMF of geometric type.** Let H_* be an **OBMF** H_* over $\mathbb{L}_*$. Let $\Phi_H \in H_*(pt)[[u, v]]$ be the image by the homomorphism $\mathbb{L}_* \to H_*(pt)$ (giving the $\mathbb{L}_*$-structure) of the power series $\Phi_{\mathbb{L}}$. Thus Φ_H is a formal group law over the ring $H_*(pt)$.

An **OBMF** H_* over L_* is said to be of *geometric type* is the following three axioms hold:

Dimension axiom **(Dim)**: For any $Y \in \mathbf{Sm}(F)$ and any family $(L_1, \dots, L_r)$ of line bundles on Y with $r > \dim Y$ one has

$$c_1(L_1) \circ \cdots \circ c_1(L_r)(1_Y) = 0 \in H_*(Y).$$

Let $L \to Y$ be a line bundle over $Y \in \mathbf{Sm}(F)$ and let $s : Y \to L$ be a section. Let $Z \subset Y$ be the closed subscheme of zeros of s. We say that *s is transverse to the zero section* if Z is a smooth divisor in Y.

Section axiom **(Sect)**: For any $Y \in \mathbf{Sm}(F)$, any line bundle $L \to Y$ and any section s of L which is transverse to the zero section of L, one has

$$c_1(L)(1_Y) = i_*(1_Z)$$

where $i : Z \to X$ is the closed embedding of the scheme of the zeros of s.

Note that for every smooth divisor $Z \subset Y$ there is a line bundle, namely, $L(Z)$ and a section having Z the scheme of zeros. If Z and Z' are two rationally equivalent smooth divisors on Y, then $L(Z) \simeq L(Z')$ and hence

$$i_*(1_Z) = c_1(L)(1_Y) = c_1(L')(1_Y) = i_*(1'_Z).$$

Formal Group Law axiom (**FGL**): For every $Y \in \mathbf{Sm}(F)$ and every two line bundles L and M on Y, one has

$$c_1(L \otimes M) = \Phi_H\big(c_1(L), c_1(M)\big)(1_Y) \in H_*(Y).$$

Remark 1.9. Suppose that H_* has pull-backs with respect to regular closed embeddings. Let $L \to Y$ be a line bundle and let s be a section of L which is transverse to the zero section of L with the scheme Z of the zeros of s. Then in the fiber square

$$\begin{array}{ccc} Z & \xrightarrow{i} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ Y & \xrightarrow{z} & L \end{array}$$

all morphisms are regular closed embeddings of codimension 1. We have

$$c_1(L)(1_Y) = s^* z_*(1_Y) = i_* i^*(1_Y) = i_*(1_Z) = [Z \to Y],$$

so that **Sect** follows.

Remark 1.10. The L.H.S. of **FGL** to make sense, we need the vanishing stated in **Dim** and commutativity of Chern classes.

Example 1.11. The functors CH_* and G_* are of geometric type.

Let the power series $\chi_H(u)$ be such that $\Phi_H(u, \chi_H(u)) = 0$. Then

$$c_1(L^\vee) = \chi_H\big(c_1(L)\big).$$

1.6. **Cohomology functors.** A morphism $f : Y \to X$ is called a *local complete intersection* (l.c.i.) if f factors $Y \xrightarrow{g} X' \xrightarrow{h} X$ where g is a regular closed embedding and h is a smooth morphism.

Let H_* be an **OBMF** with products that admits pull-backs with respect to regular closed embedding. Therefore, H^* admits pull-backs with respect to all l.c.i. morphisms.

If X is smooth connected of dimension d we set

$$H^s(X) = H_{d-s}(X)$$

In general, we write

$$H^s(X) = \sum_{i=1}^{n} H^s(X_i)$$

if $X = \coprod_{i=1}^n X_i$ disjoint union of connected components.

The pull-bach along the diagonal makes $H^*(X)$ a graded ring for every $X \in \mathbf{Sm}(F)$. Since every morphism of smooth schemes is l.c.i., we get a functor

$$H^* : \mathbf{Sm}(F)^{op} \to \mathbf{Rings}^*$$

that is called the *associated cohomology functor*.

2. Definition of algebraic cobordism

We are going to construct the cobordism functor on the category $\mathcal{V} = \mathbf{Sch}(F)$. Fix a scheme $X \in \mathbf{Sch}(F)$. A *cobordism cycle over* X is a family

$$(f : Y \to X; L_1, \dots, L_r)$$

such that f is a projective morphism of a smooth integral scheme Y and the L_i are line bundles on Y, $r \geq 0$. The dimension of this cycle is $\dim Y - r$. An isomorphism of two cobordism classes $(f : Y \to X; L_1, \dots, L_r)$ and $(f' : Y' \to X; L'_1, \dots, L'_{r'})$ over X is an isomorphism $g : Y \xrightarrow{\sim} Y'$ over X, a bijection $\sigma : \{1, \dots, r\} \xrightarrow{\sim} \{1, \dots, r'\}$ (so that $r = r'$) and isomorphisms $L_i \xrightarrow{\sim} g^* L'_{\sigma(i)}$ for every i.

For every $i \in \mathbb{Z}$, let $\mathcal{Z}_i(X)$ be the free abelian group on the set of isomorphism classes of cobordism cycles over X of dimension i. The class of the cobordism cycle $(f : Y \to X; L_1, \dots L_r)$ is denoted by $[f : Y \to X; L_1, \dots, L_r]$. If X is smooth, we write 1_X for $[\mathrm{Id} : X \to X]$.

By linearity, the definition of the cobordism cycle extends to the case of a non-connected scheme Y.

For a projective morphism $g : X \to X'$ we define the push-forward homomorphism

$$g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X')$$
$$[f : Y \to X; L_1, \dots, L_r] \mapsto [g \circ f : Y \to X'; L_1, \dots, L_r].$$

For a smooth morphism $g : X' \to X$ of relative dimension d we define the pull-back homomorphism

$$g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_{*+d}(X')$$
$$[f : Y \to X; L_1, \dots, L_r] \mapsto [p_2 : Y \times_X X' \to X'; p_1^* L_1, \dots, p_1^* L_r].$$

For a line bundle $g : L \to X$ we define the first Chern class operation

$$c_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X)$$
$$[f : Y \to X; L_1, \dots, L_r] \mapsto [f : Y \to X; L_1, \dots, L_r, f^* L].$$

We define an external product

$$\mathcal{Z}_*(X) \otimes \mathcal{Z}_*(X') \to \mathcal{Z}_*(X \times X')$$

$$[f : Y \to X; L_1, \dots, L_r] \otimes [f' : Y' \to X; L'_1, \dots, L'_{r'}] \mapsto$$
$$[f \times f' : Y \times Y' \to X; p_1^* L_1, \dots, p_1^* L_r, p_2^* L'_1, \dots, p_2^* L'_{r'}].$$

Note that

$$\begin{aligned}[f : Y \to X; L_1, \dots, L_r] &= f_* \circ [\mathrm{Id} : Y \to Y; L_1, \dots, L_r] \\ &= f_* \circ c_1(L_r) \circ \dots \circ c_1(L_1)(1_Y).\end{aligned}$$

The functor $\mathcal{Z}_*$ is an **OBMF** with products. Let H_* be another **OBMF** with products. We define a morphism $\theta : \mathcal{Z}_* \to H_*$ as follows. We set

$$\theta([f : Y \to X; L_1, \dots, L_r]) = f_* \circ c_1^H(L_r) \circ \dots \circ c_1^H(L_1)(1_Y)$$

One checks that θ is well defined and unique. Thus the functor $\mathcal{Z}_*$ is universal in the class of all **OBMF** with products.

2.0.1. *Imposing relations.* Let H_* be an **OBMF** and, for each X, let $R_*(X) \subset H_*(X)$ be a subset of homogeneous elements. We will construct a new **OBMF** H_*/R_* together with a morphism of **OBMF** $\pi : H_* \to H_*/R_*$ satisfying the following universal property: a morphism of **OBMF** $\theta : H_* \to H'_*$ such that $\theta(X)$ vanishes on $R_*(X)$ for any X, factors through π.

For $X \in \mathbf{Sch}(F)$ let $\overline{R}_*(X) \subset H_*(X)$ be the subgroup generated by elements of the form

$$f_* \circ c_1(L_1) \circ \cdots \circ c_1(L_r) \circ g^*(\rho)$$

where $f : Y \to X$ a projective morphism, $L_1, \ldots, L_r$ a family of line bundles on Y, $g : Y \to Z$ a smooth equidimensional morphism and $\rho \in R_*(Z)$. The assignment $X \mapsto H_*(X)/\overline{R}_*(X)$ has a structure of an **OBMF** that solves our problem. We denote this theory by H_*/R_*.

Assume that for every $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$ one has

$$\alpha \times \beta \in R_*(X \times Y)$$

if either $\alpha \in R_*(X)$ or $\beta \in R_*(Y)$. Then the **OBMF** H_*/R_* has products.

We construct the cobordism functor in four steps.

Step 1. For every $X \in \mathbf{Sm}(F)$ let $R_*^{Dim}(X)$ be the subset of all cobordism cycles of the form $[f : Y \to X; L_1, \ldots, L_r]$ where $r > \dim Y$. Set

$$\underline{\mathcal{Z}}_* = \mathcal{Z}_*/R_*^{Dim}.$$

By construction, for every $X \in \mathbf{Sch}(F)$ and a line bundle $L \to X$ the endomorphism $c_1(L)$ of $\underline{\mathcal{Z}}_n(X)$ is locally nilpotent, that is for each $a \in \underline{\mathcal{Z}}_n(X)$ there is an $m \in \mathbb{N}$ such that $c_1(L)^m(a) = 0$.

Step 2. Let $Y \in \mathbf{Sm}(F)$. We let $R_*^{Sect} \subset \underline{\mathcal{Z}}_*(Y)$ be the subset of all elements of the form

$$c_1(L)(1_Y) - [Z \to Y],$$

where $L \to Y$ is a line bundle and Z is the scheme of zeros of a section transverse to the zero section. Set

$$\underline{\Omega}_* = \underline{\mathcal{Z}}_*/R_*^{Sect}.$$

The **OBMF** $\underline{\Omega}_*$ with products is called *algebraic pre-cobordism.*

Step 3. We consider first the tensor product $\mathbb{L}_* \otimes \underline{\Omega}_*$. This is an **OBMF** over $\mathbb{L}_*$ satisfying **Dim** and **Sect**.

Step 4. Let $Y \in \mathbf{Sm}(F)$. We let $R_*^{FGL} \subset \underline{\mathcal{Z}}_*(Y)$ be the subset of all elements of the form

$$\Phi_{\mathbb{L}_*}\big(c_1(L), c_1(M)\big)(1_Y) - c_1(L \otimes M)(1_Y)$$

for all line bundles L and M on Y. We define *algebraic cobordism*:

$$\Omega_* = (\mathbb{L}_* \otimes \underline{\Omega}_*)/\mathbb{L}_* R_*^{FGL}.$$

Thus, Ω_* is an **OBMF** of geometric type.

Recall all steps in the definition:

$$\mathcal{Z}_* \to \underline{\mathcal{Z}}_* \to \underline{\Omega}_* \to \mathbb{L}_* \otimes \underline{\Omega}_* \to \Omega_*.$$

Theorem 2.1. *Algebraic cobordism is the universal* **OBMF** *of geometric type. More precisely, given an* **OBMF** *of geometric type, H_*, there is a unique morphism of* **OBMF** *over $\mathbb{L}_*$:*

$$\theta_H : \Omega_* \to H_*.$$

Remark 2.2. For every X, the homomorphism $\Omega_*(X) \to H_*(X)$ factors through

$$\Omega_*(X) \otimes_{\mathbb{L}} H_*(pt) \to H_*(X).$$

Example 2.3. We have canonical morphisms

$$\mathrm{CH}_* \leftarrow \Omega_* \to G_*.$$

Let L_n be the canonical line bundle on $\mathbb{P}^n$. The linear subscheme $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is the zero scheme of a section of L_n. By **Sect**,

$$c_1(L_n)(1_{\mathbb{P}^n}) = [\mathbb{P}^{n-1} \to \mathbb{P}^n],$$

and more generally,

$$c_1(L_n)^i(1_{\mathbb{P}^n}) = [\mathbb{P}^{n-i} \to \mathbb{P}^n].$$

For every $n, m \in \mathbb{N}$ consider the line bundle $L_{n,m} = L_n \otimes L_m$ on $\mathbb{P}^n \times \mathbb{P}^m$. Choose a section of $L_{n,m}$ transverse to the zero section and let $M_{n,m}$ be the scheme of zeros of that section. In the homogeneous coordinates (X, Y), $M_{n,m}$ can be given by one equation $\sum_{i=0}^m X_i Y_i = 0$ (provided $m \le n$). The hypersurface $M_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^m$ is called *Milnor hypersurface.*

Proposition 2.4. *We have in $\Omega_*(\mathbb{P}^n \times \mathbb{P}^m)$:*

$$[M_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} \, [\mathbb{P}^{n-i} \times \mathbb{P}^{m-j} \to \mathbb{P}^n \times \mathbb{P}^m],$$

where a_{ij} are the coefficients of the universal formal group law.

Proof. By the axiom **Sect**,

$$\begin{aligned}
[M_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m] &= c_1(L_{n,m})(1_{\mathbb{P}^n \times \mathbb{P}^m}) \\
&= c_1(L_n \otimes L_m) \\
&= \sum_{i \ge 0} \sum_{j \ge 0} a_{ij} \, c_1(L_n)^i c_1(L_m)^j (1_{\mathbb{P}^n \times \mathbb{P}^m}).
\end{aligned}$$

By the axiom **Sect** applied repeatedly,

$$c_1(L_n)^i c_1(L_m)^j(1_{\mathbb{P}^n\times\mathbb{P}^m}) = [\mathbb{P}^{n-i}\times\mathbb{P}^{m-j}\to\mathbb{P}^n\times\mathbb{P}^m].$$

□

For a $X\in\mathbf{Sm}(F)$ set

$$[X]=[X\to pt]\in\Omega_*(pt).$$

Corollary 2.5.

$$[M_{n,m}]=\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}\ [\mathbb{P}^{n-i}]\cdot[\mathbb{P}^{m-j}]\in\Omega_{n+m}(pt).$$

This gives an inductive formula for $a_{n,m}$ in terms of the classes $[M_{i,j}]$ and $[\mathbb{P}^k]$.

Example 2.6. Since $M_{1,1}\simeq\mathbb{P}^1$ we have $[\mathbb{P}^1]=[\mathbb{P}^1]+[\mathbb{P}^1]+a_{11}$, hence $[\mathbb{P}^1]=-a_{11}$.

Corollary 2.7. *The image of $\mathbb{L}_*$ in $\Omega_*(pt)$ is contained in the subring generated by the classes $[M_{i,j}]$ and $[\mathbb{P}^k]$.*

Lemma 2.8. *For every $X\in\mathbf{Sch}(F)$, the group $\Omega(X)$ is generated by the standard cobordism cycles*

$$[Y\to X;L_1,\dots L_r].$$

In other words, the canonical homomorphism $\underline{\Omega}_*(X)\to\Omega_*(X)$ is surjective.

Proof. The $\mathbb{L}_*$-module $\Omega_*(X)$ is generated by standard cobordism cycles. Since The $\mathbb{L}_*$-action factors through the canonical homomorphism $\mathbb{L}_*\to\Omega_*(pt)$, via the external product $\Omega_*(pt)\otimes\Omega_*(X)\to\Omega_*(X)$, it is sufficient to show that the ring homomorphism

$$\mathcal{Z}_*(pt)\to\Omega_*(pt)$$

is surjective. Since the ring homomorphism

$$\mathbb{L}_*\otimes\mathcal{Z}_*(pt)\to\Omega_*(pt)$$

is surjective by definition, it is sufficient to prove that the image of $\mathbb{L}_*\to\Omega_*(pt)$ is contained in the image of $\mathcal{Z}_*(pt)\to\Omega_*(pt)$. Since the ring $\mathbb{L}_*$ is generated by the coefficient a_{ij}, the result follows from Corollary 2.7. □

Lemma 2.9. *Let $X\in\mathbf{Sch}(F)$. Then the $\Omega(X)$ is generated by the standard cobordism cycles*

$$[Y\to X;L_1,\dots L_r]$$

such that all line bundles L_i are very ample.

Proof. By Lemma 2.8, $\Omega(X)$ is generated by the standard cobordism cycles

$$[f:Y\to X;L_1,\dots,L_r]=f_*\circ c_1(L_r)\circ\cdots\circ c_1(L_1)(1_Y)$$

Write $L_i = M_i \otimes N_i^\vee$ where M_i and N_i are very ample line bundles and use the formula

$$c_1(L_i) = \Phi_{\mathbb{L}}\big(c_1(M_i), \chi(c_1(N_i))\big).$$

□

Proposition 2.10. *For every $X \in \mathbf{Sch}(F)$, the group $\Omega_*(X)$ is generated by the classes $Y \to X$ of projective morphisms with Y smooth irreducible.*

Proof. By Lemma 2.9, $\Omega(X)$ is generated by the standard cobordism cycles

$$[Y \to X; L_1, \dots L_r]$$

such that all line bundles L_i are very ample. By Bertini theorem, there is a section s of L_1 transverse to the zero section. Let Z be the scheme of zeros of s. By the axiom **Sect**,

$$[\mathrm{Id}_Y, L_1] = c_1(1_Y) = [Z \to Y].$$

Applying the Chern classes and the push-forward f_*, we get

$$[Y \to X; L_1, L_2, \dots L_r] = [Z \to X; L_2, \dots L_r]$$

and we proceed by induction. □

Remark 2.11. The Bertini theorem holds over infinite fields. Over a finite field one considers infinite pro-p-extensions for two different prime values p.

Theorem 2.12. (LOC) *Let $X \in \mathbf{Sch}(F)$, $Z \xrightarrow{i} X$ be a closed subscheme and $j : U \to X$ the open complement. Assume that F admits resolution of singularities. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact.

Proof. Surjectivity. Let $f : Y \to U$ be a cobordism class. Since f is projective, if factors $Y \xrightarrow{k} U \times \mathbb{P}^n \xrightarrow{p} U$ where k is a closed embedding and p is the projection. Let $\overline{Y}$ be the closure of $k(Y)$ in $X \times \mathbb{P}^n$ and let $g : \widetilde{Y} \to \overline{Y}$ be a birational closed morphism with $\widetilde{Y}$ smooth and such that $g|_{g^{-1}(Y)} : g^{-1}(Y) \to Y$ is an isomorphism. If h is the composition

$$\widetilde{Y} \xrightarrow{g} \overline{Y} \hookrightarrow X \times \mathbb{P}^n \to X$$

then obviously $j^*([\widetilde{Y} \xrightarrow{h} X]) = [Y \xrightarrow{f} U]$. □

Theorem 2.13. (HI) *Let $p : E \to X$ be an affine bundle of rank r. Assume that F admits resolution of singularities. Then the pull-back homomorphism*

$$p^* : \Omega_*(X) \to \Omega_{*+r}(E).$$

is an isomorphism.

The following moving lemma is used in the proof:

Lemma 2.14. *Let W be in* $\mathbf{Sm}(F)$ *and let $i : Z \to W$ be a smooth closed subscheme. Then $\Omega_*(W)$ is generated by standard cobordism cycles of the form $[f : Y \to W]$ with f transverse to i.*

It is sufficient to prove **HI** for the morphism $p : X \times \mathbb{A}^1 \to X$. By Lemma, the group $\Omega_*(X \times \mathbb{A}^1)$ is generated by the cobordism classes $f : Y \to X \times \mathbb{A}^1$ such that $Z = f^{-1}(X \times 0)$ is smooth of codimension 1 in Y. Then one proves that

$$[f : Y \to X \times \mathbb{A}^1] = p^*([Z \to X]).$$

Theorem 2.15. (PBT) *Let $E \to X$ be a vector bundle of rank $r + 1$ and let $q : \mathbb{P}(E) \to X$ be the associated projective bundle. Assume that F admits resolution of singularities. Then every element $a \in \Omega_{k-1}\big(\mathbb{P}(E)\big)$ can be written in the form*

$$a = \sum_{i=1}^{r} (c_1(L_t)^{r-i} \circ q^*)(a_i)$$

for unique elements $a_i \in \Omega_{k-i}(X)$.

One reduces to the case of the projection $X \times \mathbb{P}^n \to X$ and proceeds by induction on n using **HI**.

3. The ring $\Omega_*(pt)$

Theorem 3.1. *Let* $\operatorname{char} F = 0$. *Then the canonical homomorphism*

$$\mathbb{L}_* \to \Omega_*(pt)$$

is an isomorphism.

To prove injectivity, we will construct a "large" **OBMF** H_* of geometric type such that the canonical composition

$$\mathbb{L}_* \to \Omega_*(pt) \to H_*(pt)$$

is injective.

A *partition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a sequence of integers (possibly empty) $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k > 0$. The *degree* of α is the integer

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k.$$

The integer k is called the *length* $l(\alpha)$ of the partition α.

We consider the polynomial ring $\mathbb{Z}[b_1, b_2, \dots] = \mathbb{Z}[\mathbf{b}]$ in infinitely many variables $b_1, b_2, \dots$ as a graded ring with $\deg b_i = i$. For every partition α set

$$b_\alpha = b_{\alpha_1} b_{\alpha_2} \dots b_{\alpha_k}.$$

The monomials b_α form a basis of the polynomial ring over $\mathbb{Z}$.

Let $\mathbb{Z}[c_1, c_2, \dots] = \mathbb{Z}[\mathbf{c}]$ be another polynomial ring with the grading $\deg c_i = i$. The elements of $\mathbb{Z}[\mathbf{c}]$ are called the *characteristic classes* and the c_n - the *Chern classes.*

For every partition α we define the "smallest" symmetric polynomial

$$P_\alpha(x_1, x_2, \dots) = \sum_{(i_1, i_2, \dots, i_k)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k} = Q_\alpha(\sigma_1, \sigma_2, \dots),$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, where the σ_i are the standard symmetric functions, and set

$$c_\alpha = Q_\alpha(c_1, c_2, \dots).$$

For example, $c_n = c_{(1,1,\dots,1)}$ (n units).

For every element (characteristic class) $c \in \mathbb{Z}[\mathbf{c}]$ and every vector bundle E over a variety $X \in \mathbf{Sm}(F)$ there is a well defined operation $c(E)$ on $\mathrm{CH}_*(X)$. In particular, for every partition α there are *generalized Chern operations* $c_\alpha(E)$.

We consider a new **OBMF** H_* on $\mathbf{Sm}(F)$ defined by

$$H_*(X) = \mathrm{CH}_*(X) \otimes \mathbb{Z}[\mathbf{b}] = \mathrm{CH}_*(X)[\mathbf{b}].$$

We define the *characteristic polynomial operation of* E by the formula

$$\mathbf{P}(E) = \sum_\alpha c_\alpha(E) b_\alpha \in H_*(X).$$

Note that the polynomial $\mathbf{P}(E)$ is invertible in $H_*(X)$.

Example 3.2. If L is a line bundle, then $\mathbf{P}^A(L) = \sum_{i \geq 0} c_1^A(L)^i b_i$.

The pull-back homomorphism $f_H^* : H_*(X) \to H_*(Y)$ associated to a smooth morphism $f : Y \to X$ is equal to $f^* \otimes \mathrm{id}_{\mathbb{Z}[\mathbf{b}]}$. The push-forward map f_*^H associated to a projective morphism $f : Y \to X$ is defined by

$$f_*^H = \mathbf{P}(T_X)^{-1} \circ f_* \circ \mathbf{P}(T_Y).$$

where T_Z is the tangent bundle of a smooth scheme Z. The first Chern is defined by the formula

$$c_1^H(L) = c_1(L) \cdot \mathbf{P}(L) = \sum_{i \geq 0} c_1(L)^{i+1} b_i.$$

Consider the power series

$$\exp t = t + b_1 t^2 + b_2 t^3 + \dots \in \mathbb{Z}[[\mathbf{b}, t]].$$

and its formal inverse

$$\log t = t + m_1 t^2 + m_2 t^3 + \dots \in \mathbb{Z}[[\mathbf{b}, t]]$$

where $m_i \in \mathbb{Z}[\mathbf{b}]$. Clearly,

$$c_1^H(L) = \exp c_1(L) \qquad \text{and} \qquad c_1(L) = \log c_1^H(L).$$

For any two line bundles L and L' on X we have

$$\begin{aligned} c_1^H(L \otimes L') &= \exp c_1(L \otimes L') \\ &= \exp\big(c_1(L) + c_1(L')\big) \\ &= \exp\big(\log c_1^H(L) + \log c_1^H(L')\big). \end{aligned}$$

Thus

$$\Phi_H(u,v) = \exp(\log u + \log v).$$

It is known that the corresponding ring homomorphisms

$$\mathbb{L} \to \mathbb{Z}[\mathbf{b}] = H_*(pt)$$

injective, hence so is

$$\mathbb{L} \to \Omega_*(pt).$$

We illustrate the proof of surjectivity on $\Omega_1(pt)$. Let $D = D_1 + D_2$ be a normal crossing divisor on a smooth surface S and let $L = L(D)$ be the corresponding line bundle. By **FGL** and **Sect**,

$$c_1(L) = c_1L(D_1) + c_1L(D_2) + a_{11}c_1L(D_1)c_1L(D_2),$$

$$c_1(L)(1_S) = [D_1 \to S] + [D_2 \to S] + a_{11}[pt \to S] \in \Omega_1(S)$$

and hence in $\Omega_1(pt)$:

$$c_1(L)[S] = [D_1] + [D_2] + a_{11} \equiv [D_1] + [D_2] \quad (mod\ L_1).$$

More generally, if $D = \sum n_i D_i$ and $D' = \sum n'_j D'_j$ are two normal crossing divisors on S such that $L(D) \simeq L(D')$ then in $\Omega_1(pt)$:

$$\sum n_i[D_i] \equiv \sum n'_j[D'_j] \quad (mod\ L_1). \tag{2}$$

Let Y be a smooth projective curve over F and let $\overline{Y} \subset \mathbb{P}^2$ be a projection of Y on the projective plane. By resolution of singularities, there is a projective morphism $f : S \to \mathbb{P}^2$ of a smooth projective surface S and a smooth projective curve $\widetilde{Y} \subset S$ such that the restriction of f on $\widetilde{Y}$ is a birational equivalence of $\widetilde{Y}$ and $\overline{Y}$. Clearly, $\widetilde{Y} \simeq Y$ Moreover, we may assume that

$$f^{-1}(\overline{Y}) = \widetilde{Y} + \sum n_i D_i$$

is a normal crossing divisor.

Let $D \subset \mathbb{P}^2$ be a smooth divisor rationally equivalent to $\overline{Y}$ such that $\widetilde{D} = f^{-1}(D)$ is isomorphic to D. Thus, $\widetilde{D}$ is a smooth divisor on S rationally equivalent to $\widetilde{Y} + \sum n_i D_i$. Hence, by (2), in $\Omega_1(pt)$,

$$[\widetilde{Y}] + \sum n_i[D_i] \equiv [\widetilde{D}].$$

Since $D_i \simeq \mathbb{P}^1$ and $[\mathbb{P}^1] = -a_{11}$, we get

$$[Y] = [\widetilde{Y}] \equiv [\widetilde{D}] = [D].$$

Let $H \subset \mathbb{P}^2$ be a projective line. We have $D \sim nH$ for some n and again by (2),

$$[D] \equiv n[H] = n[\mathbb{P}^1] \equiv 0,$$

i.e., $[Y] = [D] \in L_1$. In fact, one derives from the proof that

$$[Y] = (1-g)[\mathbb{P}^1] = (g-1)a_{11}$$

where g is the genus of Y.

In the general case the factorization theorem is used (that is why we assume $\operatorname{char} F = 0$).

4. Comparison with CH_* and the degree formula

Let X be a reduced scheme over F that has **RS** property. For every closed integral subscheme $Z \subset X$, choose a projective birational morphism $\widetilde{Z} \to Z$ with $\widetilde{Z} \in \mathbf{Sm}(F)$.

Theorem 4.1. *The $\mathbb{L}_*$-module $\Omega(X)$ is generated by the classes $\widetilde{Z} \to X$.*

Proof. Induction on $\dim X$. Let X_i, $i = 1, \dots, r$ be the irreducible components of X. Let $[f : Y \to X]$ be a class in $\Omega_*(X)$ and let $Y_i = f^{-1}(x_i)$ where $x_i \in X_i$ is the generic point. Every Y_i is a smooth scheme over $k(x_i)$. The class $[Y_i] \in \Omega_*(\operatorname{Spec} k(x_i)) = \mathbb{L}$ will be viewed as an element of $\mathbb{L}_*$.

The element

$$\alpha = [Y \to X] - \sum_{i=1}^{r} [Y_i] \cdot [\widetilde{X}_i \to X]$$

vanishes when restricted to a neighborhood $U \subset X$ of all generic points. Let $Z = X - U$ be the reduced complement. By **LOC**, there is $\beta \in \Omega_*(Z)$ such that $\alpha = i_*(\beta)$ where $i : Z \to X$ is the closed embedding. Since $\dim Z < \dim X$ we can apply the induction hypothesis to Z. □

Let $f : Y \to X$ be a projective morphism with Y smooth and X irreducible. The generic fiber $Y' \to \operatorname{Spec} F(X)$ represents an elements

$$\deg(f) \in \Omega_*\big(\operatorname{Spec} F(X)\big) = \mathbb{L}_*.$$

In particular, if f is finite, $\deg(f) \in \mathbb{L}_0 = \mathbb{Z}$ is the standard degree.

Corollary 4.2. (General degree formula) *There are elements $\alpha_Z \in \mathbb{L}_*$, almost all zero, such that*

$$[Y \to X] = \deg(f) \cdot 1_X + \sum_{Z,\ \operatorname{codim}_X Z > 0} \alpha_Z \cdot [\widetilde{Z} \to X].$$

Let $I_* = \mathbb{L}_{>0}$ be the kernel of the canonical ring homomorphism $\mathbb{L}_* \to \mathbb{Z}$.

Corollary 4.3. *Let $d = \dim X$. Then $\Omega_{>d}(X) \subset I \cdot \Omega(X)$.*

This corollary can be generalized:

Theorem 4.4. *Let F be a field of characteristic zero. Then for every $X \in \mathbf{Sch}(F)$, the canonical homomorphism*

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} = \Omega_*(X)/I_* \cdot \Omega_*(X) \to \mathrm{CH}_*(X)$$

is an isomorphism.

Corollary 4.5. *The Chow functor CH_* is the universal* **OBMF** *of geometric type with the additive formal group law.*

5. Comparison with K-theory

Theorem 5.1. *Let F admits* **RS**. *Then for every $X \in \mathbf{Sm}(F)$ the canonical homomorphism*

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K_0(X)[\beta, \beta^{-1}]$$

is an isomorphism.

Corollary 5.2. *The K-theory functor $K_0(X)[\beta, \beta^{-1}]$ on $\mathbf{Sm}(F)$ is the universal* **OBMF** *of geometric type with the multiplicative periodic formal group law.*

6. Pull-backs with respect to closed embeddings

The functor Ω_* admits pull-backs with respect to regular closed embeddings. In particular, we get the associated *cobordism cohomology* functor

$$\Omega^* : \mathbf{Sm}(F)^{op} \to \mathbf{Rings}^*.$$

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